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Research Article

The Analytic Solution of Time-Space Fractional Diffusion Equation via New Inner Product with Weighted Function

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Abstract. In this research, we determine the analytic solution of initial boundary value problem including time-space fractional differential equation with Dirichlet boundary conditions in one dimension. By using separation of variables the solution is constructed in the form of a Fourier series with respect to the eigenfunctions of a corresponding Sturm-Liouville eigenvalue problem including fractional derivative in Caputo sense. A new inner product with weighted function is defined to obtain coefficients in the Fourier series.

Keywords. Caputo fractional derivative; Space-fractional diffusion equation; Mittag-Leffler function; Initial-boundary-value problems; Spectral method

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1. Introduction

As PDEs of fractional order plays an important role in modelling for the numerous processes and systems in various scientific research areas such as applied mathematics, physics chemistry etc., the interest of this topic is increasing enourmously. Since the fractional derivative is non-local, the model with fractional derivative for physical problems turns out to be the best choice to analyze the behaviour of the complex non linear processes. That is why attracts increasing number of researchers. The derivatives in the sense of Caputo is one of the most common one since mathematical models with Caputo derivatives gives better results compare to the analysis of ones including other fractional derivatives. In literature increasing number of studies can be found supporting this conclusion ([1], [2],[3], [4], [5], [6], [7], [8], [11], [10], [11], [12], [13]). Moreover, the Caputo derivative of constant is zero which is not hold by many fractional derivatives. The solutions of fractional PDEs and ODEs are determined in terms of Mittag-Leffler function.

2. Preliminary Results

In this section, we recall fundamental definition and well known results about fractional derivative in Caputo sense.

Definition 1. The qth order fractional derivative of $u(t)$ in Caputo sense is defined as

$$D^q u(t) = \frac{1}{\Gamma(n-q)} \int_{t_0}^{t} (t-s)^{n-q-1} u^{(n)}(s)ds, \quad t \in [t_0, t_0+T], \tag{1}$$

where $u^{(n)}(t) = \frac{d^n u}{dt^n}$, $n-1 < q < n$. Note that Caputo fractional derivative is equal to integer order derivative when the order of the derivative is integer.

Definition 2. If $0 < q < 1$, the qth order Caputo fractional derivative is defined as

$$D^q u(t) = \frac{1}{\Gamma(1-q)} \int_{t_0}^{t} (t-s)^{-q} u'(s)ds, \quad t \in [t_0, t_0+T]. \tag{2}$$

The two-parameter Mittag–Leffler function which is taken into account in eigenvalue problem, is given by

$$E_{\alpha,\beta}\left(\lambda(t-t_0)^\alpha\right) = \sum_{k=0}^{\infty} \frac{(\lambda(t-t_0)^\alpha)^k}{\Gamma\left(\alpha k+\beta\right)}, \quad \alpha,\beta > 0 \tag{3}$$

including constant λ. Especially, for $t_0 = 0$, $\alpha = \beta = q$, we have

$$E_{\alpha,\beta}\left(\lambda t^q\right) = \sum_{k=0}^{\infty} \frac{(\lambda t^q)^k}{\Gamma(qk+q)}, \quad q > 0. \tag{4}$$

Mittag-Leffler function coincides with exponential function i.e., $E_{1,1}(\lambda t) = e^{\lambda t}$ for $q = 1$ (for details see [14, 15]).

Via the Mittag-Leffler function of two parameters, the following significant functions are defined as

$$\sin_q(\mu t^q) = \frac{E_{q,1}(i\mu t^q) - E_{q,1}(-i\mu t^q)}{2i} = \sum_{k=0}^{\infty} \frac{(-1)^k (\mu t^q)^{2k+1}}{\Gamma((2k+1)q+1)} \tag{5}$$

and

$$\cos_q(\mu t^q) = \frac{E_{q,1}(i\mu t^q) + E_{q,1}(-i\mu t^q)}{2} = \sum_{k=0}^{\infty} \frac{(-1)^k (\mu t^q)^{2k}}{\Gamma(2kq+1)}. \tag{6}$$

Note that for $q = 1$ these functions are usual trigonometric functions $\sin(\mu t)$ and $\cos(\mu t)$.

In this study, we deal with the following initial boundary value problem involving time and space-fractional PDE:

$$D_t^\alpha u(x,t;\alpha,\beta) = D_x^{2\beta} u(x,t;\alpha,\beta) + B D_x^\beta u(x,t;\alpha,\beta) - C u(x,t;\alpha,\beta),$$
$$0 < \alpha < 1, 1 < 2\beta < 2, 0 \le x \le l, 0 \le t \le T, B, C \in \mathbb{R} \tag{7}$$

$$u(0,t) = u(l,t;) = 0, \quad 0 \le t \le T \tag{8}$$

$$u(x,0) = f(x)e^{-\frac{B}{2}x}, \quad 0 \le x \le l \tag{9}$$

2.1 Inner product with weighted function

Let V be a vector space, produced of all linear combinations of $\sin_\beta\left(\mu\left(\frac{x}{b-a}\right)^\beta\right)E_{\beta,1}\left(-\frac{B}{2}x^\beta\right)$ and $\cos_\beta\left(\mu\left(\frac{x}{b-a}\right)^\beta\right)E_{\beta,1}\left(-\frac{B}{2}x^\beta\right)$ for fixed β where $B \in \mathbb{R}$ is fixed, $0 < \beta \le 1$ and $\mu \in \mathbb{R}$ on the interval $I = [a,b]$, i.e., $V = \operatorname{span}\left\{\sin_\beta\left(\mu\left(\frac{x}{b-a}\right)^\beta\right)E_{\beta,1}\left(-\frac{B}{2}x^\beta\right), \cos_\beta\left(\mu\left(\frac{x}{b-a}\right)^\beta\right)E_{\beta,1}\left(-\frac{B}{2}x^\beta\right)\right\}$.

Let $T : V \to \operatorname{span}\left\{\sin\left(\frac{\mu x}{b-a}\right)e^{-\frac{B}{2}x}, \cos\left(\frac{\mu x}{b-a}\right)e^{-\frac{B}{2}x}\right\}$ be a linear transformation which is one-to-one and onto. Thus it has its inverse transformation T^{-1}. The mapping $\langle \bullet, \bullet \rangle : V \times V \to \mathbb{R}$ is defined as

$$\langle u(x;\beta), v(x;\beta)\rangle = T^{-1}\int Tu(x;\beta).Tv(x;\beta)\rho(x)dx)\Big|_{x=a}^{b}, \tag{10}$$

where $Tu(x;\beta) = u(x;1)$, $Tv(x;\beta) = v(x;1)$ and $\rho(x) = e^{Bx}$.

3. Main Results

By means of separation of variables method. The generalized solution of above problem is constructed in analytical form. Thus a solution of problem (7)-(9) have the following form:

$$u(x,t;\alpha,\beta) = X(x;\beta)T(t;\alpha,\beta), \tag{11}$$

where $0 \le x \le l$, $0 \le t \le T$.

Note that the functions X and T depend on orders of fractional derivatives with respect to x and t. Plugging (11) into (7) and arranging it, we have

$$\frac{D_t^\alpha(T(t;\alpha,\beta))}{T(t;\alpha,\beta)} + C = \frac{D_x^{2\beta}(X(x;\beta)) + BD_x^\beta(X(x;\beta))}{X(x;\beta)} = -\lambda(\beta). \tag{12}$$

Note that the value of λ varies based on β. Equation (12) produces two fractional differential equations with respect to time and space. The first fractional differential equation is obtained by taking the equation on the right hand side of Eq. (12). Hence with boundary conditions (8), we have the following problem:

$$D_x^{2\beta}(X(x;\beta)) + BD_x^\beta(X(x;\beta)) + \lambda(\beta)X(x;\beta) = 0, \tag{13}$$

$$X(0;\beta) = X(l;\beta) = 0. \tag{14}$$

The solution of eigenvalue problem (13)-(14) is accomplished by making use of the Mittag-Leffler function of the following form:

$$X(x;\beta) = E_{\beta,1}(rx^\beta) \tag{15}$$

Hence the characteristic equation is computed in the following form:

$$r^2 + Br + \lambda(\beta) = 0. \tag{16}$$

Case 1. If $B^2 - 4\lambda(\beta) > 0$, the characteristic equation have two real and distinct solutions r_1, r_2 leading to the general solution of the eigenvalue problem (13)-(14) having the following form:

$$X(x;\beta) = c_1E_{\beta,1}(r_1x^\beta) + c_2E_{\beta,1}(r_2x^\beta).$$

By making use of the first boundary condition, we have

$$X(0;\beta) = c_1 + c_2 = 0 \Longrightarrow c_2 = -c_1. \tag{17}$$

Hence the solution becomes

$$X(x;\beta) = c_1(E_{\beta,1}(r_1 x^\beta) - E_{\beta,1}(r_2 x^\beta)). \tag{18}$$

Similarly, last boundary condition leads to

$$X(l;\beta) = c_1(E_{\beta,1}(r_1 l^\beta) - E_{\beta,1}(r_2 l^\beta)) = 0 \tag{19}$$

which implies that

$$E_{\beta,1}(r_1 l^\beta) \neq E_{\beta,1}(r_2 l^\beta). \tag{20}$$

Thus

$$c_1 = 0 \tag{21}$$

which means that there is no solution for the case $B^2 - 4\lambda(\beta) > 0$.

Case 2. $B^2 - 4\lambda(\beta) = 0$, the characteristic equation have two coincident roots $r_1 = r_2$, leading to the general solution of the eigenvalue problem (13)-(14) having the following form:

$$X(x;\beta) = c_1 E_{\beta,1}(r_1 x^\beta) + c_2 \frac{x^\beta}{\beta} E_{\beta,1}(r_1 x^\beta). \tag{22}$$

By making use of the first boundary condition, we have

$$X(0) = c_1 = 0 \tag{23}$$

Hence the solution becomes

$$X(x;\beta) = c_2 \frac{x^\beta}{\beta} E_{\beta,1}(r_1 x^\beta). \tag{24}$$

Similarly, second boundary condition leads to

$$X(l) = c_2 \frac{l^\beta}{\beta} E_{\beta,1}(r_1 l^\beta) \Longrightarrow c_2 = 0 \tag{25}$$

which leads to $X(x;\beta) = 0$ which means that there is no solution for $B^2 - 4\lambda(\beta) = 0$ as in the previous case.

Case 3: $B^2 - 4\lambda(\beta) < 0$, the characteristic equation have two complex roots $-\frac{B}{2} \mp i\frac{\sqrt{4\lambda(\beta)-B^2}}{2}$ leading to the general solution of the eigenvalue problem (13)-(14) having the following form:

$$X(x;\beta) = E_{\beta,1}\left(-\frac{B}{2}x^\beta\right)\left(c_1 \cos_\beta\left(\frac{\sqrt{4\lambda(\beta)-B^2}}{2}x^\beta\right) + i c_2 \sin_\beta\left(\frac{\sqrt{4\lambda(\beta)-B^2}}{2}x^\beta\right)\right). \tag{26}$$

By making use of the first boundary condition, we have

$$X(0) = c_1 = 0. \tag{27}$$

Hence the solution becomes

$$X(x;\beta) = E_{\beta,1}\left(-\frac{B}{2}x^\beta\right) i c_2 \sin_\beta\left(\frac{\sqrt{4\lambda(\beta)-B^2}}{2}x^\beta\right). \tag{28}$$

Similarly, second boundary condition leads to

$$X(l) = E_{\beta,1}\left(-\frac{B}{2}l^\beta\right) i c_2 \sin_\beta\left(\frac{\sqrt{4\lambda(\beta)-B^2}}{2}l^\beta\right) = 0 \tag{29}$$

which implies that

$$\sin_\beta\left(\frac{\sqrt{4\lambda(\beta)-B^2}}{2}l^\beta\right)=0. \tag{30}$$

Let $w_n(\beta)=\frac{\sqrt{4\lambda(\beta)-B^2}}{2}l^\beta$. Hence the eigenvalues can be represented in terms of $w_n(\beta)$ as follows:

$$\lambda_n(\beta)=\frac{4w_n^2(\beta)+(Bl^\beta)^2}{(2l^\beta)^2},\quad 0<w_1(\beta)<w_2(\beta)<w_3(\beta)<\dots. \tag{31}$$

Thus the solution of the eigenvalue problem is represented in the following form:

$$X_n(x;\beta)=c_n\sin_\beta\left(w_n(\beta)\left(\frac{x}{l}\right)^\beta\right)E_{\beta,1}\left(-\frac{B}{2}x^\beta\right),\quad n=1,2,3,\dots. \tag{32}$$

The equation on the left of (12) for each eigenvalue $\lambda_n(\beta)$ gives the following fractional differential equation:

$$\frac{D_t^\alpha(T(t;\alpha,\beta))}{T(t;\alpha,\beta)}=-(C+\lambda(\beta)). \tag{33}$$

By using the similar calculations the solution of (33) is determined in the following form:

$$\begin{aligned}T_n(t;\alpha,\beta)&=k_1E_{\alpha,1}(-(C+\lambda_n(\beta))t^\alpha)\\&=k_1E_{\alpha,1}\left(-\left(C+\frac{4w_n^2(\beta)+(Bl^\beta)^2}{(2l^\beta)^2}\right)t^\alpha\right),\quad n=1,2,3,\dots.\end{aligned} \tag{34}$$

For each eigenvalue $\lambda_n(\beta)$, we obtain the following solution:

$$\begin{aligned}u_n(x,t;\alpha,\beta)&=X_n(x;\beta)T_n(t;\alpha,\beta)\\&=d_n\sin_\beta\left(w_n(\beta)\left(\frac{x}{l}\right)^\beta\right)E_{\beta,1}\left(-\frac{B}{2}x^\beta\right)E_{\alpha,1}\left(-\left(C+\frac{4w_n^2(\beta)+\left(Bl^\beta\right)^2}{\left(2l^\beta\right)^2}\right)t^\alpha\right)\end{aligned} \tag{35}$$

and hence we have the following sum:

$$u(x,t;\alpha,\beta)=\sum_{n=1}^\infty d_n\sin_\beta\left(w_n(\beta)\left(\frac{x}{l}\right)^\beta\right)E_{\beta,1}\left(-\frac{B}{2}x^\beta\right)E_{\alpha,1}\left(-\left(C+\frac{4w_n^2(\beta)+\left(Bl^\beta\right)^2}{\left(2l^\beta\right)^2}\right)t^\alpha\right) \tag{36}$$

which satisfy both the fractional equation (7) and boundary condition (8).

In order to establish the solution which satisfies the initial condition (9), the inner product defined in (10) is used. In (36), replacing t by 0 and using the initial condition (10), we have

$$u(x,0)=f(x)e^{-\frac{B}{2}x}=\sum_{n=1}^\infty d_n\sin_\beta\left(w_n(\beta)\left(\frac{x}{l}\right)^\beta\right)E_{\beta,1}\left(-\frac{B}{2}x^\beta\right) \tag{37}$$

$$\begin{aligned}\implies\quad d_n&=\frac{2}{l}\left\langle\sin_\beta\left(w_n(\beta)\left(\frac{x}{l}\right)^\beta\right)E_{\beta,1}\left(-\frac{B}{2}x^\beta\right),f(x)e^{-\frac{B}{2}x}\right\rangle\\&=\frac{2}{l}T^{-1}\left(\int T\left[\{\sin_\beta\left(w_n(\beta)\left(\frac{x}{l}\right)^\beta\right)E_{\beta,1}\left(-\frac{B}{2}x^\beta\right)\right]f(x)e^{-\frac{B}{2}x}\rho(x)dx\right)\Bigg|_{x=0}^{x=l}\\&=\frac{2}{l}T^{-1}\left(\int T\left[\sin_\beta\left(w_n(\beta)\left(\frac{x}{l}\right)^\beta\right)E_{\beta,1}\left(-\frac{B}{2}x^\beta\right)\right]f(x)e^{-\frac{B}{2}x}e^{Bx}dx\right)\Bigg|_{x=0}^{x=l}\\&=\frac{2}{l}T^{-1}\left(\int\left[\sin\left(\frac{n\pi x}{l}\right)\ e^{-\frac{B}{2}x}\right]f(x)e^{-\frac{B}{2}x}e^{Bx}dx\right)\Bigg|_{x=0}^{x=l}\end{aligned}$$

Via the inner product (10) we obtain the coefficients d_n for $n = 1,2,3,\ldots$ as follows:

$$d_n = \frac{2}{l}T^{-1}\left(\int\left[\sin\left(\frac{n\pi x}{l}\right)f(x)\right]dx\right)\Big|_{x=0}^{x=l}, n = 1,2,3,\ldots \tag{38}$$

4. Illustrative Example

In this section, we first consider the following initial boundary value problem:

$$u_t = u_{xx} + u_x - u, 0 \le x \le 2, t \ge 0$$
$$u(0,t) = 0,\ u(2,t) = 0, t \ge 0 \tag{39}$$
$$u(x,0) = -\sin(\pi x)e^{-\frac{1}{2}x} 0 \le x \le 2$$

which has the solution in the following form:

$$u(x,t) = -\sin(\pi x)e^{-\frac{1}{2}x}e^{-(\pi^2+\frac{5}{4})t}. \tag{40}$$

Now, let us take the following fractional heat-like problem into consideration:

$$D_t^\alpha u(x,t) = D_x^{2\beta}u(x,t) + D_x^\beta u(x,t) - u(x,t),\quad 0 < \alpha < 1,\ 1 < 2\beta < 2, 0 \le x \le 1,\ 0 \le t \le T \tag{41}$$

$$u(0,t) = u(2,t) = 0,\quad 0 \le t \le T \tag{42}$$

$$u(x,0) = \sin(\pi x)e^{-\frac{1}{2}x},\quad 0 \le x \le 1. \tag{43}$$

Applying separation of the variables to (41) leads to the equation

$$\frac{D_t^\alpha(T(t;\alpha,\beta))}{T(t;\alpha,\beta)} + 1 = \frac{D_x^{2\beta}(X(x;\beta)) + D_x^\beta(X(x;\beta))}{X(x;\beta)} = -\lambda(\beta). \tag{44}$$

Equation (44) produces two fractional differential equations with respect to time and space. The first fractional differential equation is obtained by taking the equation on the right hand side of eq. (44). Hence with boundary conditions (42), we have the following problem:

$$D_x^{2\beta}(X(x;\beta)) + D_x^\beta(X(x;\beta)) + \lambda(\beta)X(x;\beta) = 0, \tag{45}$$

$$X(0) = 0,\ X(2) = 0. \tag{46}$$

Using the Mittag-Leffler function $X(x;\beta) = E_{\beta,1}(rx^\beta)$ we obtain the following characteristic equation $r^2 + r + \lambda(\beta) = 0$. Same as the problem (13)-(14). The solution becomes as follows:

$$X(x;\beta) = E_{\beta,1}\left(-\frac{1}{2}x^\beta\right)\left(c_1\cos_\beta\left(\frac{\sqrt{4\lambda(\beta)-1}}{2}x^\beta\right) + ic_2\sin_\beta\left(\frac{\sqrt{4\lambda(\beta)-1}}{2}x^\beta\right)\right). \tag{47}$$

By making use of the first boundary condition we have

$$X(0;\beta) = 0 = c_1. \tag{48}$$

Hence the solution becomes

$$X(x;\beta) = E_{\beta,1}\left(-\frac{1}{2}x^\beta\right)\ ic_2\sin_\beta\left(\frac{\sqrt{4\lambda(\beta)-1}}{2}x^\beta\right). \tag{49}$$

Similarly, second boundary condition leads to

$$X(1;\beta) = 0 = E_{\beta,1}\left(-\frac{1}{2}2^\beta\right)ic_2\sin_\beta\left(\frac{\sqrt{4\lambda(\beta)-1}}{2}2^\beta\right) \tag{50}$$

which implies that

$$\{sin_\beta\left(\frac{\sqrt{4\lambda(\beta)-1}}{2}2^\beta\right)=0. \tag{51}$$

Let $w_n(\beta)=\frac{\sqrt{4\lambda_n(\beta)-1}}{2}2^\beta$. The solutions of (44) can be denoted by means of $w_n(\beta)$ which are eigenvalues of the problem (46)-(47), as follows:

$$\lambda_n(\beta)=\frac{4w_n^2(\beta)+2^{2\beta}}{2^{2\beta+2}}, \quad 0<w_1(\beta)<w_2(\beta)<w_3(\beta)<\dots. \tag{52}$$

As a result

$$X_n(x;\beta)=c_n\sin_\beta\left(w_n(\beta)\left(\frac{x}{2}\right)^\beta\right)E_{\beta,1}\left(-\frac{1}{2}x^\beta\right) \tag{53}$$

represents the solution of the eigenvalue problem.

The equation on the left of (44) for each eigenvalue $\lambda_n(\beta)$ gives the following fractional differential equation:

$$D_t^\alpha(T_n(t;\alpha,\beta))+\left(\frac{4w_n^2(\beta)+2^{2\beta}}{2^{2\beta+2}}+1\right)T_n(t;\alpha,\beta)=0 \tag{54}$$

which has the following solutions

$$T_n(t;\alpha,\beta)=k_1E_{\alpha,1}\left(-\left(1+\frac{4w_n^2(\beta)+2^{2\beta}}{2^{2\beta+2}}\right)t^\alpha\right), \quad n=1,2,3,\dots. \tag{55}$$

As a result the specific solutions of problem (41)-(43) can be written as

$$u_n(x,t;\alpha,\beta)=d_n\sin_\beta\left(w_n(\beta)\left(\frac{x}{2}\right)^\beta\right)E_{\beta,1}\left(-\frac{1}{2}x^\beta\right)E_{\alpha,1}\left(-\left(1+\frac{4w_n^2(\beta)+2^{2\beta}}{2^{2\beta+2}}\right)t^\alpha\right) \tag{56}$$

which leads to following general solution of problem (41)-(43)

$$u(x,t;\alpha,\beta)=\sum_{n=1}^{\infty}d_n\sin_\beta\left(w_n(\beta)\left(\frac{x}{2}\right)^\beta\right)E_{\beta,1}\left(-\frac{1}{2}x^\beta\right)E_{\alpha,1}\left(-\left(1+\frac{4w_n^2(\beta)+2^{2\beta}}{2^{2\beta+2}}\right)t^\alpha\right). \tag{57}$$

Note that the general solution (57) satisfy both boundary conditions (42) and the fractional equation (41).

By making use of the inner product defined in (10), we determine the coefficients d_n in such a way that the general solution (57) satisfies the initial condition (43). Plugging $t=0$ in to the general solution (57) and making equal to the initial condition (43) we have

$$u(x,0)=-\sin(\pi x)e^{-\frac{1}{2}x}=\sum_{n=1}^{\infty}d_n\sin_\beta\left(w_n(\beta)\left(\frac{x}{2}\right)^\beta\right)E_{\beta,1}\left(-\frac{1}{2}x^\beta\right). \tag{58}$$

Via the inner product we obtain the coefficients d_n for $n=1,2,3,\dots$ as follows:

$$\begin{aligned}
d_n&=\frac{2}{2}T^{-1}\left(\int T\left[\sin_\beta\left(w_n(\beta)\left(\frac{x}{2}\right)^\beta\right)E_{\beta,1}\left(-\frac{1}{2}x^\beta\right)\right](-\sin(\pi x))e^{-\frac{1}{2}x}\rho(x)dx\right)\Bigg|_{x=0}^{x=2}\\
&=T^{-1}\left(\int\left[\sin\left(\frac{n\pi x}{2}\right)e^{-\frac{1}{2}x}\right](-\sin(\pi x))e^{-\frac{1}{2}x}e^xdx\right)\Bigg|_{x=0}^{x=2}\\
&=T^{-1}\left(\int\sin\left(\frac{n\pi x}{2}\right)(-\sin(\pi x))dx\right)\Bigg|_{x=0}^{x=2}.
\end{aligned}$$

Thus $d_n=0$ for $n\neq 2$.

For $n=2$, we get

$$\begin{aligned} d_2 &= T^{-1}\left(-\int \sin^2(\pi x)dx\right)\Big|_{x=0}^{x=2} \\ &= T^{-1}\left(-\frac{1}{2}\left(x+\frac{\sin(2\pi x)}{4\pi}\right)\right)\Big|_{x=0}^{x=2} \\ &= -\frac{1}{2}x^\beta + \frac{\sin_\beta\left(w_4(\beta)\left(\frac{x}{2}\right)^\beta\right)}{w_4(\beta)}\Bigg|_{x=0}^{x=2} = -2^{\beta-1}. \end{aligned} \tag{59}$$

Thus

$$u(x,t;\alpha,\beta) = -2^{\beta-1}\sin_\beta\left(w_2(\beta)\left(\frac{x}{2}\right)^\beta\right)E_{\beta,1}\left(-\frac{1}{2}x^\beta\right)E_{\alpha,1}\left(-\left(1+\frac{4w_2^2(\beta)+2^{2\beta}}{2^{2\beta+2}}\right)t^\alpha\right). \tag{60}$$

It is important to note that plugging $\alpha=\beta=1$ in to the solution (60) gives the solution (40) which confirm the accuracy of the method we apply.

5. Conclusion

In this research, the analytic solution of initial boundary value problem with Dirichlet boundary conditions in one dimension is constructed. By making use of separation of variables the solution is formed in the form of a Fourier series with respect to the eigenfunctions of a corresponding Sturm-Liouville eigenvalue problem including fractional derivative in Caputo sense. Because of the structure of the solution the inner product with weighted function is utilized which allows us to determine the coefficients in the series form of the solution without any difficulty.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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