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Commutativity of Involutorial Rings with Constraints on Left Multipliers

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Abstract. Let $(R,*)$ be a ring with involution and let $Z(R)$ be the center of R. The purpose of this paper is to explore the commutativity of R if it admits a left multiplier F satisfying certain identities on Lie ideals. Furthermore, some results for left multipliers in prime rings are extended to Lie ideals. Finally, examples are given to prove that the restrictions imposed on the hypothesis of the various theorems were not superfluous.

1. **Introduction**

Throughout this paper, R will represent an associative ring with center $Z(R)$. For any $x, y \in R$, the symbol $[x, y]$ denotes the commutator $xy - yx$. Recall that R is said to be 2-torsion free if whenever $2x = 0$, with $x \in R$, then $x = 0$. R is prime if $aRb = 0$ implies $a = 0$ or $b = 0$. If R is equipped with an involution $*$, then R is $*$-prime if $aRb = aRb^* = 0$ yields $a = 0$ or $b = 0$. Note that every prime ring having an involution $*$ is $*$-prime but the converse is in general not true. For example, if R^o denotes the opposite ring of a prime ring R, then $R \times R^o$, equipped with the exchange involution $*_{ex}$, defined by $*_{ex}(x, y) = (y, x)$, is $*_{ex}$-prime but not prime. This example shows that every prime ring can be injected in a $*$-prime ring and from this point of view $*$-prime rings constitute a more general class of prime rings.

An additive subgroup U of R is said to be a Lie ideal of R if $[u, r] \in U$ for all $u \in U$ and $r \in R$. A Lie ideal U which satisfies $U^* = U$ is called a $*$-Lie ideal. If U is a Lie (resp., $*$-Lie) ideal of R, then U is called a square closed Lie (resp., $*$-Lie) ideal if $u^2 \in U$ for all $u \in U$. The fact that $uv + vu = (u + v)^2 - u^2 - v^2 \in U$ together with $[u, v] \in U$, implies that $2uv \in U$ for all $u, v \in U$.

An additive mapping $F : R \to R$ is called a left multiplier if $F(xy) = F(x)y$ for all x, y in R. A derivation of R is an additive mapping d satisfying $d(xy) = d(x)y + xd(y)$ for all x, y in R. An additive mapping $F : R \to R$ is called a generalized

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derivation if there exists a derivation $d : R \to R$ such that $F(xy) = F(x)y + xd(y)$ holds for all $x, y \in R$. Hence generalized derivation covers both the concepts of derivation and generalized inner derivation. Moreover, generalized derivation with $d = 0$ covers the concept of left multipliers.

There has been an ongoing interest concerning the relationship between the commutativity of a prime ring R and the behavior of a generalized derivation of R, with associated *nonzero* derivation. Many of obtained results extend other ones proven previously just for the action of the generalized derivation on the whole ring. In this direction, it seems natural to ask what we can say about the commutativity of R if the generalized derivation is replaced by a left multiplier. Our aim in this paper is to investigate the commutativity of a ring with involution $(R, *)$ satisfying certain identities involving left multiplier acting on Lie ideals.

2. Main Results

Throughout, $(R, *)$ will be a 2-torsion free ring with involution and $Sa_*(R) := \{r \in R / r^* = \pm r\}$ the set of symmetric and skew symmetric elements of R.

Lemma 1 ([7, Lemma 4]). *If $U \not\subseteq Z(R)$ is a $*$-Lie ideal of a 2-torsion free $*$-prime ring R and $a, b \in R$ such that $aUb = a^*Ub = 0$, then $a = 0$ or $b = 0$.*

Lemma 2 ([8, Lemma 2.3]). *Let $0 \neq U$ be a $*$-Lie ideal of a 2-torsion free $*$-prime ring R. If $[U, U] = 0$, then $U \subseteq Z(R)$.*

Theorem 1. *Let U be a square closed $*$-Lie ideal of R and F a left multiplier such that $F(xy) - xy \in Z(R)$ for all $x, y \in U$. If R is $*$-prime, then F is trivial or $U \subseteq Z(R)$.*

Proof. Assume that $U \not\subseteq Z(R)$. From $F(xy) - xy \in Z(R)$ it follows that

$$F(x)y - xy \in Z(R) \quad \text{for all } x, y \in U. \tag{1}$$

Hence $[F(x)y - xy, r] = 0$ for all $r \in R$ so that

$$(F(x) - x)[y, r] + [F(x) - x, r]y = 0. \tag{2}$$

Writing $2xu$ instead of x in (2) and using (1) we find that $(F(x) - x)u[y, r] = 0$ and therefore

$$(F(x) - x)U[y, r] = 0 \ \text{ for all } x, y \in U, \ r \in R. \tag{3}$$

Since U is a $*$-ideal, from (3) it follows that

$$(F(x) - x)U[y, r]^* = 0. \tag{4}$$

As $U \not\subseteq Z(R)$, (3) together with (4) assure that

$$F(x) = x \quad \text{for all } x \in U. \tag{5}$$

Let $r \in R$, for all $x \in U$ the fact that $F([u, r]) = [u, r]$ implies that

$$(F(r) - r)u = 0 \ \text{ for all } u \in U.$$

Replacing u by $2uw$ in (5), where $w \in U$, we get $(F(r)-r)uw = 0$ and thus

$$(F(r)-r)Uw = 0 \quad \text{for all} \quad w \in U. \tag{6}$$

Since $U^* = U$, the relation (6) forces

$$(F(r)-r)Uw^* = 0 \quad \text{for all} \quad w \in U. \tag{7}$$

Applying Lemma 1, from (6) and (7) it follows that $F(r) = r$ for all $r \in R$. □

Theorem 2. *Let U be a square closed $*$-Lie ideal of R and F a left multiplier such that $F(xy)+xy \in Z(R)$ for all $x, y \in U$. If R is $*$-prime, then $-F$ is trivial or $U \subseteq Z(R)$.*

Proof. If F is a left multiplier satisfying the property $F(xy)+xy \in Z(R)$ for all $x, y \in U$, then the left multiplier $(-F)$ satisfies the condition $(-F)(xy)-xy \in Z(R)$ for all $x, y \in U$ and hence Theorem 1 forces $U \subseteq Z(R)$. □

Remark. If we choose the underlying subset as a $*$-ideal instead of a square closed $*$-Lie ideal, then Theorems 1 and 2 remain valid even without the characteristic assumption on the ring.

Corollary 1. *Let R be a $*$-prime ring and I a nonzero $*$-ideal of R. If R admits a left multiplier F such that F (resp., $(-F)$) is nontrivial and $F(xy)-xy \in Z(R)$ (resp., $F(xy)+xy \in Z(R)$) for all $x, y \in I$, then R is commutative.*

The following example proves that Theorems 1 and 2 are not true in the case of arbitrary rings.

Example 1. Let $\mathbb{R}$ be the ring of real numbers. Set $R = \left\{ \begin{pmatrix} x & y \\ 0 & z \end{pmatrix} \middle| \, x, y, z \in \mathbb{R} \right\}$ and $U = \left\{ \begin{pmatrix} 0 & y \\ 0 & 0 \end{pmatrix} \middle| \, y \in \mathbb{R} \right\}$. If we set $\begin{pmatrix} x & y \\ 0 & z \end{pmatrix}^* = \begin{pmatrix} z & -y \\ 0 & x \end{pmatrix}$ and $F\begin{pmatrix} x & y \\ 0 & z \end{pmatrix} = \begin{pmatrix} x & y-z \\ 0 & z \end{pmatrix}$, then it is straightforward to check that $*$ is an involution and F is a nontrivial left multiplier. Moreover, U is a square closed $*$-Lie ideal of R such that $F(xy)-xy \in Z(R)$ for all $x, y \in U$ and $F(xy)+xy \in Z(R)$ for all $x, y \in U$; but $U \nsubseteq Z(R)$. Accordingly, the $*$-primeness hypothesis in Theorems 1 and 2 is crucial.

The following theorem extends Theorem 3.1 and Theorem 3.2 of [1] to Lie ideals.

Theorem 3. *Let R be a 2-torsion free prime ring and U a square closed Lie ideal of R. If R admits a left multiplier F such that $F(xy)-xy \in Z(R)$ (resp., $F(xy)+xy \in Z(R)$) for all $x, y \in U$, then F (resp., $(-F)$) is trivial or $U \subseteq Z(R)$.*

Proof. Assume that $F(xy)-xy \in Z(R)$ for all $x, y \in U$. Let $\mathscr{F}$ be the left multiplier defined on the $*_{ex}$-prime ring $\mathscr{R} = R \times R^0$ by $\mathscr{F}(x, y) = (F(x), y)$. It we set $W = U \times U$, then W is a square closed $*_{ex}$-Lie ideal of $\mathscr{R}$. Moreover, from

$$\mathscr{F}((x,y)(u,v)) - (x,y)(u,v) = (F(xu)-xu, 0) \quad \text{for all } (x,y),(u,v) \in W$$

it follows that $\mathscr{F}((x,y)(u,v)) - (x,y)(u,v) \in Z(\mathscr{R})$. Applying Theorem 1, either $\mathscr{F}$ is trivial or $W \subseteq Z(\mathscr{R})$. Accordingly, F is trivial or $U \subseteq Z(R)$.

If $F(xy) + xy \in Z(R)$ for all $x, y \in U$; then $\mathscr{F}(x,y) = (F(x), -y)$ is a left multiplier on $\mathscr{R}$ such that $\mathscr{F}(xy) + yx \in Z(\mathscr{R})$ for all $x, y \in \mathscr{R}$. Reasoning as above and using Theorem 2, we get the required result. □

As an application of Theorem 3, we obtain the following results.

Corollary 2. *Let R be a prime ring and I a nonzero ideal of R. If R admits a left multiplier F such that F (resp., $(-F)$) is nontrivial and $F(xy) - xy \in Z(R)$ (resp., $F(xy) + xy \in Z(R)$) for all $x, y \in I$, then R is commutative.*

Theorem 4. *Let U be a square closed $*$-Lie ideal of R and F a left multiplier such that $F(xy) - yx \in Z(R)$ for all $x, y \in U$. If R is $*$-prime, then $U \subseteq Z(R)$.*

Proof. Assume that $U \not\subseteq Z(R)$. From $F(xy) - yx \in Z(R)$ it follows that

$$F(x)y - yx \in Z(R) \quad \text{for all} \quad x, y \in U.$$

Hence $[F(x)y - yx, u] = 0$ for all $u \in U$ so that

$$[F(x), u]y + F(x)[y, u] = y[x, u] + [y, u]x. \tag{8}$$

Replacing y by $2yu$ in (8), we conclude that

$$[y, u]xu + y[x, u]u - [y, u]ux - yu[x, u] = 0$$

and therefore

$$[y, u][x, u] + y[[x, u], u] = 0. \tag{9}$$

Substituting $2wy$ for y in (9) and employing (9) we obtain

$$[w, u]y[x, u] = 0 \quad \text{for all} \quad w, u, x, y \in U.$$

Consequently,

$$[w, u]U[x, u] = 0 \quad \text{for all} \quad w, u, x \in U. \tag{10}$$

If $u \in U \cap Sa_*(R)$; then (10) yields

$$[w, u]U[x, u]^* = 0 \quad \text{for all} \quad w, u, x \in U. \tag{11}$$

By Lemma 1, equations (10) and (11) give $[u, U] = 0$ for all $u \in U \cap Sa_*(R)$. Let $u \in U$, as $u^* - u \in U \cap Sa_*(R)$, then $[x, u^* - u] = 0$ for all $x \in U$ and therefore

$$[x, u] = [x, u^*] \quad \text{for all} \quad u, x \in U. \tag{12}$$

Writing u^* instead of u in (10) and using (12) we find that

$$[w, u]U[x, u^*] = 0 \quad \text{for all} \quad w, u, x \in U. \tag{13}$$

Since $U^* = U$, from (13) it follows that

$$[w, u]U[x, u]^* = 0 \quad \text{for all} \quad w, x \in R. \tag{14}$$

In view of Lemma 1, equations (10) and (14) give $[y, u] = 0$ for all $y, u \in U$. Hence, $[U, U] = 0$ which, in view of Lemma 2, contradicts $U \not\subseteq Z(R)$. □

Using similar arguments as above, we can prove the following:

Theorem 5. *Let U be a square closed $*$-Lie ideal of R and F a left multiplier such that $F(xy)+yx \in Z(R)$ for all $x,y \in U$. If R is $*$-prime, then $U \subseteq Z(R)$.*

Example 2. In hypothesis of Theorem 4 and Theorem 5 the $*$-primeness condition is necessary. Indeed, in Example 1 it is clear that $F(xy) - yx \in Z(R)$ and $F(xy)+yx \in Z(R)$ for all $x,y \in U$, but $U \nsubseteq Z(R)$.

Corollary 3. *Let R be a $*$-prime ring and I a nonzero $*$-ideal of R. If R admits a left multiplier F such that $F(xy)-yx \in Z(R)$ (or $F(xy)+yx \in Z(R)$) for all $x,y \in I$, then R is commutative.*

Theorem 6. *Let U be a square closed $*$-Lie ideal of R and F a left multiplier such that $F(x)F(y)-yx \in Z(R)$ for all $x,y \in U$. If R is $*$-prime, then $U \subseteq Z(R)$.*

Proof. Assume that $U \nsubseteq Z(R)$ and

$$F(x)F(y)-yx \in Z(R) \text{ for all } x,y \in U. \tag{15}$$

Replacing y by yu in (15), we obtain

$$(F(x)F(y)-yx)u + y[x,u] \in Z(R) \text{ for all } u,x,y \in U$$

so that

$$[(F(x)F(y)-yx)u + y[x,u],u] = 0 \text{ for all } u,x,y \in U,$$

which, by view of (15), leads us to $[y[x,u],u]=0$ and hence

$$[y,u][x,u] + y[[x,u],u] = 0 \text{ for all } u,x,y \in U. \tag{16}$$

Replace y by wy in (16) and then employ (16) to obtain

$$[w,u]y[x,u] = 0 \text{ for all } w,u,x,y \in U.$$

Hence

$$[w,u]U[x,u] = 0 \text{ for all } w,u,x \in U. \tag{17}$$

Since equation (17) is the same as equation (10), arguing as in the proof of Theorem 4, we obtain $[U,U]=0$ which, by view of Lemma 2, contradicts the fact that $U \nsubseteq Z(R)$. $\square$

Using similar arguments as used in the proof of Theorem 6 we can prove the following result.

Theorem 7. *Let U be a square closed $*$-Lie ideal of R and F a left multiplier such that $F(x)F(y)+yx \in Z(R)$ for all $x,y \in U$. If R is $*$-prime, then $U \subseteq Z(R)$.*

Example 3. In Theorem 6 and Theorem 7 the $*$-primeness condition cannot be omitted. Indeed, in Example 1, the left multiplier F satisfies $F(x)F(y)-yx \in Z(R)$ and $F(x)F(y)+yx \in Z(R)$ for all $x,y \in U$, but $U \nsubseteq Z(R)$.

As an application of Theorems 6 and 7, we obtain the following results.

Corollary 4. *Let R be a $*$-prime ring and I a nonzero $*$-ideal of R. If R admits a left multiplier F such that $F(x)F(y) - yx \in Z(R)$ or $F(x)F(y) + yx \in Z(R)$ for all $x, y \in I$, then R is commutative.*

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