Communications in Mathematics and Applications
Vol. 10, No. 4, pp. 659–671, 2019
ISSN 0975-8607 (online); 0976-5905 (print)
Published by RGN Publications

http://www.rgnpublications.com

DOI: 10.26713/cma.v10i4.1256

Research Article

On Ternary Monoid of All Hypersubstitutions of Type $\tau = (2)$

Nagornchat Chansuriya[1,*] and Sorasak Leeratanavalee[2]

[1] *Faculty of Science, Energy and Environment, King Mongkut's University of Technology North Bangkok, Rayong Campus, Rayong 21120, Thailand*
[2] *Department of Mathematics, Faculty of Science, Chiang Mai University, Chiang Mai 50200, Thailand*
*__Corresponding author:__ nagornchat.c@sciee.kmutnb.ac.th

Abstract. The present paper gives the concept of a ternary monoid $Hyp(2)$ and studies some algebraic-structural properties of this monoid. We consider some submonoids of $Hyp(2)$ and study the relationship between these submonoids.

Keywords. Hypersubstitutions; Ternary semigroup; Ternary ideal

MSC. 20M05

Received: May 29, 2019 **Accepted:** August 16, 2019

1. Introduction

Ternary semigroups are special case of n-ary semigroups, where $n = 3$, which firstly investigated by E. Kasner in 1904 [8]. After that, F. M. Sioson described a regular algebraic systems with respect to the m-ary operation in 1963 [14], and introduced the ideal theory in ternary semigroups in 1965 [12], [13]. In 1980, W. A. Dudek and I. Groździńska [4] introduced a new definition of regular n-semigroups and proved some theorems in that structure. Later, the idempotents in n-ary semigroups were described by W. A. Dudek in 2001 [3]. In 2007, S. Kar and B. K. Maity [7] introduced the notion of a congruence on a ternary semigroup and studied some of its interesting properties. Moreover, they also introduced the notions of cancellative congruence, group congruence and Rees congruence and characterized these congruences on ternary semigroups. After that, in 2010, M. L. Santiago and S. Sri Bala [10] introduced a regularity condition on a ternary semigroup and studied the properties of a regular semigroup.

The monoid of all hypersubstitutions of type $\tau = (2)$ was studied by K. Denecke and Sh. L. Wismath [2] in 1998. They characterized idempotent elements of the monoid of all hypersubstitutions of type $\tau = (2)$. Subsequently, in the same year, K. Denecke and J. Koppitz [1] studied the finite monoid of hypersubstitutions of type $\tau = (2)$. They determined all finite submonoids of $Hyp(2)$ and studied properties of all finite submonoids.

Let $\{f_i \mid i \in I\}$ be the set of all n_i-ary operation symbols which $n_i \in \mathbf{N}$ and $W_\tau(X_n)$ be the set of all *n-ary terms* of type τ constructed by operation symbols from $\{f_i \mid i \in I\}$ and variables from an n-element set $X_n := \{x_1,\dots,x_n\}$. A hypersubstitution of type τ is a mapping $\sigma : \{f_i \mid i \in I\} \to W_\tau(X_n)$ which preserves arities. We denoted by $Hyp(\tau)$ the set of all hypersubstitutions of type τ.

To define the binary operation on $Hyp(\tau)$, we necessarily define the extension $\hat{\sigma}$ of σ. Firstly, the concept of superposition of terms $S^n_m : W_\tau(X_n) \times W_\tau(X_m)^n \to W_\tau(X_m)$, where $W_\tau(X_n), W_\tau(X_m)$ are the sets of all n-ary and m-ary terms of type τ for $n,m \geq 1$ and $n,m \in \mathbf{N}$ is inductively defined as follows:

(i) $S^n_m(x_i, t_1,\dots,t_n) := t_i$, where $x_i \in X_n, t_1,\dots,t_n \in W_\tau(X_m)$.

(ii) $S^n_m(f_i(s_1,\dots,s_{n_i}), t_1,\dots,t_n) := f_i(S^n_m(s_1,t_1,\dots,t_n),\dots,S^n_m(s_{n_i},t_1,\dots,t_n))$, where $f_i(s_1,\dots,s_{n_i}) \in W_\tau(X_n)$.

Now, the extension $\hat{\sigma}$ of σ is a mapping $\hat{\sigma} : W_\tau(X_n) \to W_\tau(X_n)$ which defined inductively by

(1) for any variable $x \in X$, $\hat{\sigma}[x] := x$, and

(2) $\hat{\sigma}[f_i(t_1,\dots,t_{n_i})] := S^{n_i}_m(\sigma(f_i), \hat{\sigma}[t_1],\dots,\hat{\sigma}[t_{n_i}])$, where $\hat{\sigma}[f_j]; 1 \leq j \leq n_i$ are already defined.

Then the binary operation on $Hyp(\tau)$, denoted by $\circ_h$, is defined by $\sigma_1 \circ_h \sigma_2 = \hat{\sigma}_1 \circ \sigma_2$ where $\circ$ is the usual composition of mappings and $\sigma_1, \sigma_2 \in Hyp(\tau)$.

Proposition 1 ([9]). *If $\hat{\sigma}$ is the extension of a hypersubstitution σ, then for $n,m \in \mathbf{N}$,*

$$\hat{\sigma}[S^n_n(t,t_1,\dots,t_n)] = S^n_n(\hat{\sigma}[t], \hat{\sigma}[t_1],\dots,\hat{\sigma}[t_n]).$$

Proposition 2 ([9]). *Let $\sigma_1, \sigma_2 \in Hyp(\tau)$. Then $\hat{\sigma}_1 \circ \sigma_2$ is a hypersubstitution, and*

$$(\hat{\sigma}_1 \circ \sigma_2)\hat{} = \hat{\sigma}_1 \circ \hat{\sigma}_2.$$

Let σ_{id} be a hypersubstitution which is defined by $\sigma_{id}(f_i) = f_i(x_1,\dots,x_{n_i})$ for all $i \in I$. Then the set $\underline{Hyp(\tau)} := (Hyp(\tau), \circ_h, \sigma_{id})$ forms a monoid. For more detail of hypersubstitution and the monoid of hypersubstitutions of type τ (see in [9]).

In this paper, we first give the concept of a ternary monoid $Hyp(2)$ and study some algebraic-structural properties of this monoid. Finally, we consider some finite submonoids of $Hyp(2)$ and show that it is a ternary ideals.

Throughout this paper, we denote:

$\sigma_t :=$ the hypersubstitution σ of type τ which maps f to the term t,

$var(t) :=$ the set of all variables occurring in the term t,

$W_{(2)}(\{x_1,x_2\}) := \{t \in W_{(2)}(X_2) \mid x_1, x_2 \in var(t)\}$,

$W_{(2)}(\{x_1\}) := \{t \in W_{(2)}(X_2) \mid x_1 \in var(t), x_2 \notin var(t)\}$,

$W_{(2)}(\{x_2\}) := \{t \in W_{(2)}(X_2) \mid x_1 \notin var(t), x_2 \in var(t)\}$,

$E_{x_1} := \{\sigma_{f(x_1,u)} \mid u \in W_{(2)}(\{x_1\})\}$,

$E_{x_2} := \{\sigma_{f(v,x_2)} \mid v \in W_{(2)}(\{x_2\})\}$,

$op(s) :=$ the number of all operation symbols occurring in the term s.

Corollary 1 ([9]). *For any* $t \in W_\tau(X_n)$, $t_1, t_2, \dots, t_n \in W_\tau(X_m)$, *we get*

$$op(S^n_m(t, t_1 \dots t_n)) \ge op(t).$$

If $\tau = (2)$, then we have the following theorems.

Theorem 1 ([9]). *A hypersubstitution* $\sigma \in Hyp(2)$ *is an idempotent element if and only if* $\sigma \in E_{x_1} \cup E_{x_2} \cup \{\sigma_{x_1}, \sigma_{x_2}, \sigma_{id}\}$.

Theorem 2 ([9]). *If* $\sigma \in Hyp(2)$ *is a regular element, then* σ *is in* $E_{x_1} \cup E_{x_2} \cup \{\sigma_{x_1}, \sigma_{x_2}, \sigma_{id}, \sigma_{f(x_2,x_1)}\}$.

2. Ternary Semigroup

In this section, we give some important definitions of a ternary semigroup which will be used throughout of this paper. All definitions are defined in [11].

Definition 1. A nonempty set S with a ternary operation $[-,-,-] : S \times S \times S \to S$, written as $(x_1, x_2, x_3) \longmapsto [x_1x_2x_3]$, is called **a ternary semigroup** if it satisfies the following associative law:

$$[[x_1x_2x_3]x_4x_5] = [x_1[x_2x_3x_4]x_5] = [x_1x_2[x_3x_4x_5]],$$

for any $x_1, x_2, x_3, x_4, x_5 \in S$.

Let $(S, \cdot)$ be a semigroup. For $x_1, x_2, x_3 \in S$, let $[-,-,-] : S \times S \times S \to S$ be a ternary operation on S defined by

$$[x_1x_2x_3] = x_1 \cdot x_2 \cdot x_3.$$

Then $(S; [-,-,-])$ is a ternary semigroup.

Example 1 ([6]). Let

$$M = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}.$$

Then M is a ternary semigroup under a ternary operation $[-,-,-] : M \times M \times M \to M$ defined by $[M_1M_2M_3] := (M_1 \circ M_2) \circ M_3$, where $\circ$ is a usual matrix multiplication.

Definition 2. An element a of a ternary semigroup S is said to be a **ternary identity** provided that $[aas] = [asa] = [saa]$, for all $s \in S$. A ternary identity element of a ternary semigroup S is also called as **unital element**.

Definition 3. An element a of a ternary semigroup S is said to be

(i) an **idempotent** element provided that $[aaa]=a^3=a$, for all $a\in S$;

(ii) a **proper idempotent** element provided that a is an idempotent which is not the identity element of S if the identity exists.

Definition 4. An element a of a ternary semigroup S is said to be **regular** if there exist $x,y\in S$ such that $[[axa]ya]=axaya=a$.

3. Ternary Monoid $Hyp(2)$

To study algebraic-structural properties of the ternary monoid $Hyp(2)$, we first give the concept of this monoid as follows.

Defined a ternary operation $[-,-,-]:Hyp(2)\times Hyp(2)\times Hyp(2)\to Hyp(2)$ by

$$[\sigma_1\sigma_2\sigma_3]:=\sigma_1\circ_h\sigma_2\circ_h\sigma_3$$

for each $\sigma_1,\sigma_2,\sigma_3\in Hyp(2)$. Then we obtain that $\underline{Hyp(2)}:=(Hyp(2),[-,-,-])$ is a ternary semigroup.

Let $\sigma_t\in Hyp(2)$ where $t\in W(\{x_1,x_2\})$. Then we have

$$[\sigma_{id}\sigma_{id}\sigma_t](f)=(\sigma_{id}\circ_h\sigma_{id}\circ_h\sigma_t)(f)=(\sigma_{id}\circ_h\sigma_t)(f)=\sigma_t(f).$$

Similarly, we have $[\sigma_{id}\sigma_t\sigma_{id}](f)=\sigma_t(f)=[\sigma_t\sigma_{id}\sigma_{id}](f)$.

Proposition 3. $\underline{Hyp(2)}:=(Hyp(2),[-,-,-],\sigma_{id})$ *is a ternary monoid.*

Firstly, we study all idempotent elements in the ternary monoid of $Hyp(2)$. It is trivially that every idempotent element is ternary idempotent, but the converse is not true. As an example, $\sigma_{f(x_2,x_1)}$ is a ternary idempotent element, but it is not an idempotent element. Moreover, we can see that every ternary idempotent is a regular element.

Let

$$J_1:=\{\sigma_{f(x_1,t)}\mid t\in W_{(2)}(X_2)\setminus W_{(2)}(\{x_1\})\}\setminus\{\sigma_{id}\},$$
$$J_2:=\{\sigma_{f(s,x_2)}\mid s\in W_{(2)}(X_2)\setminus W_{(2)}(\{x_2\})\}\setminus\{\sigma_{id}\},$$
$$J_3:=\{\sigma_{f(s,x_1)}\mid s\in W_{(2)}(X_2)\setminus X_2\},$$
$$J_4:=\{\sigma_{f(x_2,t)}\mid t\in W_{(2)}(X_2)\setminus X_2\},$$
$$J_5:=\{\sigma_{f(s,t)}\mid s,t\in W_{(2)}(X_2)\setminus X_2\}.$$

Then we have the following proposition.

Proposition 4. *A ternary hypersubstitution σ is not a ternary idempotent if $\sigma\in\bigcup_{i=1}^5 J_i$.*

Proof. Let $\sigma\in\bigcup_{i=1}^5 J_i$. If $\sigma\in J_1$, then we consider

$$\begin{aligned}[\sigma_{f(x_1,s)}\sigma_{f(x_1,s)}\sigma_{f(x_1,t)}](f)&=(\sigma_{f(x_1,t)}\circ_h\sigma_{f(x_1,t)}\circ_h\sigma_{f(x_1,t)})(f)\\&=\hat\sigma_{f(x_1,t)}[S^2(\sigma_{f(x_1,t)}(f),x_1,\hat\sigma_{f(x_1,t)}[t])]\\&=\hat\sigma_{f(x_1,t)}[S^2(f(x_1,t),x_1,\hat\sigma_{f(x_1,t)}[t])].\end{aligned}$$

Since $t \in W_{(2)}(X_2) \setminus W_{(2)}(\{x_1\})$, so $x_2 \in var(t)$ and we have to substitute x_2 by $\hat{\sigma}_{f(x_1,t)}[t]$. Thus

$$op(\hat{\sigma}_{f(x_1,t)}[S^2(f(x_1,ts),x_1,\hat{\sigma}_{f(x_1,t)}[t])]) > op(\sigma_{f(x_1,t)}).$$

That is $op([\sigma_{f(x_1,t)}\sigma_{f(x_1,t)}\sigma_{f(x_1,t)}]) > op(\sigma_{f(x_1,t)})$.

Hence $[\sigma_{f(x_1,t)}\sigma_{f(x_1,t)}\sigma_{f(x_1,t)}] \neq \sigma_{f(x_1,t)}$. Therefore, $\sigma_{f(x_1,t)}$ is not a ternary idempotent.

The proofs of the case $\sigma \in J_l$, $l = 2,3,4,5$ are similar to the case J_1. □

Since $Hyp(2) \setminus \{\sigma_{id}\} = \sigma \in \bigcup_{i=1}^{5} J_i \cup E_{x_1} \cup E_{x_2} \cup \{\sigma x_1, \sigma x_2, \sigma_{f(x_2,x_1)}\}$, then, by the consequence of Proposition 4, we obtain the following corollary.

Corollary 2. *A hypersubstitution σ is a proper idempotent element of a ternary semigroup $Hyp(2)$ iff $\sigma \in E_{x_1} \cup E_{x_2} \cup \{\sigma_{x_1}, \sigma_{x_2}, \sigma_{f(x_2,x_1)}\}$.*

We next study all ternary regular elements of $Hyp(2)$. It is clear that every regular element is ternary regular and all ternary idempotent elements of $Hyp(2)$ are also ternary regular elements.

Proposition 5. *A ternary hypersubstitution σ is not a ternary regular if $\sigma \in \bigcup_{i=1}^{5} J_i$.*

Proof. Let $\sigma \in \bigcup_{i=1}^{5} J_i$ and $\sigma_{f(t_1,t_2)}, \sigma_{f(r_1,r_2)} \in Hyp(2)$ where $t_1,t_2,r_1,r_2 \in W_{(2)}(\{x_1,x_2\})$. If $\sigma \in J_1$, then we have

$$\begin{aligned}
&[[\sigma_{f(x_1,t)}\sigma_{f(t_1,t_2)}\sigma_{f(x_1,t)}]\sigma_{f(r_1,r_2)}\sigma_{f(x_1,t)}](f)\\
&\quad = ((\sigma_{f(x_1,t)} \circ_h \sigma_{f(t_1,t_2)} \circ_h \sigma_{f(x_1,t)}) \circ_h \sigma_{f(r_1,r_2)} \circ_h \sigma_{f(x_1,t)})(f)\\
&\quad = \hat{\sigma}_{f(x_1,t)}[\hat{\sigma}_{f(t_1,t_2)}[\hat{\sigma}_{f(x_1,t)}[\hat{\sigma}_{f(r_1,r_2)}[\sigma_{f(x_1,t)}(f)]]]]\\
&\quad = \hat{\sigma}_{f(x_1,t)}[\hat{\sigma}_{f(t_1,t_2)}[\hat{\sigma}_{f(x_1,t)}[f(S^2(r_1,x_1,\hat{\sigma}_{f(r_1,r_2)}[t]), S^2(r_2,x_1,\hat{\sigma}_{f(r_1,r_2)}[t]))]]]\\
&\quad = \hat{\sigma}_{f(x_1,t)}[\hat{\sigma}_{f(t_1,t_2)}[S^2(f(x_1,t),\hat{\sigma}_{f(x_1,t)}[S^2(r_1,x_1,\hat{\sigma}_{f(r_1,r_2)}[t])],\hat{\sigma}_{f(x_1,s)}[S^2(r_2,x_1,\hat{\sigma}_{f(r_1,r_2)}[t])])]].
\end{aligned}$$

Since $t \in W_{(2)}(X_2) \setminus W_{(2)}(\{x_1\})$, so $x_2 \in var(t)$, and we have to substitute x_2 by

$$\hat{\sigma}_{f(x_1,t)}[S^2(r_2,x_1,\hat{\sigma}_{f(r_1,r_2)}[t])].$$

Thus

$$\begin{aligned}
&op(\hat{\sigma}_{f(x_1,t)}[\hat{\sigma}_{f(t_1,t_2)}[S^2(f(x_1,t),\hat{\sigma}_{f(x_1,t)}[S^2(r_1,x_1,\hat{\sigma}_{f(r_1,r_2)}[s])],\hat{\sigma}_{f(x_1,t)}[S^2(r_2,x_1,\hat{\sigma}_{f(r_1,r_2)}[t])])]])\\
&\quad > op(\sigma_{f(x_1,t)}).
\end{aligned}$$

Therefore, $\sigma_{f(x_1,t)})$ is not a ternary regular. The proofs of the case $\sigma \in J_l$, $l = 2,3,4,5$ are similar to the case J_1. □

By the consequence of Proposition 5, we obtain that the set of all ternary idempotent elements is also the set of all ternary regular elements.

4. Ideals of Submonoid of $Hyp(2)$

We study the relationship between some submonoids of $Hyp(2)$ and give some characterizations of the ideals of these submonoids. We first recall from [5] the definition of a left (right, two sided) ideal.

Definition 5. Let S be a semigroup. A nonempty subset A of S is called a **left ideal** if $SA \subseteq A$, a **right ideal** if $AS \subseteq A$, and an **ideal** if it is both a left and a right ideal.

To study the ideals of some submonoids of $Hyp(2)$, we consider some finite submonoids of $Hyp(2)$ which characterized by K. Denecke and J. Koppitz [1] as follows.

Let

$$M_T := \{\sigma_{id}\},$$
$$M_D := \{\sigma_{id}, \sigma_{f(x_2,x_1)}\},$$
$$M_1 := \{\sigma_{x_1}, \sigma_{x_2}, \sigma_{id}, \sigma_{f(x_2,x_1)}\},$$
$$M_2 := \{\sigma_{f(x_1,x_1)}, \sigma_{f(x_2,x_2)}, \sigma_{id}\},$$
$$M_3 := M_2 \cup \{\sigma_{f(x_2,x_1)}\},$$
$$M_4 := \{\sigma_{x_1}, \sigma_{x_2}, \sigma_{f(x_1,x_1)}, \sigma_{f(x_2,x_2)}, \sigma_{id}\},$$
$$M_5 := M_4 \cup \{\sigma_{f(x_2,x_1)}\}.$$

and $P(2)$-the set of all projection hypersubstitutions of type $\tau = (2)$, i.e.

$$P(2) := \{\sigma \in Hyp(2) \mid \sigma(f) \text{is a variable}\} = \{\sigma_{x_1}, \sigma_{x_2}\}.$$

It is obviously seen that $P(2)$ is an ideal of $Hyp(2)$. The ideals of some finite submonoid of $Hyp(2)$ are following proposition.

Proposition 6. *$M_2 \setminus M_T$ is an ideal of M_2.*

Proof. Let $\sigma_{f(x_i,x_i)} \in M_2 \setminus M_T$; $i = 1,2$ and $\sigma_s \in M_2$.

Since $\sigma_s = \sigma_{id}$, it is trivially that $(\sigma_{f(x_i,x_i)} \circ_h \sigma_s)(f) = (\sigma_{id} \circ_h \sigma_{f(x_i,x_i)})(f) = \sigma_{f(x_i,x_i)}(f)$.

Since $\sigma_s = \sigma_{f(x_j,x_j)}$; $j = 1,2$, then

$$\begin{aligned}(\sigma_{f(x_i,x_i)} \circ_h \sigma_s](f) &= (\sigma_{f(x_i,x_i)} \circ_h \sigma_{f(x_j,x_j)})(f)\\ &= \hat{\sigma}_{f(x_i,x_i)}[f(x_j,x_j)]\\ &= S^2(f(x_i,x_i), x_j, x_j\\ &= f(x_j,x_j)\\ &= \sigma_{f(x_j,x_j)}(f)\end{aligned}$$

and

$$\begin{aligned}(\sigma_s \circ_h \sigma_{f(x_i,x_i)}](f) &= (\sigma_{f(x_j,x_j)} \circ_h \sigma_{f(x_i,x_i)})(f)\\ &= \hat{\sigma}_{f(x_j,x_j)}[f(x_i,x_i)]\\ &= S^2(f(x_j,x_j), x_i, x_i)\\ &= f(x_i,x_i)\\ &= \sigma_{f(x_i,x_i)}(f).\end{aligned}$$

Therefore, $M_2 \setminus M_T$ is an ideal of M_2. □

Proposition 7. *$M_3 \setminus M_D = M_2 \setminus M_T$ is an ideal of M_3.*

Proof. Let $\sigma_{f(x_i,x_i)} \in M_3 \setminus M_D = M_2 \setminus M_T$; $i = 1,2$ and $\sigma_s \in M_3$.
Since $\sigma_s \in M_2$, it is trivially from Proposition 6. We only proof in the case that $\sigma_s = \sigma_{f(x_2,x_1)}$.
Consider

$$\begin{aligned}(\sigma_{f(x_i,x_i)} \circ_h \sigma_{f(x_2,x_1)}](f) &= \hat{\sigma}_{f(x_i,x_i)}[f(x_2,x_1)]\\ &= S^2(f(x_i,x_i),x_2,x_1)\\ &= f(x_j,x_j)\\ &= \sigma_{f(x_j,x_j)}(f), \quad j=1,2\end{aligned}$$

and

$$\begin{aligned}(\sigma_{f(x_2,x_1)} \circ_h \sigma_{f(x_i,x_i)})(f) &= \hat{\sigma}_{f(x_2,x_1)}[f(x_i,x_i)]\\ &= S^2(f(x_2,x_1),x_i,x_i)\\ &= f(x_i,x_i)\\ &= \sigma_{f(x_i,x_i)}(f).\end{aligned}$$

Therefore, $M_3 \setminus M_D = M_2 \setminus M_T$ is an ideal of M_3. □

Proposition 8. *$M_4 \setminus M_T = M_5 \setminus M_D$ is an ideal of M_4.*

Proof. Let $\sigma_t \in M_4 \setminus M_T = M_5 \setminus M_D$; $i = 1,2$ and $\sigma_s \in M_4$.

Since $\sigma_t = \sigma_{x_i}$, $i = 1,2$, the proof is obviously.

Since $\sigma_t = \sigma_{f(x_i,x_i)}$, $i = 1,2$, then we consider:

Case 1: $\sigma_s = \sigma_{x_j}$, $j = 1,2$. Then it is easy to see that $(\sigma_t \circ_h \sigma_s)(f) = (\sigma_s \circ_h \sigma_t)(f) = \sigma_{x_j}(f)$.

Case 2: $\sigma_s = \sigma_{id}$. Then $(\sigma_t \circ_h \sigma_{id})(f) = (\sigma_{id} \circ_h \sigma_t)(f) = \sigma_t(f)$.

Case 3: $\sigma_s = \sigma_{f(x_j,x_j)}$, $j = 1,2$. Then, we have

$$\begin{aligned}(\sigma_{f(x_i,x_i)} \circ_h \sigma_s](f) &= (\sigma_{f(x_i,x_i)} \circ_h \sigma_{f(x_j,x_j)})(f)\\ &= \hat{\sigma}_{f(x_i,x_i)}[f(x_j,x_j)]\\ &= S^2(f(x_i,x_i),x_j,x_j)\\ &= f(x_j,x_j)\\ &= \sigma_{f(x_j,x_j)}(f)\end{aligned}$$

and

$$\begin{aligned}(\sigma_s \circ_h \sigma_{f(x_i,x_i)}](f) &= (\sigma_{f(x_j,x_j)} \circ_h \sigma_{f(x_i,x_i)})(f)\\ &= \hat{\sigma}_{f(x_j,x_j)}[f(x_i,x_i)]\\ &= S^2(f(x_j,x_j),x_i,x_i)\\ &= f(x_i,x_i)\\ &= \sigma_{f(x_i,x_i)}(f).\end{aligned}$$

Therefore, $M_4 \setminus M_T = M_5 \setminus M_D$ is an ideal of M_4. □

Proposition 9. *$M_4 \setminus M_T = M_5 \setminus M_D$ is an ideal of M_5.*

Proof. The proof of this proposition is similarly to Proposition 8. □

5. Ternary Ideals of Submonoid of $Hyp(2)$

In this section, the study focuses on some submonoids of the monoid $Hyp(2)$ and determine the ternary ideals of these submonoids. Firstly, we recall the definitions of a ternary ideal which are important in this study.

Definition 6. A nonempty subset A of a ternary semigroup T is said to be a **ternary subsemigroup** if $A^3 = [AAA] \subseteq A$.

Definition 7. A nonempty subset A of a ternary semigroup T is said to be

(i) **a left ideal** of T if $b,c \in T$, $a \in A$ implies $bca \in A$, i.e. $TTA \subseteq A$.

(ii) **a lateral ideal** of T if $b,c \in T$, $a \in A$ implies $bac \in A$, i.e. $TAT \subseteq A$.

(iii) **a right ideal** of T if $b,c \in T$, $a \in A$ implies $abc \in A$, i.e. $ATT \subseteq A$.

Definition 8. A nonempty subset A of a ternary semigroup T is said to be **a ternary ideal** of T if $b,c \in T$, $a \in A$ implies $bca \in A$, $bac \in A$, $abc \in A$. That means A is left ideal, lateral ideal and right ideal of T.

We now consider some finite submonoids of $Hyp(2)$ which characterized by K. Denecke and J. Koppitz [1]. The ternary ideals of some finite submonoids of $Hyp(2)$ are following proposition.

Proposition 10. *M_T, M_D, M_i; $1 \le i \le 5$ and $P(2) \cup \{\sigma_{id}\}$ are ternary submonoids of $Hyp(2)$.*

Proof. The proof is straightforward. □

Proposition 11. *$P(2)$ is a ternary ideal of $Hyp(2)$.*

Proof. The proof of this proposition is directly from $[\sigma_{x_i}\sigma_s\sigma_t] \in P(2)$, $[\sigma_s\sigma_{x_i}\sigma_t] \in P(2)$, and $[\sigma_s\sigma_t\sigma_{x_i}] \in P(2)$ for all $\sigma_s,\sigma_t \in Hyp(2)$, $i \in \{1,2\}$. □

Proposition 12. *$P(2)$ is a ternary ideal of M_1.*

Proof. Let $\sigma_s \in P(2)$. Then $\sigma_s(f)$ is a variable. Since

$$\begin{aligned}[\sigma_{id}\sigma_{f(x_2,x_1)}\sigma_s](f) &= (\sigma_{id} \circ_h \sigma_{f(x_2,x_1)} \circ_h \sigma_s)(f) \\ &= \hat{\sigma}_{id}[\hat{\sigma}_{f(x_2,x_1)}[\sigma_s(f)]],\end{aligned}$$

and $\sigma_s(f)$ is a variable, we have $[\sigma_{id}\sigma_{f(x_2,x_1)}\sigma_s]$ is a variable. Thus $[\sigma_{id}\sigma_{f(x_2,x_1)}\sigma_s] \in P(2)$. Similarly, we have $[\sigma_{id}\sigma_s\sigma_{f(x_2,x_1)}] \in P(2)$ and $[\sigma_s\sigma_{id}\sigma_{f(x_2,x_1)}] \in P(2)$.

Hence for every element σ_t,σ_r in M_1 we have $[\sigma_t\sigma_r\sigma_s] \in P(2)$, $[\sigma_t\sigma_s\sigma_r] \in P(2)$ and $[\sigma_s\sigma_t\sigma_r] \in P(2)$. Therefore, $P(2)$ is a ternary ideal of M_1. □

Proposition 13. *$P(2)$ is a ternary ideal of M_4.*

Proof. The proof of this proposition is similar to Proposition 12. □

Proposition 14. *$P(2)$ is a ternary ideal of M_5.*

Proof. The proof of this proposition is similar to Proposition 12. □

Proposition 15. *$P(2)$ is a ternary ideal of $M_1\setminus M_T$.*

Proof. Let $\sigma_{x_i}\in P(2)$; $i=1,2$, and $\sigma_r,\sigma_s\in M_1\setminus M_T$.
Case 1: $\sigma_r=\sigma_{x_j}$ and $\sigma_s=\sigma_{x_k}$; $j,k\in\{1,2\}$. Then $[\sigma_{x_i}\sigma_{x_j}\sigma_{x_k}](f)=[\sigma_{x_j}\sigma_{x_i}\sigma_{x_k}](f)=\sigma_{x_k}(f)$ and $[\sigma_{x_j}\sigma_{x_k}\sigma_{x_i}](f)=\sigma_{x_i}(f)$.
Case 2: $\sigma_r=\sigma_{x_j}$; $j\in\{1,2\}$, $\sigma_s=\sigma_{f_{(x_2,x_1)}}$ (or $\sigma_r=\sigma_{f_{(x_2,x_1)}},\sigma_s=\sigma_{x_j}$; $j\in\{1,2\}$). Then

$$\begin{aligned}[\sigma_{x_i}\sigma_{x_j}\sigma_{f(x_2,x_1)}](f)&=(\sigma_{x_i}\circ_h\sigma_{x_j}\circ_h\sigma_{f(x_2,x_1)})(f)\\&=\hat{\sigma}_{x_i}[S^2(x_j,x_2,x_1)]\\&=\hat{\sigma}_{x_i}[x_k]=x_k=\sigma_{x_k}(f);\quad k=1,2.\end{aligned}$$

Similarly, we have $[\sigma_{x_j}\sigma_{x_i}\sigma_{f(x_2,x_1)}](f)=\sigma_{x_k}(f)$; $k=1,2$ and $[\sigma_{x_j}\sigma_{f(x_2,x_1)}\sigma_{x_i}](f)=\sigma_{x_i}(f)$.
Case 3: $\sigma_r=\sigma_s=\sigma_{f(x_2,x_1)}$. Then $[\sigma_{x_i}\sigma_{f(x_2,x_1)}\sigma_{f(x_2,x_1)}](f)=(\sigma_{x_i}\circ_h\sigma_{id})(f)=\sigma_{x_i}(f)$,

$$\begin{aligned}[\sigma_{f(x_2,x_1)}\sigma_{x_i}\sigma_{f(x_2,x_1)}](f)&=(\sigma_{f(x_2,x_1)}\circ_h\sigma_{x_i}\circ_h\sigma_{f(x_2,x_1)})(f)\\&=\hat{\sigma}_{f(x_2,x_1)}[S^2(x_i,x_2,x_1)]=\hat{\sigma}_{f(x_2,x_1)}[x_i]=x_i=\sigma_{x_i}(f).\end{aligned}$$

$$[\sigma_{f(x_2,x_1)}\sigma_f(x_2,x_1)\sigma_{x_i}](f)=(\sigma_{id}\circ_h\sigma_{x_i})(f)=\sigma_{x_i}(f).$$

Therefore, $P(2)$ is a ternary ideal of $M_1\setminus M_T$. □

Proposition 16. *$M_2\setminus M_T$ is a ternary ideal of M_2.*

Proof. Let $\sigma_{f(x_i,x_i)}\in M_2\setminus M_T$; $i=1,2$ and $\sigma_r,\sigma_s\in M_2$.
Case 1: $\sigma_r=\sigma_s=\sigma_{id}$. Then

$$[\sigma_{f(x_i,x_i)}\sigma_{id}\sigma_{id}](f)=[\sigma_{id}\sigma_{f(x_i,x_i)}\sigma_{id}](f)=[\sigma_{id}\sigma_{id}\sigma_{f(x_i,x_i)}](f)=\sigma_{f(x_i,x_i)}(f).$$

Case 2: $\sigma_r=\sigma_{id}$, $\sigma_s=\sigma_{f(x_j,x_j)}$; $j=1,2$ (or $\sigma_r=\sigma_{f(x_j,x_j)}$; $j=1,2$, $\sigma_s=\sigma_{id}$). Then

$$\begin{aligned}[\sigma_{f(x_i,x_i)}\sigma_{id}\sigma_{f(x_j,x_j)}](f)&=(\sigma_{f(x_i,x_i)}\circ_h\sigma_{id}\circ_h\sigma_{f(x_j,x_j)})(f)\\&=(\sigma_{f(x_i,x_i)}\circ_h\sigma_{f(x_j,x_j)})(f)\\&=S^2(f(x_i,x_i),x_j,x_j)\\&=f(x_j,x_j)=\sigma_{f(x_j,x_j)}(f).\end{aligned}$$

Similarly, we have $[\sigma_{f(x_i,x_i)}\sigma_{id}\sigma_{f(x_j,x_j)}](f)=[\sigma_{f(x_i,x_i)}\sigma_{id}\sigma_{f(x_j,x_j)}](f)=\sigma_{f(x_j,x_j)}(f)$.
Case 3: $\sigma_r=\sigma_{f(x_j,x_j)}$ and $\sigma_s=\sigma_{f(x_k,x_k)}$; $j,k\in\{1,2\}$. Then

$$\begin{aligned}[\sigma_{f(x_i,x_i)}\sigma_{f(x_j,x_j)}\sigma_{f(x_k,x_k)}](f)&=(\sigma_{f(x_i,x_i)}\circ_h\sigma_{f(x_j,x_j)}\circ_h\sigma_{f(x_k,x_k)})(f)\\&=\hat{\sigma}_{f(x_i,x_i)}[S^2(f(x_j,x_j),x_k,x_k)]\\&=\hat{\sigma}_{f(x_i,x_i)}[f(x_k,x_k)]\\&=S^2(f(x_i,x_i),x_k,x_k)\\&=f(x_k,x_k)=\sigma_{f(x_k,x_k)}(f).\end{aligned}$$

By using the same method, we have

$$[\sigma_{f(x_j,x_j)}\sigma_{f(x_i,x_i)}\sigma_{f(x_k,x_k)}](f) = [\sigma_{f(x_j,x_j)}\sigma_{f(x_k,x_k)}\sigma_{f(x_i,x_i)}](f) = \sigma_{f(x_k,x_k)}(f).$$

Therefore, $M_2 \setminus M_T$ is a ternary ideal of M_2. □

Proposition 17. *$M_3 \setminus M_D$ is a ternary ideal of M_3.*

Proof. Let $\sigma_{f(x_i,x_i)} \in M_3 \setminus M_D$; $i = 1,2$ and $\sigma_{f(x_{j_1},x_{k_1})}, \sigma_{f(x_{j_2},x_{k_2})} \in M_3$; $j_l, k_l \in \{1,2\}$ and $l = 1,2$. Consider

$$\begin{aligned}[\sigma_{f(x_i,x_i)}\sigma_{f(x_{j_1},x_{k_1})}\sigma_{f(x_{j_2},x_{k_2})}](f) &= (\sigma_{f(x_i,x_i)} \circ_h \sigma_{f(x_{j_1},x_{k_1})} \circ_h \sigma_{f(x_{j_2},x_{k_2})})(f)\\ &= \hat{\sigma}_{f(x_i,x_i)}[S^2(f(x_{j_1},x_{k_1}),x_{j_2},x_{k_2})]\\ &= \hat{\sigma}_{f(x_i,x_i)}[f(x_{r_1},x_{r_2})]; \quad r_1,r_2 \in \{1,2\}\\ &= S^2(f(x_i,x_i),x_{r_1},x_{r_2})\\ &= f(x_q,x_q) = \sigma_{f(x_q,x_q)}(f); \quad q = 1,2.\end{aligned}$$

Similarly, we have

$$[\sigma_{f(x_{j_1},x_{k_1})}\sigma_{f(x_i,x_i)}\sigma_{f(x_{j_2},x_{k_2})}](f) = \sigma_{f(x_q,x_q)}(f); \quad q = 1,2$$

and

$$[\sigma_{f(x_{j_1},x_{k_1})}\sigma_{f(x_{j_2},x_{k_2})}\sigma_{f(x_i,x_i)}](f) = \sigma_{f(x_i,x_i)}.$$

Therefore, $M_3 \setminus M_D$ is a ternary ideal of M_3. □

Proposition 18. *$M_4 \setminus M_T$ is a ternary ideal of M_4.*

Proof. Let $\sigma_t \in M_4 \setminus M_T$ and $\sigma_s, \sigma_r \in M_4$. Then we consider

Case 1:: $\sigma_t = \sigma_{x_i}$; $i = 1,2$.

Case 1.1: $\sigma_s = \sigma_{x_j}, \sigma_r = \sigma_{x_k}; j,k = 1,2$.
Then $[\sigma_t\sigma_s\sigma_r](f) = [\sigma_s\sigma_t\sigma_r](f) = \sigma_{x_k}(f)$ and $[\sigma_s\sigma_r\sigma_t](f) = \sigma_{x_i}(f)$.

Case 1.2: $\sigma_s = \sigma_{x_j}, \sigma_r = \sigma_{f(x_k,x_k)}$; $j,k \in \{1,2\}$ (or $\sigma_s = \sigma_{f(x_j,x_j)}, \sigma_r = \sigma_{x_k}$; $j,k \in \{1,2\}$). Then we have

$$\begin{aligned}[\sigma_t\sigma_s\sigma_r](f) &= [\sigma_{x_i}\sigma_{x_j}\sigma_{f(x_k,x_k)}](f) = (\sigma_{x_i} \circ_h \sigma_{x_j} \circ_h \sigma_{f(x_k,x_k)})(f)\\ &= \hat{\sigma}_{x_i}[S^2(x_j,x_k,x_k)] = \hat{\sigma}_{x_i}[x_k]\\ &= x_k = \sigma_{x_k}(f).\end{aligned}$$

Similarly, we obtain that $[\sigma_s\sigma_t\sigma_r](f) = \sigma_{x_k}(f)$ and $[\sigma_s\sigma_r\sigma_t](f) = \sigma_{x_i}(f)$.

Case 1.3: $\sigma_s = \sigma_{x_j}, \sigma_r = \sigma_{id}$ (or $\sigma_s = \sigma_{id}, \sigma_r = \sigma_{x_k}; j,k \in \{1,2\}$). Then

$$\begin{aligned}[\sigma_t\sigma_s\sigma_r](f) &= [\sigma_{x_i}\sigma_{x_j}\sigma_{id}](f) = (\sigma_{x_i} \circ_h \sigma_{x_j} \circ_h \sigma_{id})(f)\\ &= (\sigma_{x_i} \circ_h \sigma_{x_j})(f) = \hat{\sigma}_{x_i}[x_j] = x_j = \sigma_{x_j}(f).\end{aligned}$$

Similarly, we have $[\sigma_s\sigma_t\sigma_r](f) = [\sigma_s\sigma_r\sigma_t](f) = \sigma_{x_i}(f)$.

Case 1.4: $\sigma_s = \sigma_{f(x_j,x_j)}$ and $\sigma_r = \sigma_{f(x_k,x_k)}$; $j,k \in \{1,2\}$. Then

$$[\sigma_t\sigma_s\sigma_r](f) = [\sigma_{x_i}\sigma_{f(x_j,x_j)}\sigma_{f(x_k,x_k)}](f) = (\sigma_{x_i} \circ_h \sigma_{f(x_j,x_j)} \circ_h \sigma_{f(x_k,x_k)})(f)$$

$$= \hat{\sigma}_{x_i}[S^2(f(x_j,x_j),x_k,x_k)] = \hat{\sigma}_{x_i}[f(x_k,x_k)]$$
$$= x_k = \sigma_{x_k}(f).$$

Similarly, we obtain that $[\sigma_s\sigma_t\sigma_r](f) = \sigma_{x_k}(f)$ and $[\sigma_s\sigma_r\sigma_t](f) = \sigma_{x_i}(f)$.

Case 1.5: $\sigma_s = \sigma_{f(x_j,x_j)}, \sigma_r = \sigma_{id}$ (or $\sigma_s = \sigma_{id}$, $\sigma_r = \sigma_{f(x_k,x_k)}$; $j,k \in \{1,2\}$ or $\sigma_s = \sigma_r = \sigma_{id}$). Then, we also have $[\sigma_t\sigma_s\sigma_r](f) = [\sigma_s\sigma_t\sigma_r](f) = [\sigma_s\sigma_r\sigma_t](f) = \sigma_{x_l}(f)$; $l \in \{1,2\}$.

Case 2: $\sigma_t = \sigma_{f(x_i,x_i)}$; $i \in \{1,2\}$.

Case 2.1: $\sigma_s = \sigma_{x_j}, \sigma_r = \sigma_{x_k}$ (or $\sigma_s = \sigma_{x_j}, \sigma_r = \sigma_{id}$ or $\sigma_s = \sigma_{id}, \sigma_r = \sigma_{x_k}$; $j,k \in \{1,2\}$). Then, $[\sigma_t\sigma_s\sigma_r](f) = [\sigma_s\sigma_t\sigma_r](f) = \sigma_{x_k}$ and $[\sigma_s\sigma_r\sigma_t](f) = \sigma_{x_i}$.

Case 2.2: $\sigma_s = \sigma_{x_j}, \sigma_r = \sigma_{f(x_k,x_k)}$ (or $\sigma_s = \sigma_{f(x_j,x_j)}, \sigma_r = \sigma_{x_k}; j,k \in \{1,2\}$). Then, we have

$$[\sigma_t\sigma_s\sigma_r](f) = [\sigma_{f(x_i,x_i)}\sigma_{x_j}\sigma_{f(x_k,x_k)}](f) = (\sigma_{f(x_i,x_i)} \circ_h \sigma_{x_j} \circ_h \sigma_{f(x_k,x_k)})(f)$$
$$= \hat{\sigma}_{f(x_i,x_i)}[S^2(x_j,x_k,x_k)] = \hat{\sigma}_{f(x_i,x_i)}[x_k]$$
$$= x_k = \sigma_{x_k}(f).$$

Similarly, we obtain that $[\sigma_s\sigma_t\sigma_r](f) = \sigma_{x_k}(f)$ and $[\sigma_s\sigma_r\sigma_t](f) = \sigma_{x_i}(f)$.

Case 2.3: $\sigma_s = \sigma_{f(x_j,x_j)}$ and $\sigma_r = \sigma_{f(x_k,x_k)}$; $j,k \in \{1,2\}$. Then

$$[\sigma_t\sigma_s\sigma_r](f) = [\sigma_{f(x_i,x_i)}\sigma_{f(x_j,x_j)}\sigma_{f(x_k,x_k)}](f) = (\sigma_{f(x_i,x_i)} \circ_h \sigma_{f(x_j,x_j)} \circ_h \sigma_{f(x_k,x_k)})(f)$$
$$= \hat{\sigma}_{f(x_i,x_i)}[S^2(f(x_j,x_j),x_k,x_k)] = \hat{\sigma}_{f(x_i,x_i)}[f(x_k,x_k)]$$
$$= S^2(f(x_i,x_i),x_k,x_k) = f(x_k,x_k) = \sigma_{f(x_k,x_k)}(f).$$

Similarly, we obtain that $[\sigma_s\sigma_t\sigma_r](f) = \sigma_{f(x_k,x_k)}(f)$ and $[\sigma_s\sigma_r\sigma_t](f) = \sigma_{f(x_i,x_i)}(f)$.

Case 2.4: $\sigma_s = \sigma_{f(x_j,x_j)}, \sigma_r = \sigma_{id}$ (or $\sigma_s = \sigma_{id}$, $\sigma_r = \sigma_{f(x_k,x_k)}$; $j,k \in \{1,2\}$ or $\sigma_s = \sigma_r = \sigma_{id}$). Then, we also have $[\sigma_t\sigma_s\sigma_r](f) = [\sigma_s\sigma_t\sigma_r](f) = [\sigma_s\sigma_r\sigma_t](f) = \sigma_{f(x_l,x_l)}(f)$; $l = 1,2$.

Therefore, $M_4 \setminus M_T$ is a ternary ideal of M_4. □

Proposition 19. *$M_5 \setminus M_D = M_4 \setminus M_T$ is a ternary ideal of M_5.*

Proof. Let $\sigma_t \in M_5 \setminus M_D$ and $\sigma_s, \sigma_r \in M_5$. We will consider

Case 1: $\sigma_t = \sigma_{x_i}$; $i \in \{1,2\}$.

Case 1.1: $\sigma_s = \sigma_{x_j}$, $\sigma_r = \sigma_{x_k}$; $j,k \in \{1,2\}$. Then it is easy to see that $[\sigma_t\sigma_s\sigma_r](f) = [\sigma_s\sigma_t\sigma_r](f) = \sigma_{x_k}(f)$ and $[\sigma_s\sigma_r\sigma_t](f) = \sigma_{x_i}$.

Case 1.2: $\sigma_s = \sigma_{x_j}$, $\sigma_r = \sigma_{f(x_k,x_l)}$ (or $\sigma_s = \sigma_{f(x_k,x_l)}$, $\sigma_r = \sigma_{x_j}$; $j,k,l \in \{1,2\}$). Then

$$[\sigma_t\sigma_s\sigma_r](f) = [\sigma_{x_i}\sigma_{x_j}\sigma_{f(x_k,x_l)}](f) = (\sigma_{x_i} \circ_h \sigma_{x_j} \circ_h \sigma_{f(x_k,x_l)})(f)$$
$$= \hat{\sigma}_{x_i}[S^2(x_j,x_k,x_l)] = \hat{\sigma}_{x_i}[x_q]; \quad q \in \{1,2\}$$
$$= x_q = \sigma_{x_q}(f).$$

By using the same method, we have $[\sigma_s\sigma_t\sigma_r](f) = \sigma_{x_q}$ and $[\sigma_s\sigma_r\sigma_t](f) = \sigma_{x_i}$.

Case 1.3: $\sigma_s = \sigma_{f(x_j,x_k)}, \sigma_{f(x_l,x_m)}$; $j,k,l,m \in \{1,2\}$. Then

$$[\sigma_t\sigma_s\sigma_r](f) = [\sigma_{x_i}\sigma_{f x_j,x_k}\sigma_{f(x_l,x_m)}](f) = (\sigma_{x_i} \circ_h \sigma_{f x_j,x_k} \circ_h \sigma_{f(x_l,x_m)})(f)$$
$$= \hat{\sigma}_{x_i}[S^2(f(x_j,x_k),x_l,x_m)] = \hat{\sigma}_{x_i}[f(x_u,x_v)]; \quad u,v \in \{1,2\}$$

$$= S^2(x_i, x_u, x_v) = x_q = \sigma_{x_q}(f); \quad q \in \{1,2\}.$$

By using the same method, we have $[\sigma_s\sigma_t\sigma_r](f) = \sigma_{x_q}$ and $[\sigma_s\sigma_r\sigma_t](f) = \sigma_{x_i}$.

Case 2: $\sigma_t = \sigma_{f(x_i,x_i)}$; $i \in \{1,2\}$.

Case 2.1: $\sigma_s = \sigma_{x_j}$, $\sigma_r = \sigma_{x_k}$; $j,k \in \{1,2\}$.
Then $[\sigma_t\sigma_s\sigma_r](f) = [\sigma_s\sigma_t\sigma_r](f) = \sigma_{x_k}$ and $[\sigma_s\sigma_r\sigma_t](f) = \sigma_{x_i}$.

Case 2.2: $\sigma_s = \sigma_{x_j}$, $\sigma_r = \sigma_{f(x_k,x_l)}$ (or $\sigma_s = \sigma_{f(x_k,x_l)}$, $\sigma_r = \sigma_{x_j}$; $j,k,l \in \{1,2\}$). Then

$$\begin{aligned}[\sigma_t\sigma_s\sigma_r](f) &= [\sigma_{f(x_i,x_i)}\sigma_{x_j}\sigma_{f(x_k,x_l)}](f) = (\sigma_{f(x_i,x_i)} \circ_h \sigma_{x_j} \circ_h \sigma_{f(x_k,x_l)})(f) \\ &= \hat{\sigma}_{f(x_i,x_i)}[S^2(x_j,x_k,x_l)] = \hat{\sigma}_{f(x_i,x_i)}[x_q]; \quad q = 1,2 \\ &= x_q = \sigma_{x_q}(f).\end{aligned}$$

By using the same method, we have $[\sigma_s\sigma_t\sigma_r](f) = \sigma_{x_q}$ and $[\sigma_s\sigma_r\sigma_t](f) = \sigma_{x_i}$.

Case 2.3: $\sigma_s = \sigma_{f(x_j,x_k)}$, $\sigma_{f(x_l,x_m)}$; $j,k,l,m \in \{1,2\}$. Then

$$\begin{aligned}[\sigma_t\sigma_s\sigma_r](f) &= [\sigma_{f(x_i,x_i)}\sigma_{fx_j,x_k}\sigma_{f(x_l,x_m)}](f) \\ &= (\sigma_{f(x_i,x_i)} \circ_h \sigma_{fx_j,x_k} \circ_h \sigma_{f(x_l,x_m)})(f) \\ &= \hat{\sigma}_{f(x_i,x_i)}[S^2(f(x_j,x_k),x_l,x_m)].\end{aligned}$$

If $j = k$, then

$$\begin{aligned}[\sigma_t\sigma_s\sigma_r](f) &= \hat{\sigma}_{f(x_i,x_i)}[f(x_q,x_q)]; \quad q = 1,2 \\ &= S^2(f(x_i,x_i),x_q,x_q) \\ &= f(x_q,x_q) = \sigma_{f(x_q,x_q)}(f).\end{aligned}$$

If $j \neq k$, then

$$\begin{aligned}[\sigma_t\sigma_s\sigma_r](f) &= \hat{\sigma}_{f(x_i,x_i)}[f(x_p,x_q)]; \quad p,q = 1,2 \\ &= S^2(f(x_i,x_i),x_p,x_q) \\ &= f(x_m,x_m) = \sigma_{f(x_m,x_m)}(f); \quad m = 1,2.\end{aligned}$$

Similarly, we have $[\sigma_s\sigma_t\sigma_r](f) = [\sigma_s\sigma_r\sigma_t](f) = \sigma_{f(x_l,x_l)}$; $l = 1,2$.
Therefore, $M_5 \setminus M_D = M_4 \setminus M_T$ is a ternary ideal of M_5. □

Acknowledgement

The authors would like to thank the referees for useful comments.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

References

[1] K. Denecke and J. Koppitz, Finite monoid of hypersubstitutions of type $\tau = (2)$, *Semigroup Forum* **56** (1998), 265 – 275, DOI: 10.1007/PL00005945.

[2] K. Denecke and Sh. L. Wismath, The monoid of hypersubstitutions of type $\tau = (2)$, *Contributions to General Algebra, Klagenfurt* **10** (1998), 109 – 127.

[3] W. A. Dudek, Idempotents in n-ary semigroups, Southeast Asian Bull. Math. **25** (2001), 97 – 104, DOI: 10.1007/s10012-001-0097-y.

[4] W. A. Dudek and I. Groúdzińska, On ideals in regular n-semigroups, *Mat. Bilten (Skopje)* **4(XXX)** (1980), 35 – 44.

[5] J. M. Howie, *An Introduction to Semigroup Theory*, Academic Press Inc., London (1976).

[6] A. Iampan, Some properties of ideal extensions in ternary semigroups, *Iranian Journal of Mathematical Sciences and Informatics* **8**(1) (2013), 67 – 74, DOI: 10.7508/ijmsi.2013.01.007.

[7] S. Kar and B. K. Maity, Congruences on ternary semigroups, *Journal of Chungcheong Mathematical Society* **20**(3) (2007), 191 – 201.

[8] E. Kasner, An extension of the group concept, (reported by Weld L. G.), *Bull. Amer. Math. Soc.* **10** (1904), 290 – 291.

[9] J. Koppitz and K. Denecke, *M-Solid Varieties of Algebras*, Springer Science+Business Media, Inc., New York (2006), DOI: 10.1007/0-387-30806-7.

[10] M. L. Santiago and S. Sri Bala, S Ternary semigroups, *Semigroup Forum* **81** (2010), 380 – 388, DOI: 10.1007/s00233-010-9254-x.

[11] Y. Sarala, A. Anjaneyulu and D. Madhusudhana Rao, Pseudo symmetric ideals in ternary semigroups, *International Refereed Journal of Engineering and Science* (IRJES) **1**(4) (2012), 33 – 43.

[12] F. M. Siosan, Ideals in $(m+1)$-semigroups, *Ann. Mat. Pura Appl.* **68** (1965), 161 – 200, DOI: 10.1007/BF02411024.

[13] F. M. Siosan, Ideal theory in ternary semigroups, *Math. Japan.* **10** (1965), 63 – 84.

[14] F. M. Siosan, On regular algebraic systems: A note on notes by Iseki, Kovacs, and Lajos, *Proc. Japan. Acad.* **39** (1963), 283 – 286, DOI: 10.3792/pja/1195523057.