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Research Article

Fixed Point Theorems for a Demicontractive Mapping and Equilibrium Problems in Hilbert Spaces

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Abstract. In this research, we introduce some properties of demicontractive mapping and the combination of equilibrium problem. Then, we prove a strong convergence for the iterative sequence converging to a common element of fixed point set of demicontractive mapping and a common solution of equilibrium problems. Finally, we give a numerical example for the main theorem to support our results.

Keywords. The combination of equilibrium problem; Fixed point; Demicontractive mapping

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1. Introduction

Let C be a nonempty closed convex subset of a real Hilbert space H. Let $F : C \times C \to \mathbb{R}$ be bifunction. *The equilibrium problem* for F is to determine its equilibrium point, i.e., the set

$$EP(F) = \{x \in C : F(x, y) \geq 0, \forall\, y \in C\}. \tag{1.1}$$

Equilibrium problems were introduced by [1] in 1994 where such problems have had a significant impact and influence in the development of several branches of pure and applied sciences. Various problems in physics, optimization, and economics are related to seeking some elements of $EP(F)$ (see [1,3]). Many authors have been investigated iterative algorithms for the equilibrium problems (see, for example, [3,5,6,15]).

In 2013, Suwannaut and Kangtunyakarn [15] introduced *the combination of equilibrium problem* which is to find $u \in C$ such that

$$\sum_{i=1}^{N} a_i F_i(x,y) \geq 0, \quad \forall\, y \in C, \tag{1.2}$$

where $F_i : C \times C \to \mathbb{R}$ be bifunctions and $a_i \in (0,1)$ with $\sum_{i=1}^{N} a_i = 1$, for every $i = 1,2,\dots,N$. The set of solution (1.2) is denoted by

$$EP\left(\sum_{i=1}^{N} a_i F_i\right) = \bigcap_{i=1}^{N} EP(F_i).$$

Remark 1.1. Very recently, in the work of Suwannaut and Kangtunyakarn [14], Khuangsatung and Kangtunyakarn [7] and Bnouhachem [2], they give the numerical examples for main theorems and show that their iteration for the combination of equilibrium problem converges faster than their iteration for the classical equilibrium problem.

The fixed point problem for the mapping $T : C \to C$ is to find $x \in C$ such that

$$x = Tx. \tag{1.3}$$

We denote the set of solutions of (1.3) by $Fix(T)$. It is well known that $Fix(T)$ is closed and convex and $P_{Fix(T)}$ is well-defined.

Definition 1.1. Let C be a nonempty closed convex subset of a real Hilbert space H.

(i) A mapping $T : C \to C$ is called *nonexpansive* if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall\, x, y \in C.$$

(ii) A mapping $T : C \to C$ is called *quasi-nonexpansive* if $Fix(T) \neq \emptyset$ and

$$\|Tx - y\| \leq \|x - y\|, \quad \forall\, x \in C \text{ and } y \in Fix(T).$$

(iii) A mapping $T : C \to C$ is said to be *κ-strictly pseudo-contractive* if there exists a constant $\kappa \in [0,1)$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + \kappa \|(I - T)x - (I - T)y\|^2, \quad \forall\, x, y \in C. \tag{1.4}$$

In a real Hilbert space, the inequality (1.4) is equivalent to

$$\langle Tx - Ty, x - y\rangle \leq \|x - y\|^2 - \frac{1-\kappa}{2}\|(I - T)x - (I - T)y\|^2, \quad \forall\, x, y \in C. \tag{1.5}$$

Definition 1.2. A mapping T is said to be *demicontractive* if $Fix(T) \neq \emptyset$ and there exists a constant $\kappa \in [0,1)$ such that

$$\|Tx - y\|^2 \leq \|x - y\|^2 + \kappa \|(I - T)x\|^2, \quad \forall\, x \in C \text{ and } y \in Fix(T). \tag{1.6}$$

Observe that the class of demicontractive mapping includes various types of nonlinear mappings such as nonexpansive mapping, quasi-nonexpansive mapping and strictly pseudo-contractive mapping.

Using the same method of proof of (1.5), we obtain that if $T : C \to C$ is demicontractive mapping, then (1.6) is equivalent to the following inequality

$$\langle Tx - y, x - y\rangle \le \|x - y\|^2 - \frac{1-\kappa}{2}\|(I - T)x\|^2, \quad \forall\, x \in C \text{ and } y \in Fix(T). \tag{1.7}$$

In 1977, Maruster [10] introduced the condition (A) of a mapping T.

Definition 1.3 ([10])**.** The mapping T is said to satisfy the condition (A) if $Fix(T)$ is nonempty and if there exists a real positive number λ such that

$$\langle x - Tx, x - y\rangle \ge \lambda\|x - Tx\|^2, \quad \forall\, x \in C, y \in Fix(T).$$

In 2015, Maruster [9] studied a strong convergence theorem of a κ-demicontractive mapping as follows:

Theorem 1.2 ([9])**.** *Suppose that T is κ-demicontractive on C and satisfies the condition* (A)*. Then the sequence $\{x_n\}$ generated by the Mann iteration with control sequence t_k satisfying the condition $0 < a \le t_n \le b < 1-\kappa$, and for suitable starting point x_0, converges strongly to p.*

In 2013, Mongkolkeeha, Cho and Kumam [11] defined the new iterative scheme for two κ-demicontractive mapings as follows:

$$\begin{cases} x_1 \in C \text{ arbitrary chosen,} \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n)(\beta_n S x_n + (1-\beta_n)T x_n), \quad \forall\, n \in \mathbb{N}, \end{cases}$$

where $S, T : C \to C$ be two κ-demicontractive mappings such that $I - S$ is demiclosed at zero with $Fix(S) \cap Fix(T) \neq \emptyset$, $\{\alpha_n\} \subset [\kappa, 1]$ and $\{\beta_n\} \subset [0, 1]$ are the sequences satisfying some control conditions. Then the sequence $\{x_n\}$ converges strongly to a point $v \in Fix(S) \cap Fix(T)$.

Question. Is it possible to prove a strong convergence theorem for a demicontractive mapping and equilibrium problems without using the condition (A) and the control sequences that are not depended on the constant κ?

Motivated by the related research described above, we introduce the Halpern's iterative method modified for demicontractive mapping and a finite family of equilibrium problems. Then, under some appropriate conditions, we prove a strong convergence theorem for the combination of equilibrium problem and a fixed point set of demicontractive mapping. Finally, we give a numerical example for our main result in space of real numbers.

2. Preliminaries

Let H be a real Hilbert space and C be a nonempty closed convex subset of H. We denote weak convergence and strong convergence by notations "$\rightharpoonup$" and "$\to$", respectively. For every $x \in H$, there is a unique nearest point $P_C x$ in C such that

$$\|x - P_C x\| \le \|x - y\|,'' y \in C.$$

Such an operator P_C is called the metric projection of H onto C.

Lemma 2.1 ([16]). *For a given $z \in H$ and $u \in C$,*

$$u = P_C z \Leftrightarrow \langle u - z, v - u \rangle \geq 0,'' v \in C.$$

Furthermore, P_C is a firmly nonexpansive mapping of H onto C and satisfies

$$\|P_C x - P_C y\|^2 \leq \langle P_C x - P_C y, x - y \rangle,'' x, y \in H.$$

Lemma 2.2 ([12]). *Each Hilbert space H satisfies Opial's condition, i.e., for any sequence $\{x_n\} \subset H$ with $x_n \rightharpoonup x$, the inequality*

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|$$

holds for every $y \in H$ with $y \neq x$.

Lemma 2.3 ([17]). *Let $\{s_n\}$ be a sequence of nonnegative real numbers satisfying*

$$s_{n+1} \leq (1 - \alpha_n)s_n + \delta_n, \quad \forall\ n \geq 0,$$

where α_n is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence such that

(1) $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(2) $\limsup_{n\to\infty} \frac{\delta_n}{\alpha_n} \leq 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then, $\lim_{n\to\infty} s_n = 0$.

Lemma 2.4 ([13]). *Let H be a real Hilbert space. Then the following results hold:*

(i) *For all $x, y \in H$ and $\alpha \in [0,1]$,*

$$\|\alpha x + (1-\alpha)y\|^2 = \alpha\|x\|^2 + (1-\alpha)\|y\|^2 - \alpha(1-\alpha)\|x-y\|^2.$$

(ii) $\|x+y\|^2 \leq \|x\|^2 + 2\langle y, x+y \rangle$, *for each $x, y \in H$.*

Lemma 2.5. *Let $T : C \to C$ be a κ-demicontractive mapping with $\kappa \leq \delta$ and $Fix(T) \neq \emptyset$. Define $S : C \to C$ by $Sx := \lambda Tx + (1-\lambda)x$, where $\lambda \in (0,\sigma)$ and $\delta + \sigma < 1$. Then, there hold the following statement:*

(i) $Fix(T) = Fix(S)$;

(ii) *S is a quasi-nonexpansive mapping, that is,*

$$\|Sx - y\| \leq \|x - y\|, \quad \text{for every } x \in C \text{ and } y \in Fix(T).$$

Proof. It is obvious that $Fix(T) = Fix(S)$ due to the fact that $Sx - x = \lambda(Tx - x)$, $\forall\ x \in C$. To prove (ii), let $x \in C$ and $y \in Fix(T)$. Then, by (1.6) and (1.7), we obtain

$$\begin{aligned}
\|Sx - y\|^2 &= \|\lambda(Tx - y) + (1-\lambda)(x-y)\|^2 \\
&\leq \lambda^2\|Tx - y\|^2 + (1-\lambda)^2\|x-y\|^2 + 2\lambda(1-\lambda)\langle Tx - y, x - y \rangle \\
&\leq \lambda^2\left(\|x-y\|^2 + \kappa\|x - Tx\|^2\right) + (1-\lambda)^2\|x-y\|^2 + 2\lambda(1-\lambda)\left(\|x-y\|^2 - \frac{1-\kappa}{2}\|x - Tx\|^2\right) \\
&= \left(\lambda^2 + (1-\lambda)^2 + 2\lambda(1-\lambda)\right)\|x-y\|^2 + \left(\lambda^2\kappa - \lambda(1-\lambda)(1-\kappa)\right)\|x - Tx\|^2 \\
&= \|x-y\|^2 + \lambda(\kappa + \lambda - 1)\|x - Tx\|^2 \\
&\leq \|x-y\|^2 + \lambda(\delta + \sigma - 1)\|x - Tx\|^2 \\
&\leq \|x-y\|^2.
\end{aligned}$$

Therefore, S is a quasi-nonexpansive mapping. □

For solving the equilibrium problem for a bifunction $F: C\times C\to\mathbb{R}$, let us assume that F and C satisfy the following conditions:

(A1) $F(x,x)=0$ for all $x\in C$;

(A2) F is monotone, i.e., $F(x,y)+F(y,x)\le 0$ for all $x,y\in C$;

(A3) For each $x,y,z\in C$, $\lim\limits_{t\to 0^+} F(tz+(1-t)x,y)\le F(x,y)$;

(A4) For each $x\in C, y\mapsto F(x,y)$ is convex and lower semicontinuous.

Remark 2.6. Let C be a nonempty closed convex subset of a real Hilbert space H. For $i=1,2,\dots,N$, let $F_i: C\times C\to\mathbb{R}$ be bifunctions satisfying (A1)-(A4). Then, $\sum\limits_{i=1}^N a_iF_i$ satisfies (A1)-(A4), where $a_i\in(0,1)$ for every $i=1,2,\dots,N$ and $\sum\limits_{i=1}^N a_i=1$.

Proof. For every $i=1,2,\dots,N$, let $F_i: C\times C\to\mathbb{R}$ be bifunctions satisfying (A1)-(A4) and let $x,y,z\in C$ and $a_i\in(0,1)$ for all $i=1,2,\dots,N$ and $\sum\limits_{i=1}^N a_i=1$.
To prove (A1), we get

$$\sum_{i=1}^N a_iF_i(x,x)=a_1F_1(x,x)+a_2F_2(x,x)+\ldots+a_NF_N(x,x)=0.$$

Since

$$\sum_{i=1}^N a_iF_i(x,y)+\sum_{i=1}^N a_iF_i(y,x)=\sum_{i=1}^N a_i(F_i(x,y)+F_i(y,x))\le 0,$$

we have $\sum\limits_{i=1}^N a_iF_i$ satisfies (A2).
Let $t\in[0,1]$, then we have

$$\lim_{t\to 0^+}\sum_{i=1}^N a_iF_i(tz+(1-t)x,y)=\sum_{i=1}^N a_i\lim_{t\to 0^+}F_i(tz+(1-t)x,y)=\sum_{i=1}^N a_iF_i(x,y).$$

Thus (A3) holds.
To prove (A4), we first let $\alpha\in(0,1)$. Therefore, we get

$$\begin{aligned}\sum_{i=1}^N a_iF_i(x,\alpha z+(1-\alpha)y)&\le\sum_{i=1}^N a_i(\alpha F_i(x,z)+(1-\alpha)F_i(x,y))\\&=\alpha\sum_{i=1}^N a_iF_i(x,z)+(1-\alpha)\sum_{i=1}^N a_iF_i(x,y).\end{aligned}$$

It follows that $\sum\limits_{i=1}^N a_iF_i$ is convex. Next, let $\{y_n\}\subset C$ with $y_n\to y$ as $n\to\infty$. Thus we obtain

$$\liminf_{n\to\infty}\sum_{i=1}^N a_iF_i(x,y_n)\ge\sum_{i=1}^N a_i\liminf_{n\to\infty}F_i(x,y_n)\ge\sum_{i=1}^N a_iF_i(x,y).$$

Then $\sum\limits_{i=1}^N a_iF_i$ is lower semicontinuous. This implies that (A4) holds.
We can conclude that $\sum\limits_{i=1}^N a_iF_i$ satisfies (A1)-(A4). □

Lemma 2.7 ([15]). *Let C be a nonempty closed convex subset of a real Hilbert space H. For $i = 1,2,\dots,N$, let $F_i : C \times C \to \mathbb{R}$ be bifunctions satisfying* (A1)-(A4) *with $\bigcap_{i=1}^{N} EP(F_i) \neq \emptyset$. Then,*

$$EP\left(\sum_{i=1}^{N} a_i F_i\right) = \bigcap_{i=1}^{N} EP(F_i),$$

where $a_i \in (0,1)$ for every $i = 1,2,\dots,N$ and $\sum_{i=1}^{N} a_i = 1$.

Lemma 2.8. *Let C be a nonempty closed convex subset of a real Hilbert space H. For $i = 1,2,\dots,N$, let $F_i : C \times C \to \mathbb{R}$ be bifunctions satisfying* (A1)-(A4). *Let the sequences $\{x_n\} \subseteq H$, $\{u_n\} \subseteq C$ and $\{r_n\} \subseteq (0,1)$ satisfying the following condition:*

$$\sum_{i=1}^{N} a_i F_i(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall\ y \in C.$$

Therefore, if $u_{n_k} \rightharpoonup \omega$ as $k \to \infty$ and $\|u_n - x_n\| \to 0$ as $n \to \infty$, then $\omega \in \bigcap_{i=1}^{N} EP(F_i)$.

Proof. Due to the fact that F_i is bifunctions satisfying (A1)-(A4), for all $i = 1,2,\dots,N$, then, by Remark 2.6, we have $\sum_{i=1}^{N} a_i F_i$ satisfies the conditions (A1)-(A4). Since

$$\sum_{i=1}^{N} a_i F_i(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \forall y \in C,$$

and $\sum_{i=1}^{N} a_i F_i$ satisfies the conditions (A1)-(A4), we obtain

$$\frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq \sum_{i=1}^{N} a_i F_i(y, u_n), \quad \forall\ y \in C.$$

In particular, it follows that

$$\left\langle y - u_{n_k}, \frac{u_{n_k} - x_{n_k}}{r_{n_k}}\right\rangle \geq \sum_{i=1}^{N} a_i F_i\left(y, u_{n_k}\right), \quad \forall\ y \in C. \tag{2.1}$$

From $\|u_n - x_n\| \to 0$ as $n \to \infty$, (2.1) and (A4), we have

$$\sum_{i=1}^{N} a_i F_i(y, \omega) \leq 0, \quad \forall\ y \in C. \tag{2.2}$$

Put $y_t := ty + (1-t)\omega$, $t \in (0,1]$, we have $y_t \in C$. By using (A1), (A4) and (2.2), we have

$$\begin{aligned}
0 &= \sum_{i=1}^{N} a_i F_i(y_t, y_t) \\
&= \sum_{i=1}^{N} a_i F_i(y_t, ty + (1-t)\omega) \\
&\leq t\sum_{i=1}^{N} a_i F_i(y_t, y) + (1-t)\sum_{i=1}^{N} a_i F_i(y_t, \omega) \\
&\leq t\sum_{i=1}^{N} a_i F_i(y_t, y).
\end{aligned}$$

It implies that

$$\sum_{i=1}^{N} a_i F_i(ty+(1-t)\omega, y) \geq 0, \quad \forall\, t \in (0,1] \text{ and } \forall y \in C. \tag{2.3}$$

From (2.3), taking $t \to 0^+$ and using (A3), we can conclude that

$$0 \leq \sum_{i=1}^{N} a_i F_i(\omega, y), \quad \forall\, y \in C.$$

Therefore, $\omega \in EP\left(\sum_{i=1}^{N} a_i F_i\right)$. By Lemma 2.7, we obtain $EP\left(\sum_{i=1}^{N} a_i F_i\right) = \bigcap_{i=1}^{N} EP(F_i)$. It follows that

$$\omega \in \bigcap_{i=1}^{N} EP(F_i). \tag{2.4}$$

□

Lemma 2.9 ([1]). *Let C be a nonempty closed convex subset of H and let F be a bifunction of $C \times C$ into $\mathbb{R}$ satisfying* (A1)-(A4). *Let $r > 0$ and $x \in H$. Then, there exists $z \in C$ such that*

$$F(z,y) + \frac{1}{r}\langle y-z, z-x\rangle \geq 0, \quad \forall\, y \in C.$$

Lemma 2.10 ([3]). *Assume that $F : C \times C \to \mathbb{R}$ satisfies* (A1)-(A4). *For $r > 0$, define a mapping $T_r : H \to C$ as follows:*

$$T_r(x) = \left\{z \in C : F(z,y) + \frac{1}{r}\langle y-z, z-x\rangle \geq 0, \quad \forall\, y \in C\right\}$$

for all $x \in H$. Then, the following hold:

(i) *T_r is single-valued;*

(ii) *T_r is firmly nonexpansive, i.e., for any $x, y \in H$,*

$$\|T_r(x) - T_r(y)\|^2 \leq \langle T_r(x) - T_r(y), x-y\rangle;$$

(iii) *$Fix(T_r) = EP(F)$;*

(iv) *$EP(F)$ is closed and convex.*

Remark 2.11 ([15]). By Remark 2.6, we have $\sum_{i=1}^{N} a_i F_i$ satisfies (A1)-(A4). By using Lemma 2.7 and Lemma 2.10, we obtain

$$Fix(T_r) = EP\left(\sum_{i=1}^{N} a_i F_i\right) = \bigcap_{i=1}^{N} EP(F_i),$$

where $a_i \in (0,1)$, for each $i = 1,2,\ldots,N$, and $\sum_{i=1}^{N} a_i = 1$.

3. Strong Convergence Theorem

Theorem 3.1. *Let C be a nonempty closed convex subset of a real Hilbert space H. For $i = 1,2,\ldots,N$, let $F_i : C \times C \to \mathbb{R}$ be a bifunction satisfying* (A1)-(A4). *Let $T : C \to C$ be a demicontractive mapping with coefficient $\kappa \leq \theta_1$ and let a mapping $S_n : C \to C$ be defined by $S_n x := (1-\lambda_n)x + \lambda_n Tx$ with $\lambda_n < \theta_2$ and $\theta_1 + \theta_2 < 1$. Assume that $\Theta = \bigcap_{i=1}^{N} EP(F_i) \cap Fix(T) \neq \emptyset$.*

Let the sequences $\{x_n\}$ *and* $\{u_n\}$ *be generated by* $x_1, u \in C$ *and*

$$\begin{cases} \sum_{i=1}^{N} a_i F_i(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, & \forall\, y \in C, \\ x_{n+1} = \beta_n(\alpha_n u + (1-\alpha_n)u_n) + \left(1-\beta_n\right)S_n x_n, & \forall\, n \geq 1, \end{cases} \tag{3.1}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\lambda_n\} \subseteq (0,1)$ *and* $0 \leq a_i \leq 1$ *for every* $i = 1, 2, \dots, N$, *satisfying the following conditions:*

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(ii) $0 < \tau \leq \beta_n \leq \upsilon < 1$, *for some* $\tau, \upsilon > 0$;

(iii) $\sum_{n=1}^{\infty} \lambda_n < \infty$;

(iv) $0 < \epsilon \leq r_n \leq \eta < \infty$, *for some* $\epsilon, \eta > 0$;

(v) $\sum_{i=1}^{N} a_i = 1$;

(vi) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$, $\sum_{n=1}^{\infty} \left|\beta_{n+1} - \beta_n\right| < \infty$, $\sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty$, $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$.

Then $\{x_n\}$ *and* $\{u_n\}$ *converge strongly to* $q = P_\Theta u$.

Proof. The proof of this theorem will be divided into five steps.

Step 1. We show that $\{x_n\}$ is bounded.

Since $\sum_{i=1}^{N} a_i F_i$ satisfies (A1)-(A4) and

$$\sum_{i=1}^{N} a_i F_i(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall\, y \in C,$$

by Lemma 2.10 and Remark 2.11, we have $u_n = T_{r_n} x_n$ and $Fix(T_{r_n}) = \bigcap_{i=1}^{N} EP(F_i)$.

Let $z \in \Theta$. From Lemma 2.5 and Lemma 2.10 , we obtain

$$\begin{aligned} \|x_{n+1} - z\| &= \left\|\beta_n(\alpha_n u + (1-\alpha_n)u_n) + \left(1-\beta_n\right)S_n x_n - z\right\| \\ &= \left\|\beta_n(\alpha_n(u-z) + (1-\alpha_n)(u_n - z)) + \left(1-\beta_n\right)(S_n x_n - z)\right\| \\ &\leq \beta_n \|\alpha_n(u-z) + (1-\alpha_n)(u_n - z)\| + \left(1-\beta_n\right)\|S_n x_n - z\| \\ &\leq \beta_n(\alpha_n\|u-z\| + (1-\alpha_n)\|u_n - z\|) + \left(1-\beta_n\right)\|x_n - z\| \\ &\leq \beta_n(\alpha_n\|u-z\| + (1-\alpha_n)\|x_n - z\|) + \left(1-\beta_n\right)\|x_n - z\| \\ &\leq \max\{\|u-z\|, \|x_1 - z\|\}. \end{aligned}$$

By induction, we get $\|x_n - z\| \leq \max\{\|u-z\|, \|x_1 - z\|\}, \forall n \in \mathbb{N}$. This implies that $\{x_n\}$ is bounded and so is $\{u_n\}$.

Step 2. We will show that $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$.

Since $u_n = T_{r_n} x_n$, by utilizing the definition of T_{r_n}, we obtain

$$\sum_{i=1}^{N} a_i F_i\left(T_{r_n} x_n, y\right) + \frac{1}{r_n}\left\langle y - T_{r_n} x_n, T_{r_n} x_n - x_n\right\rangle \geq 0, \quad \forall\, y \in C, \tag{3.2}$$

and

$$\sum_{i=1}^{N} a_i F_i\left(T_{r_{n+1}}x_{n+1}, y\right) + \frac{1}{r_{n+1}}\left\langle y - T_{r_{n+1}}x_{n+1}, T_{r_{n+1}}x_{n+1} - x_{n+1}\right\rangle \ge 0, \quad \forall\, y \in C. \tag{3.3}$$

From (3.2) and (3.3), it follows that

$$\sum_{i=1}^{N} a_i F_i\left(T_{r_n}x_n, T_{r_{n+1}}x_{n+1}\right) + \frac{1}{r_n}\left\langle T_{r_{n+1}}x_{n+1} - T_{r_n}x_n, T_{r_n}x_n - x_n\right\rangle \ge 0, \tag{3.4}$$

and

$$\sum_{i=1}^{N} a_i F_i\left(T_{r_{n+1}}x_{n+1}, T_{r_n}x_n\right) + \frac{1}{r_{n+1}}\left\langle T_{r_n}x_n - T_{r_{n+1}}x_{n+1}, T_{r_{n+1}}x_{n+1} - x_{n+1}\right\rangle \ge 0. \tag{3.5}$$

From (3.4) and (3.5) and the fact that $\sum_{i=1}^{N} a_i F_i$ satisfies (A2), we have

$$\frac{1}{r_n}\left\langle T_{r_{n+1}}x_{n+1} - T_{r_n}x_n, T_{r_n}x_n - x_n\right\rangle + \frac{1}{r_{n+1}}\left\langle T_{r_n}x_n - T_{r_{n+1}}x_{n+1}, T_{r_{n+1}}x_{n+1} - x_{n+1}\right\rangle \ge 0,$$

which implies that

$$\left\langle T_{r_n}x_n - T_{r_{n+1}}x_{n+1}, \frac{T_{r_{n+1}}x_{n+1} - x_{n+1}}{r_{n+1}} - \frac{T_{r_n}x_n - x_n}{r_n}\right\rangle \ge 0.$$

It follows that

$$\left\langle T_{r_{n+1}}x_{n+1} - T_{r_n}x_n, T_{r_n}x_n - T_{r_{n+1}}x_{n+1} + T_{r_{n+1}}x_{n+1} - x_n - \frac{r_n}{r_{n+1}}\left(T_{r_{n+1}}x_{n+1} - x_{n+1}\right)\right\rangle \ge 0. \tag{3.6}$$

From (3.6), we obtain

$$\begin{aligned}
\left\|T_{r_{n+1}}x_{n+1} - T_{r_n}x_n\right\|^2 &\le \left\langle T_{r_{n+1}}x_{n+1} - T_{r_n}x_n, T_{r_{n+1}}x_{n+1} - x_n - \frac{r_n}{r_{n+1}}\left(T_{r_{n+1}}x_{n+1} - x_{n+1}\right)\right\rangle \\
&= \left\langle T_{r_{n+1}}x_{n+1} - T_{r_n}x_n, x_{n+1} - x_n + \left(1 - \frac{r_n}{r_{n+1}}\right)\left(T_{r_{n+1}}x_{n+1} - x_{n+1}\right)\right\rangle \\
&\le \left\|T_{r_{n+1}}x_{n+1} - T_{r_n}x_n\right\| \left\|x_{n+1} - x_n + \left(1 - \frac{r_n}{r_{n+1}}\right)\left(T_{r_{n+1}}x_{n+1} - x_{n+1}\right)\right\| \\
&\le \left\|T_{r_{n+1}}x_{n+1} - T_{r_n}x_n\right\| \left[\|x_{n+1} - x_n\| + \left|1 - \frac{r_n}{r_{n+1}}\right| \left\|T_{r_{n+1}}x_{n+1} - x_{n+1}\right\|\right] \\
&= \left\|T_{r_{n+1}}x_{n+1} - T_{r_n}x_n\right\| \left[\|x_{n+1} - x_n\| + \frac{1}{r_{n+1}}|r_{n+1} - r_n| \left\|T_{r_{n+1}}x_{n+1} - x_{n+1}\right\|\right] \\
&\le \left\|T_{r_{n+1}}x_{n+1} - T_{r_n}x_n\right\| \left[\|x_{n+1} - x_n\| + \frac{1}{d}|r_{n+1} - r_n| \left\|T_{r_{n+1}}x_{n+1} - x_{n+1}\right\|\right],
\end{aligned}$$

which follows that

$$\|u_{n+1} - u_n\| \le \|x_{n+1} - x_n\| + \frac{1}{\epsilon}|r_{n+1} - r_n| \|u_{n+1} - x_{n+1}\|. \tag{3.7}$$

From (3.7), we have

$$\|u_n - u_{n-1}\| \le \|x_n - x_{n-1}\| + \frac{1}{\epsilon}|r_n - r_{n-1}| \|u_n - x_n\|. \tag{3.8}$$

First, we let $y_n = \alpha_n u + (1 - \alpha_n)u_n$. From (3.8), we derive that

$$\begin{aligned}
\|x_{n+1} - x_n\| \le{}& \beta_n \|y_n - y_{n-1}\| + \left|\beta_n - \beta_{n-1}\right| \|y_{n-1}\| + \left(1 - \beta_n\right) \|S_n x_n - S_{n-1}x_{n-1}\| \\
&+ \left|\beta_n - \beta_{n-1}\right| \|S_{n-1}x_{n-1}\|
\end{aligned}$$

$$
\begin{aligned}
&\leq \beta_n \Big[|\alpha_n - \alpha_{n-1}| \|u\| + (1-\alpha_n)\|u_n - u_{n-1}\| + |\alpha_n - \alpha_{n-1}| \|u_{n-1}\| \Big] \\
&\quad + \left|\beta_n - \beta_{n-1}\right| \|y_{n-1}\| + \left(1-\beta_n\right) \Big[(1-\lambda_n)\|x_n - x_{n-1}\| + |\lambda_n - \lambda_{n-1}| \|x_{n-1}\| \\
&\quad + \lambda_n \|Tx_n - Tx_{n-1}\| + |\lambda_n - \lambda_{n-1}| \|Tx_{n-1}\| \Big] + \left|\beta_n - \beta_{n-1}\right| \|S_{n-1}x_{n-1}\| \\
&\leq \beta_n(1-\alpha_n) \left[\|x_n - x_{n-1}\| + \frac{1}{\epsilon}|r_n - r_{n-1}| \|u_n - x_n\| \right] + \left(1-\beta_n\right)\|x_n - x_{n-1}\| \\
&\quad + |\alpha_n - \alpha_{n-1}|(\|u\| + \|u_{n-1}\|) + \left|\beta_n - \beta_{n-1}\right|(\|y_{n-1}\| + \|S_{n-1}x_{n-1}\|) \\
&\quad + |\lambda_n - \lambda_{n-1}|(\|x_{n-1}\| + \|Tx_{n-1}\|) + \lambda_n \|Tx_n - Tx_{n-1}\| \\
&\leq \left(1 - \alpha_n\beta_n\right)\|x_n - x_{n-1}\| + \frac{1}{\epsilon}|r_n - r_{n-1}| \|u_n - x_n\| + |\alpha_n - \alpha_{n-1}|(\|u\| + \|u_{n-1}\|) \\
&\quad + \left|\beta_n - \beta_{n-1}\right|(\|y_{n-1}\| + \|S_{n-1}x_{n-1}\|) + |\lambda_n - \lambda_{n-1}|(\|x_{n-1}\| + \|Tx_{n-1}\|) \\
&\quad + \lambda_n \|Tx_n - Tx_{n-1}\|.
\end{aligned}
$$

By Lemma 2.3 and the conditions (i), (ii), (vi), we obtain

$$
\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{3.9}
$$

Step 3. Prove that $\lim_{n\to\infty} \|u_n - x_n\| = 0$ and $\lim_{n\to\infty} \|S_n x_n - x_n\| = 0$.

To show this, let $z \in \Theta$. Since $u_n = T_{r_n}x_n$ and T_{r_n} is firmly nonexpansive mapping, then we obtain

$$
\begin{aligned}
\left\|z - T_{r_n}x_n\right\|^2 &= \left\|T_{r_n}z - T_{r_n}x_n\right\|^2 \\
&\leq \left\langle T_{r_n}z - T_{r_n}x_n, z - x_n \right\rangle \\
&= \frac{1}{2}\left(\left\|T_{r_n}x_n - z\right\|^2 + \|x_n - z\|^2 - \left\|T_{r_n}x_n - x_n\right\|^2 \right),
\end{aligned}
$$

which follows that

$$
\left\|T_{r_n}x_n - z\right\|^2 \leq \|x_n - z\|^2 - \left\|T_{r_n}x_n - x_n\right\|^2.
$$

That is,

$$
\|u_n - z\|^2 \leq \|x_n - z\|^2 - \|u_n - x_n\|^2. \tag{3.10}
$$

By the definition of x_n, we have

$$
\begin{aligned}
\|x_{n+1} - z\|^2 &= \left\|\beta_n(\alpha_n(u-z) + (1-\alpha_n)(u_n - z)) + \left(1-\beta_n\right)(S_n x_n - z)\right\|^2 \\
&\leq \beta_n \|\alpha_n(u-z) + (1-\alpha_n)(u_n - z)\|^2 + \left(1-\beta_n\right)\|S_n x_n - z\|^2 \\
&\leq \beta_n \left[\alpha_n \|u - z\|^2 + (1-\alpha_n)\|u_n - z\|^2\right] + \left(1-\beta_n\right)\|x_n - z\|^2 \\
&\leq \beta_n \left[\alpha_n \|u - z\|^2 + (1-\alpha_n)\left(\|x_n - z\|^2 - \|u_n - x_n\|^2\right)\right] + \left(1-\beta_n\right)\|x_n - z\|^2 \\
&\leq \alpha_n \|u - z\|^2 + \|x_n - z\|^2 - \beta_n(1-\alpha_n)\|u_n - x_n\|^2,
\end{aligned}
$$

which implies that

$$
\begin{aligned}
\beta_n(1-\alpha_n)\|u_n - x_n\|^2 &\leq \alpha_n \|u - z\|^2 + \|x_n - z\|^2 - \|x_{n+1} - z\|^2 \\
&\leq \alpha_n \|u - z\|^2 + (\|x_n - z\| + \|x_{n+1} - z\|)\|x_{n+1} - x_n\|.
\end{aligned}
$$

From (i), (ii) and 3.9, we have

$$\lim_{n\to\infty} \|u_n - x_n\| = 0. \tag{3.11}$$

Since

$$x_{n+1} - x_n = \beta_n(\alpha_n(u - x_n) + (1-\alpha_n)(u_n - x_n)) + \left(1-\beta_n\right)(S_n x_n - x_n),$$

then we get

$$\left(1-\beta_n\right)\|S_n x_n - x_n\| \le \beta_n \alpha_n \|u - x_n\| + \beta_n(1-\alpha_n)\|u_n - x_n\| + \|x_{n+1} - x_n\|.$$

This follows by (i), (ii) and 3.11 that

$$\lim_{n\to\infty} \|S_n x_n - x_n\| = 0. \tag{3.12}$$

Step 4. We will show that $\limsup_{n\to\infty} \langle u - q, x_n - q\rangle \le 0$, where $q = P_\Theta u$.
To show this, choose a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty} \langle u - z, x_n - z\rangle = \lim_{k\to\infty} \left\langle u - z, x_{n_k} - z\right\rangle. \tag{3.13}$$

Without loss of generality, we can assume that $x_{n_k} \rightharpoonup \omega$ as $k \to \infty$ where $\omega \in C$. From (3.11), we obtain $u_{n_k} \rightharpoonup \omega$ as $k \to \infty$.
From

$$\sum_{i=1}^{N} a_i F_i(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \ge 0, \quad \forall\ y \in C,$$

and $u_{n_k} \rightharpoonup \omega$ as $k \to \infty$, by Lemma 2.8, we can conclude that

$$\omega \in EP\left(\sum_{i=1}^{N} a_i F_i\right) = \bigcap_{i=1}^{N} EP(F_i). \tag{3.14}$$

Next, we will show that $\omega \in Fix(T)$.

By Lemma 2.5, we have $Fix(S_n) = Fix(T)$. Assume that $\omega \ne S_n\omega$. Using Opial's condition, (3.12) and the condition (iii), then we obtain

$$\begin{aligned}
\liminf_{k\to\infty}\left\|x_{n_k} - \omega\right\| &< \liminf_{k\to\infty}\left\|x_{n_k} - S_{n_k}\omega\right\| \\
&\le \liminf_{k\to\infty}\left(\left\|x_{n_k} - S_{n_k}x_{n_k}\right\| + \left\|S_{n_k}\omega - S\omega\right\|\right) \\
&\le \liminf_{k\to\infty}\left(\left\|x_{n_k} - S_{n_k}x_{n_k}\right\| + \left\|x_{n_k} - \omega\right\| + \lambda_{n_k}\left\|(I-T)x_{n_k} - (I-T)\omega\right\|\right) \\
&\le \liminf_{k\to\infty}\left\|x_{n_k} - \omega\right\|.
\end{aligned}$$

This is a contradiction. Then we have

$$\omega \in Fix(S_n) = Fix(T). \tag{3.15}$$

From (3.14) and (3.15), we can deduce that $\omega \in \Theta$.

Since $x_{n_k} \rightharpoonup \omega$ as $k \to \infty$, $q = P_\Theta u$ and $\omega \in \Theta$, then, by Lemma 2.1, we can conclude that

$$\begin{aligned}
\limsup_{n\to\infty} \langle u - q, x_n - q\rangle &= \lim_{k\to\infty} \left\langle u - q, x_{n_k} - z\right\rangle \\
&= \langle u - q, \omega - q\rangle \\
&\le 0.
\end{aligned} \tag{3.16}$$

Step 5. Finally, we will show that the sequence $\{x_n\}$ converges strongly to $q = P_\Theta u$.
By the definition of x_n and Lemma 2.4, we have

$$\begin{aligned}
\|x_{n+1}-q\|^2 &= \left\|\beta_n(\alpha_n(u-q)+(1-\alpha_n)(u_n-q))+\left(1-\beta_n\right)(S_nx_n-q)\right\|^2 \\
&\le \left\|\beta_n(1-\alpha_n)(u_n-q)+\left(1-\beta_n\right)(S_nx_n-q)\right\|^2+2\alpha_n\beta_n\langle u-q,x_{n+1}-q\rangle \\
&\le \left(\beta_n(1-\alpha_n)\|u_n-q\|+\left(1-\beta_n\right)\|S_nx_n-q\|\right)^2+2\alpha_n\beta_n\langle u-q,x_{n+1}-q\rangle \\
&\le \left(\beta_n(1-\alpha_n)\|x_n-q\|+\left(1-\beta_n\right)\|x_n-q\|\right)^2+2\alpha_n\beta_n\langle u-q,x_{n+1}-q\rangle \\
&= \left(1-\alpha_n\beta_n\right)^2\|x_n-q\|^2+2\alpha_n\beta_n\langle u-q,x_{n+1}-q\rangle \\
&\le \left(1-\alpha_n\beta_n\right)\|x_n-q\|^2+2\alpha_n\beta_n\langle u-q,x_{n+1}-q\rangle.
\end{aligned}$$

From (3.16), the conditions (i), (ii) and Lemma 2.3, we can conclude that $\{x_n\}$ converges strongly to $q = P_\Theta u$. By (3.11), we have $\{u_n\}$ converges strongly to $q = P_\Theta u$. This completes the proof. □

The following corollary is a direct consequence of Theorem 3.1.

Corollary 3.2. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $F : C \times C \to \mathbb{R}$ be a bifunction satisfying* (A1)-(A4)*. Let $T : C \to C$ be a demicontractive mapping with coefficient $\kappa \le \theta_1$ and let a mapping $S_n : C \to C$ be defined by $S_nx := (1-\lambda_n)x+\lambda_nTx$ with $\lambda_n < \theta_2$ and $\theta_1+\theta_2 < 1$. Assume that $\Theta = EP(F)\cap Fix(T) \neq \emptyset$. Let the sequences $\{x_n\}$ and $\{u_n\}$ be generated by $x_1, u \in C$ and*

$$\begin{cases} F(u_n,y)+\frac{1}{r_n}\langle y-u_n,u_n-x_n\rangle \ge 0, & \forall\, y\in C, \\ x_{n+1}=\beta_n(\alpha_nu+(1-\alpha_n)u_n)+\left(1-\beta_n\right)S_nx_n, & \forall\, n\ge 1, \end{cases} \tag{3.17}$$

where $\{\alpha_n\},\{\beta_n\},\{\lambda_n\} \subseteq (0,1)$ and $0 \le a_i \le 1$ for every $i = 1,2,\dots,N$, satisfying the following conditions:

(i) $\lim_{n\to\infty}\alpha_n = 0$ *and* $\sum_{n=1}^{\infty}\alpha_n = \infty$*;*

(ii) $0 < \tau \le \beta_n \le \upsilon < 1$*, for some* $\tau,\upsilon > 0$*;*

(iii) $0 < \rho \le \lambda_n < \theta_2 < 1$*, for some* $\rho,\theta_2 > 0$ *and* $\sum_{n=1}^{\infty}\lambda_n < \infty$*;*

(iv) $0 < \epsilon \le r_n \le \eta < \infty$*, for some* $\epsilon,\eta > 0$*;*

(v) $\sum_{n=1}^{\infty}|\alpha_{n+1}-\alpha_n| < \infty$, $\sum_{n=1}^{\infty}\left|\beta_{n+1}-\beta_n\right| < \infty$, $\sum_{n=1}^{\infty}|\lambda_{n+1}-\lambda_n| < \infty$, $\sum_{n=1}^{\infty}|r_{n+1}-r_n| < \infty$.

Then $\{x_n\}$ and $\{u_n\}$ converge strongly to $q = P_\Theta u$.

Proof. Take $F = F_i$, $\forall\, i = 1,2,\dots,N$ in Theorem 3.1. Then we obtain the desired result. □

4. Applications

In this section, we obtain our additional results for fixed point problem of a nonspreading mapping and a quasi-nonexpansive mapping.

In 2008, Kohsaka and Takahashi [8] introduced *the nonspreading mapping T* in Hilbert space H as follows:

$$2\|Tu-Tv\|^2 \le \|Tu-v\|^2+\|u-Tv\|^2, \quad \forall\, u,v\in C. \tag{4.1}$$

In 2009, it is shown by Iemoto and Takahashi [4] that (4.1) is equivalent to the following equation.

$$\|Tu - Tv\|^2 \le \|u-v\|^2 + 2\langle u - Tu, v - Tv\rangle, \quad \forall\, u,v \in C.$$

In 2014, Suwannaut and Kangtunyakarn [14] obtain the following main results for a nonspraeding mapping on C.

Lemma 4.1 ([14]). *Let C be a nonempty closed convex subset of a real Hilbert space H and let $T : C \to C$ be a nonspreading mapping with $Fix(T) \neq \emptyset$. Then there hold the following statement:*

(i) $Fix(T) = VI(C, I-T)$;

(ii) *For every $u \in C$ and $v \in Fix(T)$,*

$$\|P_C(I - \lambda(I-T))u - v\| \le \|u - v\|, \text{ where } \lambda \in (0,1),$$

that is, a mapping $P_C(I-\lambda(I-T))$ is quasi-nonexpansive.

Theorem 4.2. *Let C be a nonempty closed convex subset of a real Hilbert space H. For $i = 1,2,\dots,N$, let $F_i : C \times C \to \mathbb{R}$ be a bifunction satisfying* (A1)-(A4). *Let $T : C \to C$ be a nonspreading mapping and let a mapping $W_n : C \to C$ be defined by $W_n x := (1-\rho_n)x + \rho_n Tx$. Assume that $\Theta = \bigcap_{i=1}^{N} EP(F_i) \cap Fix(T) \neq \emptyset$. Let the sequences $\{x_n\}$ and $\{u_n\}$ be generated by $x_1, u \in C$ and*

$$\begin{cases} \sum\limits_{i=1}^{N} a_i F_i(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \ge 0, & \forall\, y \in C, \\ x_{n+1} = \beta_n(\alpha_n u + (1-\alpha_n)u_n) + \big(1-\beta_n\big) W_n x_n, & \forall\, n \ge 1, \end{cases} \tag{4.2}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\rho_n\} \subseteq (0,1)$ and $0 \le a_i \le 1$ for every $i = 1,2,\dots,N$, satisfying the following conditions:

(i) $\lim\limits_{n\to\infty} \alpha_n = 0$ *and* $\sum\limits_{n=1}^{\infty} \alpha_n = \infty$;

(ii) $0 < \tau \le \beta_n \le \upsilon < 1$, *for some* $\tau, \upsilon > 0$;

(iii) $\sum\limits_{n=1}^{\infty} \rho_n < \infty$;

(iv) $0 < \epsilon \le r_n \le \eta < \infty$, *for some* $\epsilon, \eta > 0$;

(v) $\sum\limits_{i=1}^{N} a_i = 1$;

(vi) $\sum\limits_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$, $\sum\limits_{n=1}^{\infty} \big|\beta_{n+1} - \beta_n\big| < \infty$, $\sum\limits_{n=1}^{\infty} \big|\rho_{n+1} - \rho_n\big| < \infty$, $\sum\limits_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$.

Then $\{x_n\}$ and $\{u_n\}$ converge strongly to $q = P_\Theta u$.

Proof. Applying Lemma 4.1 and the same proof of Theorem 3.1, we obtain the desired results. □

Observe that every a nonspreading mapping T with $Fix(T) \neq \emptyset$ is quasi-nonexpansive. Then we also have the following result.

Lemma 4.3. *Let C be a nonempty closed convex subset of a real Hilbert space H and let $T : C \to C$ be a quasi-nonexpansive mapping with $Fix(T) \neq \emptyset$. Then the following results are true:*

(i) $Fix(T) = VI(C, I-T)$;

(ii) *For every $u \in C$ and $v \in Fix(T)$,*

$$\|P_C(I-\lambda(I-T))u - v\| \le \|u-v\|, \text{ where } \lambda \in (0,1).$$

Theorem 4.4. *Let C be a nonempty closed convex subset of a real Hilbert space H. For $i = 1,2,\dots,N$, let $F_i : C\times C \to \mathbb{R}$ be a bifunction satisfying* (A1)-(A4). *Let $T : C \to C$ be a quasi-nonexpansive mapping and let a mapping $W_n : C \to C$ be defined by $W_n x := (1-\rho_n)x + \rho_n Tx$. Assume that $\Theta = \bigcap_{i=1}^N EP(F_i) \cap Fix(T) \neq \emptyset$. Let the sequences $\{x_n\}$ and $\{u_n\}$ be generated by $x_1, u \in C$ and*

$$\begin{cases} \sum_{i=1}^N a_i F_i(u_n, y) + \frac{1}{r_n}\langle y-u_n, u_n - x_n\rangle \ge 0, & \forall\, y \in C, \\ x_{n+1} = \beta_n(\alpha_n u + (1-\alpha_n)u_n) + \big(1-\beta_n\big)W_n x_n, & \forall\, n \ge 1, \end{cases} \tag{4.3}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\rho_n\} \subseteq (0,1)$ and $0 \le a_i \le 1$ for every $i = 1,2,\dots,N$, satisfying the following conditions:

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^\infty \alpha_n = \infty$;

(ii) $0 < \tau \le \beta_n \le \upsilon < 1$, *for some* $\tau, \upsilon > 0$;

(iii) $\sum_{n=1}^\infty \rho_n < \infty$;

(iv) $0 < \epsilon \le r_n \le \eta < \infty$, *for some* $\epsilon, \eta > 0$;

(v) $\sum_{i=1}^N a_i = 1$;

(vi) $\sum_{n=1}^\infty |\alpha_{n+1} - \alpha_n| < \infty$, $\sum_{n=1}^\infty \big|\beta_{n+1} - \beta_n\big| < \infty$, $\sum_{n=1}^\infty \big|\rho_{n+1} - \rho_n\big| < \infty$, $\sum_{n=1}^\infty |r_{n+1} - r_n| < \infty$.

Then $\{x_n\}$ and $\{u_n\}$ converge strongly to $q = P_\Theta u$.

Proof. Using Lemma 4.3 and Theorem 3.1, we get the result of Theorem 4.4. □

5. A Numerical Example

In this section, we give numerical examples to support our main theorem.

Example 5.1. Let $\mathbb{R}$ be the set of real numbers. For every $i = 1,2,\dots,N$, let $F_i : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ and $T : \mathbb{R} \to \mathbb{R}$ be defined by

$$Tx = \frac{-7x}{5},$$
$$F_i(x,y) = i(-5x^2 + xy + 4y^2), \text{ for all } x, y \in \mathbb{R}.$$

Put $a_i = \frac{2}{7^i} + \frac{1}{N7^N}$, for every $i = 1,2,\dots,N$. Let $\alpha_n = \frac{1}{100n}$, $\beta_n = \frac{3n}{5n+3}$, $r_n = \frac{5n+6}{8n+9}$ and $\lambda_n = \frac{1}{n^2+2}$ for every $n \in \mathbb{N}$. Then, the sequences $\{x_n\}$ and $\{u_n\}$ converge strongly to 0.

Solution. Obviously, T is κ-demicontractive mapping with $\kappa = \frac{1}{6}$ and $Fix(T) = \{0\}$. If we choose $\theta_1 = \frac{1}{5}$ and $\theta_2 = \frac{1}{2}$, then we obtain $\theta_1 + \theta_2 = \frac{7}{10} < 1$. This implies by Lemma 2.5 that a mapping S_n is quasi-nonexpansive mapping.

Since $a_i = \frac{2}{7^i} + \frac{1}{N7^N}$, we obtain

$$\begin{aligned}\sum_{i=1}^{N} a_i F_i(x,y) &= \sum_{i=1}^{N}\left(\frac{2}{7^i} + \frac{1}{N7^N}\right) i(-5x^2 + xy + 4y^2)\\ &= \xi(-5x^2 + xy + 4y^2),\end{aligned}$$

where $\xi = \sum_{i=1}^{N}\left(\frac{2}{7^i} + \frac{1}{N7^N}\right) i$. It is clear to check that $\sum_{i=1}^{N} a_i F_i$ satisfies all conditions (A1)-(A4) and $0 \in EP\left(\sum_{i=1}^{N} a_i \Phi_i\right) = \bigcap_{i=1}^{N} EP(\Phi_i)$. Then we have

$$Fix(T) \cap \bigcap_{i=1}^{N} EP(F_i) = \{0\}.$$

Observe that

$$\begin{aligned}0 &\le \sum_{i=1}^{N} a_i F_i(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle\\ &= \xi(-5u_n^2 + u_n y + 4y^2) + \frac{1}{r_n}(y - u_n)(u_n - x_n)\\ &\Leftrightarrow\\ 0 &\le r_n\xi(-5u_n^2 + u_n y + 4y^2) + (y - u_n)(u_n - x_n)\\ &= 4\xi r_n y^2 + (u_n + r_n u_n \xi - x_n) y - u_n^2 - 5\xi r_n (u_n)^2 + u_n x_n. \end{aligned} \tag{5.1}$$

Let $G(y) = 4\xi r_n y^2 + (u_n + r_n u_n \xi - x_n) y - u_n^2 - 5\xi r_n (u_n)^2 + u_n x_n$. Then $G(y)$ is a quadratic function of y with coefficients $a = 4\xi r_n$, $b = u_n + r_n u_n \xi - x_n$, and $c = -u_n^2 - 5\xi r_n (u_n)^2 + u_n x_n$. Determine the discriminant Δ of G as follows:

$$\begin{aligned}\Delta &= b^2 - 4ac\\ &= (u_n + r_n u_n \xi - x_n)^2 - 4(4\xi r_n)\left(-u_n^2 - 5\xi r_n (u_n)^2 + u_n x_n\right)\\ &= (u_n)^2 + 18\xi r_n (u_n)^2 + 81\xi^2 (r_n)^2 (u_n)^2 - 18\xi r_n u_n x_n + x_n^2\\ &= (u_n + 9\xi r_n u_n - x_n)^2.\end{aligned}$$

From (5.1), we have $G(y) \ge 0$, for every $y \in \mathbb{R}$. If $G(y)$ has most one solution in $\mathbb{R}$, thus we have $\Delta \le 0$. This implies that

$$u_n = \frac{x_n}{1 + 9\xi r_n}, \tag{5.2}$$

where $\xi = \sum_{i=1}^{N}\left(\frac{2}{7^i} + \frac{1}{N7^N}\right) i$.

It is clear to see that the sequences $\{\alpha_n\}, \{\beta_n\}, \{r_n\}$ and $\{\lambda_n\}$ satisfy all the conditions of Theorem 3.1. From Theorem 3.1, we can conclude that the sequences $\{x_n\}$ and $\{u_n\}$ converge strongly to 0.

Table 1 and Figures 1-2 show the values of sequences $\{x_n\}$ and $\{u_n\}$ where $u = x_1 = -5$ and $u = x_1 = 8$ and $n = N = 20$.

Remark 5.2. From the previous example, we can conclude that

(i) Table 1, Figure 1 and Figure 2 show that the sequences $\{u_n\}$ and $\{x_n\}$ converge to 0, where $\{0\} = Fix(T) \cap \bigcap_{i=1}^{N} EP(F_i)$.

(ii) The convergence of $\{u_n\}$ and $\{x_n\}$ can be guaranteed by Theorem 3.1.

Table 1. The values of $\{u_n\}$ and $\{x_n\}$ with $n = N = 20$

	$u = x_1 = -5$		$u = x_1 = 8$	
n	u_n	x_n	u_n	x_n
1	−1.531532	−5.000000	2.450450	8.000000
2	−0.374176	−1.212331	0.598682	1.939730
3	−0.178184	−0.575048	0.285094	0.920077
4	−0.099990	−0.321920	0.159984	0.515072
5	−0.059731	−0.191994	0.095570	0.307190
⋮	⋮	⋮	⋮	⋮
10	−0.007125	−0.022817	0.011401	0.036508
⋮	⋮	⋮	⋮	⋮
16	−0.001899	−0.006073	0.003039	0.009716
17	−0.001702	−0.005442	0.002724	0.008707
18	−0.001552	−0.004960	0.002483	0.007936
19	−0.001433	−0.004578	0.002292	0.007325
20	−0.001335	−0.004266	0.002136	0.006826

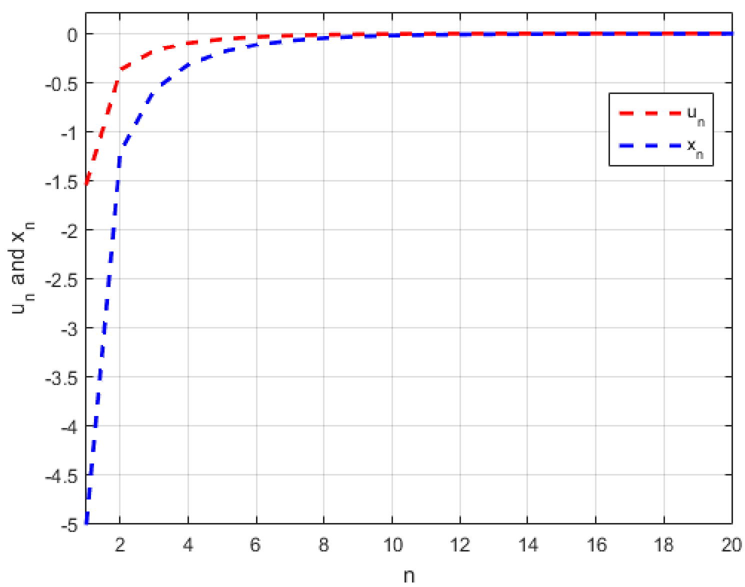

Figure 1. The convergence comparison of $\{u_n\}$ and $\{x_n\}$ with different $u = x_1 = -5$

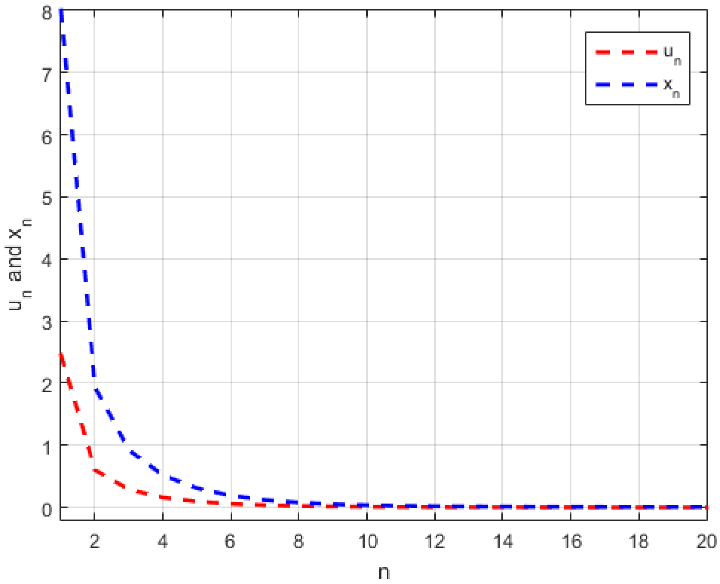

Figure 2. The convergence comparison of $\{u_n\}$ and $\{x_n\}$ with different $u = x_1 = 8$

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Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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