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Research Article

New Inequalities for Nielsen's Beta Function

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Abstract. By employing the classical mean value theorem, Hermite-Hadamard inequality and some other analytical techniques, we establish some new inequalities for Nielsen's beta function. Some of these inequalities provide bounds for certain ratios of the gamma function.

Keywords. Nielsen's beta function; Gamma function; Hermite-Hadamard inequality; Mean value theorem; Inequality

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1. Introduction

Inequalities are found in every aspect of mathematics. Their role in mathematics and its related disciplines is invaluable. Despite this, it was only in the 1930s that the first work [8] was published and it is this classic work transformed the field of inequalities from a collection of isolated formulas into a systematic and attractive discipline. Other famous works in this field include [1], [5], [6], [9] and [11]. In recent years, the theory of inequalities has developed into an active and independent area of research, necessitating the emergence of new journals devoted solely to inequalities and their applications. A particular attention has been on inequalities involving special functions and this is where the present work lies. We focus our study on inequalities involving the Nielsen's beta function.

Let $\Gamma(x)$ be the classical gamma function and $\psi(x)=\frac{d}{dx}\ln\Gamma(x)$ be the digamma function. Then the function $\beta:(0,\infty)\to(0,\infty)$ which is defined as [19, p. 16]

$$\beta(x)=\frac{1}{2}\left\{\psi\left(\frac{x}{2}+\frac{1}{2}\right)-\psi\left(\frac{x}{2}\right)\right\}, \tag{1}$$

is called Nielsen's beta function. It has the integral representations [7], [12]

$$\beta(x)=\int_0^\infty \frac{e^{-xt}}{1+e^{-t}}\,dt \tag{2}$$

$$=\int_0^1 \frac{t^{x-1}}{1+t}\,dt \tag{3}$$

and satisfies the identities

$$\beta(x+1)=\frac{1}{x}-\beta(x), \tag{4}$$

$$\beta(x)+\beta(1-x)=\frac{\pi}{\sin\pi x}. \tag{5}$$

A few specific values of the function are $\beta(1)=\ln 2$, $\beta\left(\frac{1}{2}\right)=\frac{\pi}{2}$, $\beta\left(\frac{3}{2}\right)=2-\frac{\pi}{2}$ and $\beta(2)=1-\ln 2$. Additional properties of the function can be found in [2], [4], [7] and [10].

Lately, this special function has been investigated in diverse ways. For example, in [12], [13], [17] and [18], the authors discovered some interesting properties and inequalities of the function. Also, in [14], [15], [16] and [23], the authors gave some generalizations of the function.

In this paper, we continue to explore the function. By using largely the classical Hermite-Hadamard inequality, the mean value theorem and some other analytical techniques, we establish some new inequalities involving the function. Some of these inequalities provide bounds for certain ratios of the gamma function. We present our results in the following section.

2. Results and Discussion

Theorem 2.1. *The inequalities*

$$\frac{1}{x}-1<\beta(x+1)<\frac{1}{x}-\ln 2, \quad x\in(0,1), \tag{6}$$

$$\frac{1}{2x}<\beta(x)<\frac{1}{x}, \quad x\in(0,\infty), \tag{7}$$

$$-\frac{1}{x^2}<\beta'(x)<-\frac{1}{2x^2}, \quad x\in(0,\infty), \tag{8}$$

$$-\frac{\pi^2}{6}<\beta'(x+1)e^{\beta(x+1)}, \quad x\in(0,\infty), \tag{9}$$

are satisfied.

Proof. It is known from [18, Theorem 1] that the function $\Psi(x)=x\beta(x)$ is decreasing for $x\in(0,\infty)$. Then for $x\in(0,1)$ and by virtue of identity (4), we obtain

$$\ln 2=\lim_{x\to 1}\Psi(x)<\frac{1}{x}-\beta(x+1)<\lim_{x\to 0}\Psi(x)=1,$$

which gives (6).

Next, let $\phi(x) = x\left[\beta(x)+\frac{1}{x}\right]$ for $x \in (0,\infty)$. Then $\phi(x)$ is decreasing and consequently, we obtain

$$\frac{3}{2} = \lim_{x\to\infty} \phi(x) < \phi(x) < \lim_{x\to 0} \phi(x) = 2,$$

which gives inequality (7). Alternatively, since $\beta(x)$ is positive and decreasing for all $x \in (0,\infty)$, then by (4), we obtain

$$\frac{1}{x} - \beta(x) > 0 \quad \text{and} \quad \frac{1}{x} - 2\beta(x) < 0,$$

which when put together yields (7). Also, since $\beta'(x)$ is negative and increasing for all $x \in (0,\infty)$, then by (4) we obtain

$$-\frac{1}{x^2} - \beta'(x) < 0 \quad \text{and} \quad -\frac{1}{x^2} - 2\beta'(x) > 0,$$

which gives (8). Furthermore, let $F(x) = \beta'(x+1)e^{\beta(x+1)}$ for $x \in (0,\infty)$.
Then $F'(x) = \left[\beta''(x+1) + (\beta'(x+1))^2\right] e^{\beta(x+1)} > 0$, which implies that $F(x)$ is increasing. Hence,

$$-\frac{\pi^2}{6} = \lim_{x\to 0} F(x) < F(x) < \lim_{x\to\infty} F(x) = 0,$$

which gives (9). □

Theorem 2.2. *The inequality*

$$\frac{s(1-\ln 2)^s}{1-s\beta(s)} \le \frac{\beta(x+1)^s}{\beta(sx+1)} \le (\ln 2)^{s-1}, \quad s \ge 1, \tag{10}$$

holds for $x \in (0,1)$, with equality when $s = 1$.

Proof. Let $s \ge 1$, $\alpha(x) = \frac{\beta(x+1)^s}{\beta(sx+1)}$ and $h(x) = \ln \alpha(x)$. Then

$$h'(x) = s\left[\frac{\beta'(x+1)}{\beta(x+1)} - \frac{\beta'(sx+1)}{\beta(sx+1)}\right] < 0,$$

since $\frac{\beta'(x)}{\beta(x)}$ is increasing. Thus, $\alpha(x)$ is decreasing and for $x \in (0,1)$, we have $\alpha(1) < \alpha(x) < \alpha(0)$ which give rise to (10). □

Theorem 2.3. *For $x > 0$ and $0 \le r < s$, the inequality*

$$\frac{s-r}{2(x+s)^2} < \beta(x+r) - \beta(x+s) < \frac{s-r}{(x+r)^2}, \tag{11}$$

is satisfied.

Proof. Let $x > 0$ and $0 \le r < s$. Then consider the function $\beta(x)$ on the interval $(x+r, x+s)$. By the mean value theorem, there exist $c \in (x+r, x+s)$ such that

$$\frac{\beta(x+s)-\beta(x+r)}{s-r} = \beta'(c).$$

Since $\beta'(z)$ is increasing for all $z \in (0,\infty)$, then by inequality (8) we obtain

$$-\frac{1}{(x+r)^2} < \beta'(x+r) < \frac{\beta(x+s)-\beta(x+r)}{s-r} < \beta'(x+s) < -\frac{1}{2(x+s)^2},$$

which yields (11) upon rearrangement. □

Corollary 2.4. *For $x>0$, the inequality*

$$\frac{1}{4(x+1)^2}+\frac{1}{2x}<\beta(x)<\frac{1}{2x}+\frac{1}{2x^2}, \tag{12}$$

is satisfied.

Proof. By letting $r=0$ and $s=1$ in Theorem 2.3, we obtain

$$\frac{1}{2(x+1)^2}<\beta(x)-\beta(x+1)<\frac{1}{x^2},$$

and by applying (4), we obtain (12). □

Theorem 2.5. *For $x>0$, the function $F(x)=e^{\frac{1}{x+1}}-\beta(x+1)$ is decreasing and consequently, the inequality*

$$1+\frac{1}{x}-e^{\frac{1}{x+1}}<\beta(x)<e-\ln 2+\frac{1}{x}-e^{\frac{1}{x+1}}, \tag{13}$$

is satisfied.

Proof. By using the identity $\frac{m!}{x^{m+1}}=\int_0^\infty t^m e^{-xt}\,dt$, where $m\in\mathbb{N}$, $x>0$ and (2), we obtain

$$\begin{aligned}
F'(x)&=-\left[\frac{e^{\frac{1}{x+1}}}{(x+1)^2}+\beta'(x+1)\right]\\
&=-\left[e^{\frac{1}{x+1}}\int_0^\infty te^{-(x+1)t}\,dt-\int_0^\infty\frac{te^{-(x+1)t}}{1+e^{-t}}\,dt\right]\\
&=-\int_0^\infty\left[e^{\frac{1}{x+1}}-\frac{1}{1+e^{-t}}\right]te^{-(x+1)t}\,dt\\
&<0,
\end{aligned}$$

which implies that $F(x)$ is decreasing. Hence,

$$1=\lim_{x\to\infty}F(x)<F(x)<\lim_{x\to0}F(x)=e-\ln 2,$$

and by using (4), we obtain inequality (13). □

Theorem 2.6. *For $x>0$, the function $T(x)=x(\beta(x)-\ln(1+e^{-x}))$ is decreasing and consequently, the inequality*

$$\frac{1}{2x}+\ln(1+e^{-x})<\beta(x)<\frac{1}{x}+\ln(1+e^{-x}), \tag{14}$$

is satisfied.

Proof. Since $x\beta(x)$ and e^{-x} are decreasing for $x>0$, then we have

$$T(x+1)-T(x)=(x+1)\beta(x+1)-x\beta(x)-\ln(1+e^{-(x+1)})-x\left[\ln(1+e^{-x})-\ln(1+e^{-(x+1)})\right]<0,$$

which implies that $T(x)$ is decreasing. Hence,

$$\frac{1}{2}=\lim_{x\to\infty}T(x)<T(x)<\lim_{x\to0}T(x)=1,$$

which gives inequality (14). □

Remark 2.7. The lower bound of (12) is better than that of (13) if $x > 0.449395$, and the upper bound of (12) is better than that of (13) if $x > 0.734862$.

Remark 2.8. The lower bound of (12) is better than that of (14) if $x > 4.949310$, and the upper bound of (12) is better than that of (14) if $x > 0.647255$.

Remark 2.9. The lower bound of (14) is better than that of (13) if $x > 0.288703$, and the upper bound of (14) is better than that of (13) if $x > 0.911556$.

In the following theorem, we provide bounds for the Nielsen's beta function in terms of the trigamma function $\psi'(x)$.

Theorem 2.10. *For $x > 0$, the inequalities*

$$\frac{1}{4}\psi'\left(\frac{x}{2}+\frac{1}{2}\right) < \beta(x) < \frac{1}{4}\psi'\left(\frac{x}{2}\right), \tag{15}$$

$$\frac{1}{4}\psi'\left(\frac{x}{2}+\frac{1}{4}\right) < \beta(x) < \frac{1}{8}\left\{\psi'\left(\frac{x}{2}\right)+\psi'\left(\frac{x}{2}+\frac{1}{2}\right)\right\}, \tag{16}$$

are satisfied.

Proof. Consider the function $\psi(x)$ on the interval $(\frac{x}{2}, \frac{x}{2}+\frac{1}{2})$. Then the mean value theorem implies that there exists $c \in (\frac{x}{2}, \frac{x}{2}+\frac{1}{2})$ such that

$$2\left\{\psi\left(\frac{x}{2}+\frac{1}{2}\right)-\psi\left(\frac{x}{2}\right)\right\} = \psi'(c),$$

and since $\psi'(s)$ is decreasing for $s > 0$, we have

$$\frac{1}{4}\psi'\left(\frac{x}{2}+\frac{1}{2}\right) < \frac{1}{4}\psi'(c) < \frac{1}{4}\psi'\left(\frac{x}{2}\right),$$

which gives (15). Next consider the function $\psi'(x)$ on the interval $(\frac{x}{2}, \frac{x}{2}+\frac{1}{2})$ and recall that $\psi'(x)$ is convex for all $x > 0$. Then by the Hermite-Hadamard inequality, we obtain

$$\psi'\left(\frac{x}{2}+\frac{1}{4}\right) \le 2\int_{\frac{x}{2}}^{\frac{x}{2}+\frac{1}{2}} \psi'(x)\,dx \le \frac{1}{2}\left\{\psi'\left(\frac{x}{2}\right)+\psi'\left(\frac{x}{2}+\frac{1}{2}\right)\right\},$$

which implies (16). □

Remark 2.11. Since $\psi'(s)$ is decreasing, then

$$\frac{1}{4}\psi'\left(\frac{x}{2}+\frac{1}{2}\right) < \frac{1}{4}\psi'\left(\frac{x}{2}+\frac{1}{4}\right),$$

and

$$\frac{1}{8}\left\{\psi'\left(\frac{x}{2}\right)+\psi'\left(\frac{x}{2}+\frac{1}{2}\right)\right\} < \frac{1}{8}\left\{\psi'\left(\frac{x}{2}\right)+\psi'\left(\frac{x}{2}\right)\right\} = \frac{1}{4}\psi'\left(\frac{x}{2}\right).$$

Thus, the bounds in (16) are better than those of (15).

In the following theorems, we provide bounds for certain ratios of the Gamma function. The bounds involve the Nielsen's beta function.

Theorem 2.12. *For $x > 0$, the inequality*

$$\exp\left\{\beta\left(x+\frac{1}{2}\right)\right\} \leq \frac{\Gamma(\frac{x}{2})\Gamma(\frac{x}{2}+1)}{\Gamma^2(\frac{x}{2}+\frac{1}{2})} \leq \exp\left\{\frac{1}{2x}\right\}, \tag{17}$$

is satisfied.

Proof. It is known that $\beta(x)$ is convex for all $x > 0$. Now consider $\beta(x)$ on the interval $(x, x+1)$. Then by applying the Hermite-Hadamard inequality in conjunction with (4), we obtain

$$\beta\left(x+\frac{1}{2}\right) \leq \int_x^{x+1} \beta(x)\,dx \leq \frac{1}{2x},$$

which implies that

$$\beta\left(x+\frac{1}{2}\right) \leq \left[\ln\Gamma\left(\frac{x}{2}+\frac{1}{2}\right) - \ln\Gamma\left(\frac{x}{2}\right)\right]_x^{x+1} \leq \frac{1}{2x}$$

and this further implies that

$$\beta\left(x+\frac{1}{2}\right) \leq \ln\frac{\Gamma(\frac{x}{2})\Gamma(\frac{x}{2}+1)}{\Gamma^2(\frac{x}{2}+\frac{1}{2})} \leq \frac{1}{2x}.$$

Then by taking exponents we obtain (17). □

Remark 2.13. Inequality (17) can be rearranged as

$$\sqrt{\frac{x}{2}}\exp\left\{\frac{1}{2}\beta\left(x+\frac{1}{2}\right)\right\} \leq \frac{\Gamma(\frac{x}{2}+1)}{\Gamma(\frac{x}{2}+\frac{1}{2})} \leq \sqrt{\frac{x}{2}}\exp\left\{\frac{1}{4x}\right\}. \tag{18}$$

Upon replacing x by $\frac{x}{2}$ in inequality (3) of Sandor's work [22], one obtains

$$\sqrt{\frac{x}{2}} \leq \frac{\Gamma(\frac{x}{2}+1)}{\Gamma(\frac{x}{2}+\frac{1}{2})} \leq \sqrt{\frac{x+1}{2}}, \quad x > 0, \tag{19}$$

and it is worth noting that, for $x > 0$, the lower bound of (18) is better than the lower bound of (19). Also, for $x > 0.397953$, the upper bound of (18) is better than the upper bound of (19) .

Theorem 2.14. *For $x > 0$, the inequality*

$$\exp\left\{\beta(x+1)\right\} \leq \frac{\Gamma(\frac{x}{2})\Gamma(\frac{x}{2}+1)}{\Gamma^2(\frac{x}{2}+\frac{1}{2})} \leq \exp\left\{\beta(x)\right\}, \tag{20}$$

is satisfied.

Proof. Consider the function $f(x) = \ln\Gamma(\frac{x}{2}+\frac{1}{2}) - \ln\Gamma(\frac{x}{2})$ on the interval $(x, x+1)$. Then by the mean value theorem, there exist a $c \in (x, x+1)$ such that

$$\ln\frac{\Gamma(\frac{x}{2})\Gamma(\frac{x}{2}+1)}{\Gamma^2(\frac{x}{2}+\frac{1}{2})} = f'(c) = \beta(c).$$

Since $\beta(x)$ is decreasing, then for $c \in (x, x+1)$, we have

$$\beta(x+1) < \beta(c) < \beta(x),$$

which is

$$\beta(x+1) < \ln\frac{\Gamma(\frac{x}{2})\Gamma(\frac{x}{2}+1)}{\Gamma^2(\frac{x}{2}+\frac{1}{2})} < \beta(x).$$

Then by taking exponents we obtain (20). □

Remark 2.15. Since $\beta(x)$ is decreasing, then $\exp\{\beta(x+1)\} < \exp\{\beta\left(x+\frac{1}{2}\right)\}$ for all $x>0$. Also, by (7), we have $\frac{1}{2x} < \beta(x)$ for all $x>0$. These imply that (17) is sharper than (20).

Remark 2.16. In [3], the authors established the inequality

$$\Gamma(z+a)\Gamma(z+b) < \Gamma(z)\Gamma(z+a+b), \quad z>0, a, b \geq 0, \tag{21}$$

among other things. If $z = \frac{x}{2}$, $a = b = \frac{1}{2}$, then (21) reduduces to

$$\Gamma^2\left(\frac{x}{2}+\frac{1}{2}\right) < \Gamma\left(\frac{x}{2}\right)\Gamma\left(\frac{x}{2}+1\right), \quad x>0.$$

This is however is weaker than the left side of (20) since

$$\Gamma^2\left(\frac{x}{2}+\frac{1}{2}\right) < \exp\{\beta(x+1)\}\Gamma^2\left(\frac{x}{2}+\frac{1}{2}\right) < \Gamma\left(\frac{x}{2}\right)\Gamma\left(\frac{x}{2}+1\right),$$

for all $x>0$.

Remark 2.17. By Squeeze's theorem, inequality (17) or (20) implies that

$$\lim_{x\to\infty} \frac{\Gamma(\frac{x}{2})\Gamma(\frac{x}{2}+1)}{\Gamma^2(\frac{x}{2}+\frac{1}{2})} = 1. \tag{22}$$

Theorem 2.18. *For $x>0$, the inequality*

$$\frac{1}{\sqrt{\pi}}\exp\{x\beta(x+1)\} \leq \frac{\Gamma(\frac{x}{2}+1)}{\Gamma(\frac{x}{2}+\frac{1}{2})} \leq \frac{2^x}{\sqrt{\pi}}, \tag{23}$$

is satisfied.

Proof. The proof is similar to that of Theorem 2.14. We consider the function $f(x) = \ln\Gamma(\frac{x}{2}+\frac{1}{2}) - \ln\Gamma(\frac{x}{2})$ on the interval $(1, x+1)$, and by applying the mean value theorem, we obtain (23). □

Theorem 2.19. *For $x>0$, the inequality*

$$\frac{1}{\sqrt{\pi}}\exp\left\{x\beta\left(\frac{x}{2}+1\right)\right\} \leq \frac{\Gamma(\frac{x}{2}+1)}{\Gamma(\frac{x}{2}+\frac{1}{2})} \leq \frac{2^{\frac{x}{2}}}{\sqrt{\pi}}\exp\left\{\frac{x}{2}\beta(x+1)\right\}, \tag{24}$$

is satisfied.

Proof. Similarly, consider $\beta(x)$ on the interval $(1, x+1)$. Then by the Hermite-Hadamard inequality, we obtain

$$\beta(\frac{x}{2}+1) \leq \frac{1}{x}\int_1^{x+1}\beta(x)\,dx \leq \frac{\beta(1)+\beta(x+1)}{2},$$

which implies that

$$x\beta(\frac{x}{2}+1) \leq \ln\frac{\sqrt{\pi}\Gamma(\frac{x}{2}+1)}{\Gamma(\frac{x}{2}+\frac{1}{2})} \leq \frac{x}{2}\ln 2 + \frac{x}{2}\beta(x+1).$$

Now by taking exponents we obtain (24). □

Remark 2.20. Upon replacing x by $\frac{x}{2}$ in inequality (8) of Sandor's work [22], one obtains

$$\frac{2}{\sqrt{\pi}} \leq \frac{\Gamma(\frac{x}{2}+1)}{\Gamma(\frac{x}{2}+\frac{1}{2})} \leq \frac{2^{\frac{x}{2}}}{\sqrt{\pi}}, \quad x > 3, \tag{25}$$

and we note that, for $x > 0$, the lower bounds of (23) and (24) are better than the lower bound of (25). However, the upper bounds of (23) and (24) are weaker than the upper bound of (25) for $x > 0$.

Remark 2.21. There exists an extensive literature on inequalities for ratios of the gamma function. For a detailed account on such inequalities, one may refer to the survey articles [20] and [21], and the reference in there.

3. Conclusion

We have established some new inequalities for Nielsen's beta function by employing the classical mean value theorem, Hermite-Hadamard inequality and some other analytical techniques. Some of the inequalities give bounds for certain ratios of the gamma function. These new results lay a good foundation for further studies of the function.

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Competing Interests

The author declares that he has no competing interests.

Authors' Contributions

The author wrote, read and approved the final manuscript.

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