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Research Article

Hölder Type Inequalities Involving Tracy-Singh Products and Khatri-Rao Products of Hilbert Space Operators

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Abstract. This paper generalizes the famous Hölder type inequalities for positive real numbers to positive operators on an arbitrary complex Hilbert space. We use appropriate integral representations of certain operator-monotone functions to deduce the concavity and convexity of certain maps involving Tracy-Singh products of operators. These results lead to Hölder type inequalities for operators concerning Tracy-Singh products, Khatri-Rao products, Tracy-Singh sums, and Khatri-Rao sums. In particular, we obtain Cauchy-Schwarz type inequalities for operators involving Tracy-Singh products and Khatri-Rao products.

Keywords. Hölder inequality; Tracy-Singh product; Khatri-Rao product; Hilbert space operator

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1. Introduction

One of the most important inequalities in mathematics is the famous Hölder inequality. This inequality plays an important role in real/complex analysis, numerical analysis, probability and statistics, differential equations and related fields. For any positive real number a_i and b_i, Hölder inequality states that

$$\sum_{i=1}^{k} a_i b_i \leqslant \Big(\sum_{i=1}^{k} a_i^p\Big)^{\frac{1}{p}} \Big(\sum_{i=1}^{k} b_i^q\Big)^{\frac{1}{q}}, \tag{1}$$

where $\frac{1}{p}+\frac{1}{q}=1$ with $p>1$. If $p>0$ and $0<q<1$ with $\frac{1}{q}-\frac{1}{p}=1$, then

$$\sum_{i=1}^{k} a_i b_i \geqslant \Big(\sum_{i=1}^{k} a_i^{-p}\Big)^{-\frac{1}{p}} \Big(\sum_{i=1}^{k} b_i^{q}\Big)^{\frac{1}{q}}. \tag{2}$$

Jensen [5] proved the following generalization of (1). For any positive real numbers a_{ij} and p_j $(1 \leqslant i \leqslant k,\ 1 \leqslant j \leqslant r)$ such that $\frac{1}{p_1}+\cdots+\frac{1}{p_r}=1$, we have

$$\sum_{i=1}^{k} a_{i1}\ldots a_{ir} \leqslant \Big(\sum_{i=1}^{k} a_{i1}^{p_1}\Big)^{\frac{1}{p_1}} \ldots \Big(\sum_{i=1}^{k} a_{ir}^{p_r}\Big)^{\frac{1}{p_1}}. \tag{3}$$

Ando [2] generalized (1) and (2) to the context of positive definite matrices in which the product is given by the Hadamard product (i.e. the entrywise product). Al-Zhour [1] established Hölder type inequalities for Tracy-Singh and Khatri-Rao products of positive definite matrices.

It is natural to extend Hölder type inequalities to the context of bounded linear operators on a Hilbert space. Thus, this work generalizes the inequalities (1) and (3) to the case of positive operators in which the products are given by Tracy-Singh products (see [7]) and Khatri-Rao products ([11]). We also obtain Cauchy-Schwarz type inequalities involving these products, and some bounds of Tracy-Singh sums and Khatri-Rao sums as special cases. Furthermore, we provide another versions of Cauchy-Schwarz inequality which are generalizations of Cauchy-Schwarz inequality in $\mathbb{C}^n$. In particular, our results include the matrix results in [1,2] and operator results in [3].

This paper is organized as follows. In Section 2, we explain the notions of Tracy-Singh product, Khatri-Rao product, Tracy-Singh sum, and Khatri-Rao sum. In Section 3, we establish Höder type inequalities involving Tracy-Singh products and Khatri-Rao products, and as a consequence obtain some bounds for Tracy-Singh sums and Khatri-Rao sums. Operator versions of Cauchy-Schwarz type inequalities involving Tracy-Singh products and Khatri-Rao products are presented in Section 4. Finally, we conclude the paper in Section 5.

2. Preliminaries

Throughout, let $\mathbb{H}$ and $\mathbb{K}$ be Hilbert spaces over the complex field. Whenever X and Y are Hilbert spaces, we denote by $\mathfrak{B}(X,Y)$ the Banach space of bounded linear operators from X into Y, and abbreviate $\mathfrak{B}(X,X)$ to $\mathfrak{B}(X)$. The identity operator on a space X is written by I_X or I if there is no ambiguity. Recall that an operator $T \in \mathfrak{B}(X)$ is said to be positive if $\langle Tx,x\rangle > 0$ for all $x \in X-\{0\}$. For self-adjoint operators $A,B \in \mathfrak{B}(X)$, the partial order $A \geqslant B$ means that the difference $A-B$ is positive. We denote the set of positive (invertible positive) operators on X by $\mathfrak{B}(X)^+$ ($\mathfrak{B}(X)^{++}$, respectively).

We decompose the Hilbert spaces $\mathbb{H}$ and $\mathbb{K}$ as direct sums of certain Hilbert spaces as follows:

$$\mathbb{H} = \bigoplus_{i=1}^{m} \mathbb{H}_i, \quad \mathbb{K} = \bigoplus_{j=1}^{n} \mathbb{K}_j.$$

Thus, any operator $A \in \mathfrak{B}(\mathbb{H})$ and $B \in \mathfrak{B}(\mathbb{K})$ can be expressed uniquely as operator matrices

$$A = [A_{ij}]_{i,j=1}^{m,m} \quad \text{and} \quad B = [B_{kl}]_{k,l=1}^{n,n}$$

where $A_{ij} \in \mathfrak{B}(\mathbb{H}_j,\mathbb{H}_i)$ and $B_{kl} \in \mathfrak{B}(\mathbb{K}_l,\mathbb{K}_k)$ for each $i,j=1,\ldots,m$ and $k,l=1,\ldots,n$.

Recall that the tensor product of $A \in \mathfrak{B}(\mathbb{H})$ and $B \in \mathfrak{B}(\mathbb{K})$, in a viewpoint of the universal mapping property, is the unique bounded linear operator from $A \otimes B \in \mathfrak{B}(\mathbb{H} \otimes \mathbb{K})$ such that

$$(A \otimes B)(x \otimes y) = Ax \otimes By, \quad \text{for all } x \in \mathbb{H}, y \in \mathbb{K}.$$

The tensor product was generalized to the Tracy-Singh product [7] and the Khatri-Rao product [11] as follows:

Definition 1. Let $A = [A_{ij}]_{i,j=1}^{m,m} \in \mathfrak{B}(\mathbb{H})$ and $B = [B_{kl}]_{k,l=1}^{n,n} \in \mathfrak{B}(\mathbb{K})$ be operator matrices defined as above. The Tracy-Singh product of A and B is defined to be the bounded linear operator

$$A \boxtimes B = \left[[A_{ij} \otimes B_{kl}]_{kl} \right]_{ij} : \bigoplus_{i,j=1}^{m,n} \mathbb{H}_i \otimes \mathbb{K}_j \to \bigoplus_{i,j=1}^{m,n} \mathbb{H}_i \otimes \mathbb{K}_j. \tag{4}$$

When $m = n$, the Khatri-Rao product of A and B is defined to be the bounded linear operator

$$A \boxdot B = \left[A_{ij} \otimes B_{ij} \right]_{i,j} : \bigoplus_{i=1}^{n} \mathbb{H}_i \otimes \mathbb{K}_i \to \bigoplus_{i=1}^{n} \mathbb{H}_i \otimes \mathbb{K}_i. \tag{5}$$

Note that if $m = n = 1$, then $A \boxtimes B = A \boxdot B = A \otimes B$. When $\mathbb{H}$ and $\mathbb{K}$ are finite-dimensional inner product spaces, these constructions reduce to the Tracy-Singh product and the Khatri-Rao product of matrices, respectively.

Lemma 1 ([7, 8])**.** *Let $A, C \in \mathfrak{B}(\mathbb{H})$ and $B, D \in \mathfrak{B}(\mathbb{K})$ be compatible operator matrices. Then*

(i) *The map $(A, B) \to A \boxtimes B$ is bilinear and jointly continuous.*

(ii) $(A \boxtimes B)^* = A^* \boxtimes B^*$.

(iii) $(A \boxtimes B)(C \boxtimes D) = AC \boxtimes BD$.

(iv) *If A and B are invertible, then $(A \boxtimes B)^{-1} = A^{-1} \boxtimes B^{-1}$.*

(v) *If $A, B \geqslant 0$, then $(A \boxtimes B)^\alpha = A^\alpha \boxtimes B^\alpha$ for any $\alpha \geqslant 0$.*

(vi) *If $A, B \geqslant 0$, then $A \boxtimes B \geqslant 0$.*

For each $i = 1, \ldots, r$, let $\mathbb{H}_i$ be a Hilbert space and decompose $\mathbb{H}_i = \bigoplus_{j=1}^{n_i} \mathbb{H}_{i,j}$ where all $\mathbb{H}_{i,j}$ are Hilbert spaces. We set $\boxtimes_{i=1}^{1} A_i = A_1 = \boxdot_{i=1}^{1} A_i$. For $r \in \mathbb{N} - \{0\}$ and a finite number of operators $A_i \in \mathfrak{B}(\mathbb{H}_i)$ for $i = 1, \ldots, r$, we denote

$$\mathop{\boxtimes}_{i=1}^{r} A_i = ((A_1 \boxtimes A_2) \boxtimes \cdots \boxtimes A_{r-1}) \boxtimes A_r, \quad \mathop{\boxdot}_{i=1}^{r} A_i = ((A_1 \boxdot A_2) \boxdot \cdots \boxdot A_{r-1}) \boxdot A_r.$$

Lemma 2 ([10])**.** *There exists an isometry Z such that*

$$\mathop{\boxdot}_{i=1}^{r} A_i = Z^* \Big(\mathop{\boxtimes}_{i=1}^{r} A_i \Big) Z$$

for any $A_i \in \mathfrak{B}(\mathbb{H}_i)$, $i = 1, \ldots, r$.

The notions of the Tracy-Singh sum and the Khatri-Rao sum, introduced in [6, 12], are defined as follows:

Definition 2. Let $A = [A_{ij}]_{i,j=1}^{n,n} \in \mathfrak{B}(\mathbb{H})$ and $B = [B_{kl}]_{k,l=1}^{m,m} \in \mathfrak{B}(\mathbb{K})$. We define the Tracy-Singh sum of A and B to be the bounded linear operator

$$A \boxplus B = A \boxtimes I_{\mathbb{K}} + I_{\mathbb{H}} \boxtimes B : \bigoplus_{i,j=1}^{m,n} \mathbb{H}_i \otimes \mathbb{K}_j \to \bigoplus_{i,j=1}^{m,n} \mathbb{H}_i \otimes \mathbb{K}_j. \tag{6}$$

When $m = n$, we define the Khatri-Rao sum of A and B to be the bounded linear operator

$$A \circledast B = A \boxdot I_{\mathbb{K}} + I_{\mathbb{H}} \boxdot B : \bigoplus_{i=1}^{n} \mathbb{H}_i \otimes \mathbb{K}_i \to \bigoplus_{i=1}^{n} \mathbb{H}_i \otimes \mathbb{K}_i. \tag{7}$$

If $m = n = 1$, the Tracy-Singh sum reduces to the tensor sum.

3. Hölder Type Inequalities Involving Tracy-Singh Products and Khatri-Rao Products

Recall that the harmonic mean of $A, B \in \mathfrak{B}(\mathbb{H})^{++}$ is defined by

$$A \,!\, B = 2\left(A^{-1} + B^{-1}\right)^{-1}.$$

Lemma 3 (see e.g. [4])**.** *The map $(A,B) \mapsto A\,!\,B$ is concave on $\mathfrak{B}(\mathbb{H})^{++} \times \mathfrak{B}(\mathbb{H})^{++}$.*

Lemma 4. *For each $r \in (0,1)$, the following map is concave on $\mathfrak{B}(\mathbb{H})^{+} \times \mathfrak{B}(\mathbb{K})^{+}$:*

$$(A,B) \mapsto A^r \boxtimes B^{1-r}. \tag{8}$$

Proof. Recall that the operator monotone function x^r has an integral representation

$$x^r = \frac{\sin r\pi}{\pi} \int_{[0,\infty]} \frac{x t^{r-1}}{x+t} dt.$$

By continuity, we may assume that $A, B \in \mathfrak{B}(\mathbb{H})^{++}$. We have by Lemma 1 that

$$A^r \boxtimes B^{1-r} = (A^r \boxtimes B^{-r})(I \boxtimes B) = (A \boxtimes B^{-1})^r (I \boxtimes B).$$

Using the functional calculus for $A \boxtimes B^{-1}$ and Lemma 1, we have

$$\begin{aligned}
A^r \boxtimes B^{1-r} &= \left\{ \frac{\sin r\pi}{\pi} \int_{[0,\infty]} (A \boxtimes B^{-1})(tI \boxtimes I)^{r-1}(A \boxtimes B^{-1} + tI \boxtimes I)^{-1} dt \right\} (I \boxtimes B) \\
&= \frac{\sin r\pi}{\pi} \int_{[0,\infty]} t^{r-1} (A^{-1} \boxtimes B)^{-1} (A \boxtimes B^{-1} + tI \boxtimes I)^{-1} (I \boxtimes B^{-1})^{-1} dt \\
&= \frac{\sin r\pi}{\pi} \int_{[0,\infty]} t^{r-1} \left[(I \boxtimes B^{-1})(A \boxtimes B^{-1} + tI \boxtimes I)(A^{-1} \boxtimes B) \right]^{-1} dt \\
&= \frac{\sin r\pi}{\pi} \int_{[0,\infty]} t^{r-1} \left[(I \boxtimes B)^{-1} + (t^{-1} A \boxtimes I)^{-1} \right]^{-1} dt \\
&= \frac{\sin r\pi}{2\pi} \int_{[0,\infty]} t^{r-1} \left[(t^{-1} A \boxtimes I)\,!\,(I \boxtimes B) \right] dt.
\end{aligned}$$

By Lemma 3, the map $(A \boxtimes I, I \boxtimes B) \mapsto (t^{-1} A \boxtimes I)\,!\,(I \boxtimes B)$ is concave. Since the map $(A,B) \mapsto (A \boxtimes I, I \boxtimes B)$ is linear, the map $(A,B) \mapsto (t^{-1} A \boxtimes I)\,!\,(I \boxtimes B)$ is concave. It is well-known that any nonnegative linear combination of concave maps is concave. As the integral is the limit of nonnegative linear combinations, the map $(A,B) \mapsto A^r \boxtimes B^{1-r}$ is concave. Since the Tracy-Singh product is jointly continuous, this map is also concave on $\mathfrak{B}(\mathbb{H})^{+} \times \mathfrak{B}(\mathbb{K})^{+}$. □

We obtain Hölder type inequality for positive operators as follows.

Theorem 1. *For each $i = 1,\dots,k$, let $A_i \in \mathfrak{B}(\mathbb{H})^+$ and $B_i \in \mathfrak{B}(\mathbb{K})^+$. If $p,q \geqslant 1$ and $\frac{1}{p}+\frac{1}{q}=1$, then*

$$\sum_{i=1}^{k}(A_i \boxtimes B_i) \leqslant \Big(\sum_{i=1}^{k} A_i^p\Big)^{\frac{1}{p}} \boxtimes \Big(\sum_{i=1}^{k} B_i^q\Big)^{\frac{1}{q}}, \tag{9}$$

$$\sum_{i=1}^{k}(A_i \boxdot B_i) \leqslant \Big(\sum_{i=1}^{k} A_i^p\Big)^{\frac{1}{p}} \boxdot \Big(\sum_{i=1}^{k} B_i^q\Big)^{\frac{1}{q}}. \tag{10}$$

Proof. Let us prove (9) by induction on k. Clearly, (9) holds for $k=1$. Now, assume that

$$\Big(\sum_{i=1}^{k} A_i^p\Big)^{\frac{1}{p}} \boxtimes \Big(\sum_{i=1}^{k} B_i^q\Big)^{\frac{1}{q}} \geqslant \sum_{i=1}^{k}(A_i \boxtimes B_i).$$

Consider $X_1,X_2 \in \mathfrak{B}(\mathbb{H})^+$ and $Y_1,Y_2 \in \mathfrak{B}(\mathbb{K})^+$. By Lemma 4, we have that for any $\alpha, r \in (0,1)$,

$$(\alpha X_1+(1-\alpha)X_2)^r \boxtimes (\alpha Y_1+(1-\alpha)Y_2)^{1-r} \geqslant \alpha\big(X_1^r \boxtimes Y_1^{1-r}\big)+(1-\alpha)\big(X_2^r \boxtimes Y_2^{1-r}\big)$$

Setting $\alpha = 1/2$ and $r = 1/p$, we have

$$(X_1+X_2)^{\frac{1}{p}} \boxtimes (Y_1+Y_2)^{\frac{1}{q}} \geqslant X_1^{\frac{1}{p}} \boxtimes Y_1^{\frac{1}{q}} + X_2^{\frac{1}{p}} \boxtimes Y_2^{\frac{1}{q}}.$$

Replacing X_i by X_i^p and Y_i by Y_i^q, we get

$$(X_1^p+X_2^p)^{\frac{1}{p}} \boxtimes (Y_1^q+Y_2^q)^{\frac{1}{q}} \geqslant X_1 \boxtimes Y_1 + X_2 \boxtimes Y_2. \tag{11}$$

Applying (11) and the inductive hypothesis, we have

$$\begin{aligned}
\Big(\sum_{i=1}^{k+1} A_i^p\Big)^{\frac{1}{p}} \boxtimes \Big(\sum_{i=1}^{k+1} B_i^q\Big)^{\frac{1}{q}} &= \Big\{\Big(\sum_{i=1}^{k} A_i^p\Big)+A_{k+1}^p\Big\}^{\frac{1}{p}} \boxtimes \Big\{\Big(\sum_{i=1}^{k} B_i^q\Big)+B_{k+1}^q\Big\}^{\frac{1}{q}} \\
&\geqslant \Big(\sum_{i=1}^{k} A_i^p\Big)^{\frac{1}{p}} \boxtimes \Big(\sum_{i=1}^{k} B_i^q\Big)^{\frac{1}{q}} + (A_{k+1}^p)^{\frac{1}{p}} \boxtimes (B_{k+1}^q)^{\frac{1}{q}} \\
&\geqslant \sum_{i=1}^{k}(A_i \boxtimes B_i) + A_{k+1} \boxtimes B_{k+1} \\
&= \sum_{i=1}^{k+1}(A_i \boxtimes B_i).
\end{aligned}$$

Thus, (9) holds for any $k \in \mathbb{N}$. Using Lemma 2 together with (9), we have

$$\begin{aligned}
\sum_{i=1}^{k}(A_i \boxdot B_i) &= Z^*\Big\{\sum_{i=1}^{k}(A_i \boxtimes B_i)\Big\}Z \\
&\leqslant Z^*\left\{\left(\sum_{i=1}^{k} A_i^p\right)^{\frac{1}{p}} \boxtimes \left(\sum_{i=1}^{k} B_i^q\right)^{\frac{1}{q}}\right\}Z \\
&= \left(\sum_{i=1}^{k} A_i^p\right)^{\frac{1}{p}} \boxdot \left(\sum_{i=1}^{k} B_i^q\right)^{\frac{1}{q}}.
\end{aligned}$$ □

Notice that Theorem 1 can be viewed generalization of [1, Theorem 1 and Corollary 2] and [2, Theorem 14] to the case of operators.

In the next corollary, we generalize Hölder's type inequality of real numbers (3) and matrices [1, Corollaries 3 and 4] to the case of operators.

Corollary 1. *For each* $1 \leqslant i \leqslant k, 1 \leqslant j \leqslant r$, *let* $A_{ij} \in \mathfrak{B}(\mathbb{H})^+$ *and* $p_j \geqslant 1$ *with* $\sum_{j=1}^{r} \frac{1}{p_j} = 1$. *Then*

$$\sum_{i=1}^{k}\Big(\boxtimes_{j=1}^{r} A_{ir}\Big) \leqslant \boxtimes_{j=1}^{r}\Big(\sum_{i=1}^{k} A_i^{p_j}\Big)^{\frac{1}{p_j}}, \tag{12}$$

$$\sum_{i=1}^{k}\Big(\boxdot_{j=1}^{r} A_{ir}\Big) \leqslant \boxdot_{j=1}^{r}\Big(\sum_{i=1}^{k} A_i^{p_j}\Big)^{\frac{1}{p_j}}. \tag{13}$$

Proof. Let us prove (12) by induction on r. Clearly, (12) is true in the case $r = 1$. Suppose

$$\sum_{i=1}^{k}\Big(\boxtimes_{j=1}^{r} A_{ij}\Big) \leqslant \boxtimes_{j=1}^{r}\Big(\sum_{i=1}^{k} A_i^{\alpha_j}\Big)^{\frac{1}{\alpha_j}},$$

where $\alpha_j \geqslant 1$ for $j = 1,\dots,r$ with $\sum_{j=1}^{r} \frac{1}{\alpha_j} = 1$. Set $p = \frac{p_{r+1}}{p_{r+1}-1}$ and $q_j = \frac{p_j}{p}$ for $j = 1,\dots,r$. We have by Theorem 1 and Lemma 1 that

$$\begin{aligned}\sum_{i=1}^{k}\Big(\boxtimes_{j=1}^{r+1} A_{ij}\Big) &= \sum_{i=1}^{k}\Big[\Big(\boxtimes_{j=1}^{r} A_{ij}\Big)\boxtimes A_{i(r+1)}\Big]\\ &\leqslant \Big[\sum_{i=1}^{k}\Big(\boxtimes_{j=1}^{r} A_{ij}^{p}\Big)\Big]^{\frac{1}{p}} \boxtimes \Big[\sum_{i=1}^{k} A_{i(r+1)}^{p_{r+1}}\Big]^{\frac{1}{p_{r+1}}}.\end{aligned}$$

Since $\sum_{j=1}^{r} \frac{1}{q_j} = 1$, we have by the inductive hypothesis that

$$\begin{aligned}\sum_{i=1}^{k}\Big(\boxtimes_{j=1}^{r+1} A_{ij}\Big) &\leqslant \Big[\boxtimes_{j=1}^{r}\Big(\sum_{i=1}^{k} (A_{ij}^{p})^{q_j}\Big)^{\frac{1}{q_j}}\Big]^{\frac{1}{p}} \boxtimes \Big[\sum_{i=1}^{k} A_{i(r+1)}^{p_{r+1}}\Big]^{\frac{1}{p_{r+1}}}\\ &= \Big[\boxtimes_{j=1}^{r}\Big(\sum_{i=1}^{k} A_{ij}^{pq_j}\Big)^{\frac{1}{pq_j}}\Big] \boxtimes \Big[\sum_{i=1}^{k} A_{i(r+1)}^{p_{r+1}}\Big]^{\frac{1}{p_{r+1}}}\\ &= \Big[\boxtimes_{j=1}^{r}\Big(\sum_{i=1}^{k} A_{ij}^{p_j}\Big)^{\frac{1}{p_j}}\Big] \boxtimes \Big[\sum_{i=1}^{k} A_{i(r+1)}^{p_{r+1}}\Big]^{\frac{1}{p_{r+1}}}\\ &= \boxtimes_{j=1}^{r+1}\Big(\sum_{i=1}^{k} A_{ij}^{p_j}\Big)^{\frac{1}{p_j}}.\end{aligned}$$

By Lemma 2, we reach the second inequality. □

In the next result, we provide upper bounds for the Tracy-Singh sum and Khatri-Rao sum.

Corollary 2. *Let* $A \in \mathfrak{B}(\mathbb{H})^+$ *and* $B \in \mathfrak{B}(\mathbb{K})^+$. *If* $p, q \geqslant 1$ *and* $\dfrac{1}{p} + \dfrac{1}{q} = 1$, *then*

$$A \boxplus B \leqslant (A^p + I)^{\frac{1}{p}} \boxtimes (B^q + I)^{\frac{1}{q}}, \tag{14}$$

$$A \circledast B \leqslant (A^p + I)^{\frac{1}{p}} \boxdot (B^q + I)^{\frac{1}{q}}. \tag{15}$$

Proof. Setting $k = 2$ and taking $A_1 = A$, $A_2 = I$, $B_1 = I$ and $B_2 = B$ in Theorem 1, we reach the results. □

Lemma 5. *For each $r \in (0,1)$, the following map is convex on $\mathfrak{B}(\mathbb{H})^{++} \times \mathfrak{B}(\mathbb{K})^{++}$:*

$$(A,B) \mapsto A^{-r} \boxtimes B^{1+r}. \tag{16}$$

Proof. We have by Lemma 1 that

$$A^{-r} \boxtimes B^{1+r} = \left(A^{-r} \boxtimes B^{r}\right)(I \boxtimes B) = \left(A^{-1} \boxtimes B\right)^{r}(I \boxtimes B).$$

We have

$$\begin{aligned} A^{-r} \boxtimes B^{1+r} &= \left\{\frac{\sin r\pi}{\pi} \int_{[0,\infty]} (A^{-1} \boxtimes B)(tI \boxtimes I)^{r-1}(A^{-1} \boxtimes B + tI \boxtimes I)^{-1}\, dt\right\}(I \boxtimes B) \\ &= \frac{\sin r\pi}{\pi} \int_{[0,\infty]} t^{r-1} \left[(A^{-1} \boxtimes B + tI \boxtimes I)(A \boxtimes B^{-1})\right]^{-1}(I \boxtimes B)\, dt \\ &= \frac{\sin r\pi}{\pi} \int_{[0,\infty]} t^{r-1} \left[I \boxtimes I + tA \boxtimes B^{-1}\right]^{-1}(I \boxtimes B)\, dt \\ &= \frac{\sin r\pi}{\pi} \int_{[0,\infty]} t^{r-1} \left[(I \boxtimes B + tA \boxtimes I)(I \boxtimes B^{-1})\right]^{-1}(I \boxtimes B)\, dt \\ &= \frac{\sin r\pi}{\pi} \int_{[0,\infty]} t^{r-1}(I \boxtimes B)[I \boxtimes B + tA \boxtimes I]^{-1}(I \boxtimes B)\, dt. \end{aligned}$$

Since the map $A \mapsto A^{-1}$ is convex and the map $(A,B) \mapsto tA \boxtimes I + I \boxtimes B$ is affine, the map

$$(A,B) \mapsto (I \boxtimes B)[tA \boxtimes I + I \boxtimes B]^{-1}(I \boxtimes B)$$

is convex. Thus, the map $(A,B) \mapsto A^{-r} \boxtimes B^{1+r}$ is convex. □

Theorem 2. *For each $i = 1,\dots,k$, let $A_i \in \mathfrak{B}(\mathbb{H})^{++}$ and $B_i \in \mathfrak{B}(\mathbb{K})^{++}$. If $p \geqslant 1 \geqslant q \geqslant \frac{1}{2}$ and $\frac{1}{q} - \frac{1}{p} = 1$, then*

$$\sum_{i=1}^{k}(A_i \boxtimes B_i) \geqslant \left(\sum_{i=1}^{k} A_i^{-p}\right)^{-\frac{1}{p}} \boxtimes \left(\sum_{i=1}^{k} B_i^{q}\right)^{\frac{1}{q}}, \tag{17}$$

$$\sum_{i=1}^{k}(A_i \boxdot B_i) \geqslant \left(\sum_{i=1}^{k} A_i^{-p}\right)^{-\frac{1}{p}} \boxdot \left(\sum_{i=1}^{k} B_i^{q}\right)^{\frac{1}{q}}. \tag{18}$$

Proof. Let us prove this theorem by induction on k. It is obvious that (17) is true for $k = 1$. For the inductive step, assume that

$$\left(\sum_{i=1}^{k} A_i^{-p}\right)^{-\frac{1}{p}} \boxtimes \left(\sum_{i=1}^{k} B_i^{q}\right)^{\frac{1}{q}} \leqslant \sum_{i=1}^{k}(A_i \boxtimes B_i).$$

Consider $X_1, X_2 \in \mathfrak{B}(\mathbb{H})^{++}$ and $Y_1, Y_2 \in \mathfrak{B}(\mathbb{K})^{++}$. By Lemma 5, the map $(X,Y) \mapsto X^{-\frac{1}{p}} \boxtimes Y^{\frac{1}{q}}$ is convex. Then

$$(X_1 + X_2)^{-\frac{1}{p}} \boxtimes (Y_1 + Y_2)^{\frac{1}{q}} \leqslant X_1^{-\frac{1}{p}} \boxtimes Y_1^{\frac{1}{q}} + X_2^{-\frac{1}{p}} \boxtimes Y_2^{\frac{1}{q}}.$$

Replacing X_i by X_i^{-p} and Y_i by Y_i^{q}, we have

$$(X_1^{-p} + X_2^{-p})^{-\frac{1}{p}} \boxtimes (Y_1^{q} + Y_2^{q})^{\frac{1}{q}} \leqslant X_1 \boxtimes Y_1 + X_2 \boxtimes Y_2. \tag{19}$$

It follows from (19) and inductive hypothesis that

$$\begin{aligned}\left(\sum_{i=1}^{k+1} A_i^{-p}\right)^{-\frac{1}{p}} \boxtimes \left(\sum_{i=1}^{k+1} B_i^{q}\right)^{\frac{1}{q}} &= \left\{\left(\sum_{i=1}^{k} A_i^{-p}\right) + A_{k+1}^{-p}\right\}^{-\frac{1}{p}} \boxtimes \left\{\left(\sum_{i=1}^{k} B_i^{q}\right) + B_{k+1}^{q}\right\}^{\frac{1}{q}} \\ &\leqslant \left(\sum_{i=1}^{k} A_i^{-p}\right)^{-\frac{1}{p}} \boxtimes \left(\sum_{i=1}^{k} B_i^{q}\right)^{\frac{1}{q}} + (A_{k+1}^{-p})^{-\frac{1}{p}} \boxtimes (B_{k+1}^{q})^{\frac{1}{q}} \\ &\leqslant \sum_{i=1}^{k} (A_i \boxtimes B_i) + A_{k+1} \boxtimes B_{k+1} \\ &= \sum_{i=1}^{k+1} (A_i \boxtimes B_i).\end{aligned}$$

Therefore, (12) holds for any $k \in \mathbb{N}$. We reach (18) by applying (17) and Lemma 2. □

Notice that Theorem 2 is an operator extension of [1, Theorem 2 and Corollary 5] and [2, Theorem 14].

Corollary 3. *Let $A \in \mathfrak{B}(\mathbb{H})^{++}$ and $B \in \mathfrak{B}(\mathbb{K})^{++}$. If $p \geqslant 1 \geqslant q \geqslant \frac{1}{2}$ and $\frac{1}{q} - \frac{1}{p} = 1$, then*

$$A \boxplus B \geqslant (A^{-p} + I)^{-\frac{1}{p}} \boxtimes (B^{q} + I)^{\frac{1}{q}}, \tag{20}$$

$$A \circledast B \geqslant (A^{-p} + I)^{-\frac{1}{p}} \boxdot (B^{q} + I)^{\frac{1}{q}}. \tag{21}$$

4. Cauchy-Schwarz Type Inequalities Involving Tracy-Singh Products and Khatri-Rao Products

The Cauchy-Schwarz inequality is a special case of Hölder's inequality (1). This inequality states that for any real numbers a_i and b_i,

$$\sum_{i=1}^{k} a_i b_i \leqslant \left(\sum_{i=1}^{k} a_i^2\right)^{\frac{1}{2}} \left(\sum_{i=1}^{k} b_i^2\right)^{\frac{1}{2}}. \tag{22}$$

Taking $p = q = 2$ in Theorem 1, we obtain Cauchy-Schwarz inequalities involving Tracy-Singh products and Khatri-Rao products as the following.

Corollary 4. *For each $i = 1, \ldots, k$, let $A_i \in \mathfrak{B}(\mathbb{H})^{+}$ and $B_i \in \mathfrak{B}(\mathbb{K})^{+}$. Then*

$$\sum_{i=1}^{k} (A_i \boxtimes B_i) \leqslant \left(\sum_{i=1}^{k} A_i^2\right)^{\frac{1}{2}} \boxtimes \left(\sum_{i=1}^{k} B_i^2\right)^{\frac{1}{2}}, \tag{23}$$

$$\sum_{i=1}^{k} (A_i \boxdot B_i) \leqslant \left(\sum_{i=1}^{k} A_i^2\right)^{\frac{1}{2}} \boxdot \left(\sum_{i=1}^{k} B_i^2\right)^{\frac{1}{2}}. \tag{24}$$

In any Hilbert space $\mathbb{H}$, the Cauchy-Schwarz inequality states that

$$|\langle x, y\rangle| \leqslant \|x\| \|y\| \tag{25}$$

for every $x, y \in \mathbb{H}$. We can rewrite (25) to

$$\langle x, y\rangle\langle y, x\rangle + \langle y, x\rangle\langle x, y\rangle \leqslant \langle x, x\rangle\langle y, y\rangle + \langle y, y\rangle\langle x, x\rangle.$$

For any $x, y \in \mathbb{C}^n$, we have

$$(x^* y)(y^* x) + (y^* x)(x^* y) \leqslant (x^* x)(y^* y) + (y^* y)(x^* x). \tag{26}$$

Fujii [3] gave operator extensions of (26) in which the products are given by the tensor product and the Hadamard product. In the next result, we generalize (26) to the Tracy-Singh product and the Khatri-Rao product of operators.

Proposition 1. *Let* $A, B \in \mathfrak{B}(\mathbb{H}, \mathbb{K})$. *Then*

$$(A^* B) \boxtimes (B^* A) + (B^* A) \boxtimes (A^* B) \leqslant (A^* A) \boxtimes (B^* B) + (B^* B) \boxtimes (A^* A), \tag{27}$$

$$(A^* B) \boxdot (B^* A) + (B^* A) \boxdot (A^* B) \leqslant (A^* A) \boxdot (B^* B) + (B^* B) \boxdot (A^* A). \tag{28}$$

Proof. This proof is quite similar to [3, Theorem 2.2]. Applying Lemma 1 we have

$$\begin{aligned} 0 &\leqslant (A \boxtimes B - B \boxtimes A)^* (A \boxtimes B - B \boxtimes A) \\ &= (A^* \boxtimes B^* - B^* \boxtimes A^*)(A \boxtimes B - B \boxtimes A) \\ &= (A^* \boxtimes B^*)(A \boxtimes B) - (A^* \boxtimes B^*)(B \boxtimes A) - (B^* \boxtimes A^*)(A \boxtimes B) + (B^* \boxtimes A^*)(B \boxtimes A) \\ &= (A^* A) \boxtimes (B^* B) - (A^* B) \boxtimes (B^* A) - (B^* A) \boxtimes (A^* B) + (B^* B) \boxtimes (A^* A). \end{aligned}$$

We reach the second inequality by using Lemma 2. □

5. Conclusion

We extend Hölder type inequalities for positive real numbers to the context of positive operators on a Hilbert space. The concavity and convexity of certain maps are established via suitable integral representations of the associated operator-monotone functions. We obtain Hölder type inequalities for Hilbert space operators concerning Tracy-Singh products and Khatri-Rao products via these maps. We also establish Caucy-Schwarz inequalities concerning Tracy-Singh products and Khatri-Rao products. Consequently, we get lower bounds and upper bounds for Tracy-Singh sums and Khatri-Rao sums of operators. Furthermore, we provide another versions of Cauchy-Schwarz inequality involving Tracy-Sing products and Khatri-Rao products. The results in this paper concerning the Tracy-Singh product include operator results concerning the tensor product, and matrix results concerning the Tracy-Singh product and the Kronecker product. Our results involving the Khatri-Rao product include operator results involving the tensor product, and matrix results involving the Khatri-Rao product, the Kronecker product and the Hadamard product. Our results regarding the Tracy-Singh sum include operator results regarding the tensor sum, and matrix results regarding the Kronecker sum. Our results concerning the Khatri-Rao sum include operator results regarding the tensor sum, and matrix results regarding the Khatri-Rao sum, the Kronecker sum and the Hadamard sum. In particular, our works include Hölder/Cauchy-Schwarz type inequalities in [1–3].

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Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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