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Research Article

Binomial Transform of the Generalized *k*-Fibonacci Numbers

Sergio Falcón

Department of Mathematics, University of Las Palmas de Gran Canaria, 35017 – Las Palmas de Gran Canaria, Spain
sergio.falcon@ulpgc.es

Abstract. We recall the concept and some properties of the generalized k-Fibonacci numbers and then apply the binomial transform to these sequences. As consequence, we obtain new integer sequences related to the generalized k-Fibonacci numbers. Finally, we find the recurrence relation of these new sequences and the formulas for their sums.

Keywords. k-Fibonacci numbers; Generalization of the k-Fibonacci numbers; Generating function; Binomial transform

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1. Introduction

Classical Fibonacci numbers have been generalized in different ways [12, 13, 16, 17]. One of these generalizations that greater interest lately among mathematical researchers is that leads to the k-Fibonacci numbers [7, 8].

Then, we define the k-Fibonacci numbers.

Definition 1. For a natural number k, the k-Fibonacci sequence, say $\{F_{k,n}\}$, is defined by the recurrence relation

$$F_{k,n+1} = k\,F_{k,n} + F_{k,n-1} \quad \text{for } n \geq 1$$

with the initial conditions $F_{k,0} = 0$; $F_{k,1} = 1$.

From this definition, the k-Fibonacci sequence is

$$F_k = \{0, 1, k, k^2+1, k^3+2k, k^4+3k^2+1, k^5+4k^3+3k, k^6+5k^4+6k^2+1\ldots\}.$$

If $k=1$ the classical Fibonacci sequence is obtained $F = \{0,1,1,2,3,5,8,\ldots\}$ and if $k=2$ that is the Pell sequence $F_2 = P = \{0,1,2,5,12,29,70,169,\ldots\}$.

On the same way, we define the k-Lucas numbers [3] by mean of the recurrence relation $L_{k,n+1} = kL_{k,n} + L_{k,n-1}$ with the initial conditions, $L_{k,0} = 2$ and $L_{k,1} = k$.

From this definition, the k-Lucas sequence is

$$L_k = \{2, k, k^2+2, k^3+3k, k^4+4k^2+2, k^5+5k^3+5k, \ldots\}.$$

In particular, for $k=1$, the classical Lucas sequence $\{2,1,3,4,7,11,18,29,\ldots\}$ is obtained.

It is worthy to be noted that the coefficients arising in the previous list can be written in triangular position, in such a way that every side of the triangle is double. This triangle will be called *Lucas triangle* [2, 4, 15]:

Table 1. The Lucas Triangle

0					2				
1					1				
2				1		2			
3				1		3			
4			1		4		2		
5			1		5		5		
6		1		6		9		2	
7		1		7		14		7	
8	1		8		20		16		2

The row sums are the classical Lucas numbers.

If we write the diagonals of this triangle as rows of a new triangle, then obtain the classical Lucas triangle:

Table 2. The Classical Lucas Triangle

							2							
						1		2						
					1		3		2					
				1		4		5		2				
			1		5		9		7		2			
		1		6		14		16		9		2		
	1		7		20		30		25		11		2	
1		8		27		50		55		36		13		2

In this triangle, if $L_j(r)$ is the j-th entry of the r-th row for $j = 0,1,2,\ldots$, then

$$L_j(r) = \frac{r+j}{j}\binom{r-1}{j-1} \quad \text{for } j = 1,2,\ldots r+1, \text{ with } L_0(r) = 1 \text{ for } r \geq 1.$$

The elements of the Lucas Triangle verify the Addition Property:

$$L_j(r) = L_{j-1}(r-1) + L_j(r-1)$$

that we can prove by induction.

Moreover, for $r \geq 2$, it is verified $\sum_{j=0}^{r} (-1)^j L_j(r) = 0$.

1.1 Generalized k-Fibonacci numbers

For the natural numbers $k \geq 1$, $n \geq 0$, $r \geq 1$, the generalized k-Fibonacci numbers [5], $F_{k,n}(r)$, are defined by the recurrence relation

$$F_{k,n}(r) = k\, F_{k,n-r}(r) + F_{k,n-2}(r) \quad \text{for } n > r$$

with initial conditions $F_{k,n}(r) = 1$, $n = 0,1,2,\dots r-1$, $F_{k,1}(1) = k$.

So, if $F_k(r) = \{F_{k,n}(r)/n \in \mathcal{N}\}$, the sequences obtained for $r = 1,2,\dots 7$ are:

$$F_k(1) = \{1,k,1+k^2,2k+k^3,1+3k^2+k^4,3k+4k^3+k^5,1+6k^2+5k^4+k^6,\dots\}$$

$$F_k(2) = \{1,1,1+k,1+k,(1+k)^2,(1+k)^2,(1+k)^3,(1+k)^3,(1+k)^4,(1+k)^4,\dots\}$$

$$F_k(3) = \{1,1,1,1+k,1+k,1+2k,1+2k+k^2,1+3k+k^2,1+3k+3k^2,\dots\}$$

$$F_k(4) = \{1,1,1,1,1+k,1+k,1+2k,1+2k,1+3k+k^2,1+3k+k^2,1+4k+3k^2,\dots\}$$

$$F_k(5) = \{1,1,1,1,1,1+k,1+k,1+2k,1+2k,1+3k,1+3k+k^2,1+4k+k^2,\dots\}$$

$$F_k(6) = \{1,1,1,1,1,1,1+k,1+k,1+2k,1+2k,1+3k,1+3k,1+4k+k^2,\dots\}$$

$$F_k(7) = \{1,1,1,1,1,1,1,1+k,1+k,1+2k,1+2k,1+3k,1+3k,1+4k,1+4k+k^2,\dots\}$$

Evidently, $F_k(1)$ is the k-Fibonacci sequence.

1.2 Generalized k-Fibonacci numbers for $k = 1,2,3$

If we particularize the previous sequences for $k = 1,2,3,\dots$ obtain distinct integer sequences. For $k = 1$, the following sequences appear [18]:

Table 3. Generalized 1-Fibonacci numbers

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$F_{1,n}(1)$	1	1	2	3	5	8	13	21	34	55	89	144	233	377	610	987	$\cdots$
$F_{1,n}(2)$	1	1	2	2	4	4	8	8	16	16	32	32	64	64	128	128	256
$F_{1,n}(3)$	1	1	1	2	2	3	4	5	7	9	12	16	21	28	37	49	65
$F_{1,n}(4)$	1	1	1	1	2	2	3	3	5	5	8	8	13	13	21	21	34
$F_{1,n}(5)$	1	1	1	1	1	2	2	3	3	4	5	6	8	9	12	14	18
$F_{1,n}(6)$	1	1	1	1	1	1	2	2	3	3	4	4	6	6	9	9	13
$F_{1,n}(7)$	1	1	1	1	1	1	1	2	2	3	3	4	4	5	6	7	9

Some of these sequences are cited in [14]. Moreover, $F_1(1)$ is the classical Fibonacci sequence (A000045) and $F_1(3)$ is the Padovan sequence (A000931).

For $k = 2$, it is:

Table 4. Generalized 2-Fibonacci numbers

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$F_{2,n}(1)$	1	2	5	12	29	70	169	408	985	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
$F_{2,n}(2)$	1	1	3	3	9	9	27	27	81	81	243	243	729	729	$\cdots$	$\cdots$
$F_{2,n}(3)$	1	1	1	3	3	5	9	11	19	29	41	67	99	149	233	347
$F_{2,n}(4)$	1	1	1	1	3	3	5	5	11	11	21	21	43	43	85	85
$F_{2,n}(5)$	1	1	1	1	1	3	3	5	5	7	11	13	21	23	35	45
$F_{2,n}(6)$	1	1	1	1	1	1	3	3	5	5	7	7	13	13	23	23

$F_2(1)$ is the Pell sequence.

Finally, for $k = 3$ it is the following table:

Table 5. Generalized 3-Fibonacci numbers

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$F_{3,n}(1)$	1	3	10	33	109	360	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
$F_{3,n}(2)$	1	1	4	4	16	16	64	64	256	256	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
$F_{3,n}(3)$	1	1	1	4	4	7	16	19	37	67	94	178	295	460	829	$\cdots$
$F_{3,n}(4)$	1	1	1	1	4	4	7	7	19	19	40	40	97	97	217	217
$F_{3,n}(5)$	1	1	1	1	1	4	4	7	7	10	19	22	40	43	70	100
$F_{3,n}(6)$	1	1	1	1	1	1	4	4	7	7	10	10	22	22	43	43

The following proposition shows the formulas used to calculate the general term of the sequence $F_k(r) = \{F_{k,n}(r)\}$, according to that $r \ge 2$ is odd or even (see [7,8] for $r = 1$).

Theorem 1. (1) *If r is even, $r = 2p$:*

$$F_{k,2n}(2p) = F_{k,2n+1}(2p) = \sum_{j=0}^{n/p} \binom{n-(p-1)j}{j} k^j .$$

(2) *If r is odd, $r = 2p+1 \ge 3$:*

$$F_{k,2n}(2p+1) = \sum_{j=0} \left[\binom{n-(2p-1)j}{2j} k^{2j} + \binom{n-p-(2p-1)j}{2j+1} k^{2j+1} \right], \tag{1}$$

$$F_{k,2n+1}(2p+1) = \sum_{j=0} \left[\binom{n-(2p-1)j}{2j} k^{2j} + \binom{n-(p-1)-(2p-1)j}{2j+1} k^{2j+1} \right]. \tag{2}$$

If you want to implement formulas (1) and (2) in Mathematica, the limits of the sums would be as follows:

$$F_{k,2n}(2p+1)=\sum_{j=0}^{n/r}\binom{n-(2p-1)j}{2j}k^{2j}+\sum_{j=0}^{(n-p-1)/r}\binom{n-p-(2p-1)j}{2j+1}k^{2j+1},$$

$$F_{k,2n+1}(2p+1)=\sum_{j=0}^{n/r}\binom{n-(2p-1)j}{2j}k^{2j}+\sum_{j=0}^{(n-p)/r}\binom{n-(p-1)-(2p-1)j}{2j+1}k^{2j+1}.$$

Proposition 1 (Sum of the terms of the sequence $F_k(r)$). *The sum of the first n terms of the sequence $F_k(r)$ is given by the formula*

$$S_{k,n}(r)=\sum_{j=0}^{n}F_{k,j}(r)=\frac{1}{k}\Big(F_{k,n+r-1}(r)+F_{k,n+r}(r)-2\Big).$$

In particular, for $r=1: S_{k,n}(1)=S_{k,n}=\frac{1}{k}\Big(F_{k,n}+F_{k,n+1}-2\Big)$ (see [7–9]).

1.2.1 Special sums

The formulas for the sum of both even and odd terms of the above sequences are, respectively,

$$\sum_{j=0}^{n}F_{k,2j}(r)=\frac{1}{k}\big(F_{k,2n+r}-1\big)\quad\text{and}\quad\sum_{j=0}^{n}F_{k,2j+1}(r)=\frac{1}{k}\big(F_{k,2n+1+r}-1\big).$$

Proposition 2 (Generating function). *If $r>1$, the generating function of the sequence $F_k(r)=\{F_{k,n}(r)\}$ is*

$$f_r(x)=\frac{x+x^2}{1-x^2-k\,x^r}.$$

Proof. If $r=1$, then $f_1(x)=\frac{x}{1-x-x^2}$ and the proof is in [7].

Let $f_r(x)$ be the generating function of the sequence $\{F_{k,0}(r),F_{k,1}(r),F_{k,2}(r),\ldots\}$. Then:

$$\begin{aligned}
f_r(x)&=F_{k,0}(r)+F_{k,1}(r)x+F_{k,2}(r)x^2+F_{k,3}(r)x^3+\cdots+F_{k,r}(r)x^r+F_{k,r+1}(r)x^{r+1}+\cdots\\
x^2f_r(x)&=F_{k,0}(r)x^2+F_{k,1}(r)x^3+\cdots+F_{k,r-2}(r)x^r+F_{k,r-1}(r)x^{r+1}+\cdots\\
kx^rf_r(x)&=kF_{k,0}(r)x^r+kF_{k,1}(r)x^{r+1}+\cdots
\end{aligned}$$

Hence

$$(1-x^2-kx^r)f_r(x)=F_{k,0}(r)+F_{k,1}(r)x+\left[F_{k,2}(r)-kF_{k,0}(r)\right]x^2=x+x^2,$$

because

$$F_{k,j}(r)=1,\quad 1\le j<r$$

and

$$F_{k,j}(r)=kF_{k,j-1}(r)+F_{k,n-2}(r),\quad n\ge r.$$

Finally,

$$f_r(x)=\frac{x+x^2}{1-x^2-kx^r}\quad\text{for } r>1.$$

2. Binomial Transform of the Generalized k-Fibonacci Numbers

In this section, we will apply the Binomial transform to the preceding sequences and will obtain new integer sequences [11].

Definition 2. Binomial transform of the generalized k-Fibonacci sequence [10] is defined in the classical form as

$$BF_{k,n}(r) = \sum_{j=0}^{n} \binom{n}{j} F_{k,j}(r).$$

For $r = 1$ (see [6]).
So, for $r = 2, 3, \ldots$ the sequences obtained by applying this transform are:

$$BF_k(2) = \{1, 2, 4+k, 8+4k, 16+12k+k^2, 32+32k+6k^2,$$
$$64+80k+24k^2+k^3, 128+192k+80k^2+8k^3, \ldots\},$$

$$BF_k(3) = \{1, 2, 4, 8+k, 16+5k, 32+17k, 64+49k+k^2, 128+129k+8k^2,$$
$$256+321k+39k^2, 512+769k+150k^2+k^3, 1024+1793k+501k^2+11k^3, \ldots\}$$

$$BF_k(4) = \{1, 2, 4, 8, 16+k, 32+6k, 64+23k, 128+72k, 256+201k+k^2,$$
$$512+522k+10k^2, 1024+1291k+58k^2, 2048+3084k+256k^2, \ldots\}$$

For example, the three first sequences for $k = 2, 3, 4$ and their references in [14], are:

$$BF_1(2) = \{1, 2, 5, 12, 29, 70, 169, 408, 985, 2378, 5741, 13860, 33461, \ldots\} : A000129$$

$$BF_2(2) = \{1, 2, 6, 16, 44, 120, 328, 896, 2448, 6688, 18272, 49920, \ldots\} : A002605$$

$$BF_3(2) = \{1, 2, 7, 20, 61, 182, 547, 1640, 4921, 14762, 44287, 132860, \ldots\} : A015518$$

$$BF_1(3) = \{1, 2, 4, 9, 21, 49, 114, 265, 616, 1432, 3329, 7739, 17991, 41824, \ldots\} : A052921$$

$$BF_2(3) = \{1, 2, 4, 10, 26, 66, 166, 418, 1054, 2658, 6702, 16898, 42606, \ldots\}$$

$$BF_3(3) = \{1, 2, 4, 11, 31, 83, 220, 587, 1570, 4196, 11209, 29945, 80005, \ldots\}$$

$$BF_1(4) = \{1, 2, 4, 8, 17, 38, 87, 200, 458, 1044, 2373, 5388, 12233, 27782, \ldots\} : A114199$$

$$BF_2(4) = \{1, 2, 4, 8, 18, 44, 110, 272, 662, 1596, 3838, 9240, 22286, 53812, \ldots\} : A114203$$

$$BF_3(4) = \{1, 2, 4, 8, 19, 50, 133, 344, 868, 2168, 5419, 13604, 34261, 86366, \ldots\}$$

Curiously, $BF_1(2)$ is the classical Pell sequence.

2.1 Generating function of the sequences $BF_k(r)$

P. Barry shows in [1] that if $A(x)$ is the generating function of the sequence $\{a_n\}$, then

$$S(x) = \frac{1}{1-x} A\left(\frac{x}{1-x}\right)$$

is the generating function of the sequence $\{b_n\}$ with $b_n = \sum_j \binom{n}{j} a_n$.

So, we can deduce the generating function of the $F_k(r)$ sequence, for $r = 2, 3, \dots$ is

$$g_r(x) = \frac{1}{1-x} f_r\left(\frac{x}{1-x}\right),$$

that is,

$$g_r(x) = \frac{(1-x)^{r-2}}{(1-2x)(1-x)^{r-2} - k\,x^r}.$$

If we expanding this formula, then can find

$$g_r(x) = \frac{(1-x)^{r-2}}{\sum_{j=0}^{r-1} (-1)^j L_j(r) x^j - k\,x^r},$$

where $L_j(r)$ are the coefficients of the Lucas Triangle (2). That is,

$$L_j(r) = \frac{r+j-1}{j}\binom{r-2}{j-1} \quad \text{for } j = 1, 2, \dots r+1, \text{ with } L_0(r) = 1 \text{ for all } r.$$

2.2 Recurrence relation of the sequences $BF_k(r)$

Taking into account $g_r(x)$ is the generating function of the sequences $BF_k(r)$, the coefficients of the denominator of this function shows the recurrence relation of the sequences $BF_k(r)$.

For clarity, we indicate as $b_n(r)$ the elements of the sequence $BF_k(r)$. The expansion of the denominator $D(x) = (1-2x)(1-x)^{r-2} - k\,x^r$ is

$$D(x) = \sum_{j=1}^{r} (-1)^{j-1} L_j(r) x^{j-1} - k\,x^r$$

and its coefficients shown the recurrence relation of the sequences $\{b_n(r)\}$. For instance, for $r = 2, 3, 4, 5$,

$$r = 2 \to \{1, -2, -k\} \to b_n(2) = 2b_{n-1}(2) + k\,b_{n-2}(2) \tag{3}$$

$$r = 3 \to \{1, -3, 2, -k\} \to b_n(3) = 3b_{n-1}(3) - 2b_{n-2}(3) + k\,b_{n-3}(3) \tag{4}$$

$$r = 4 \to \{1, -4, 5, -2, -k\} \to b_n(4) = 4b_{n-1}(4) - 5b_{n-2}(4) + 2b_{n-3}(4) + k\,b_{n-4}(4)$$

$$r = 5 \to \{1, -5, 9, -7, 2, -k\} \to b_n(5) = 5b_{n-1}(5) - 9b_{n-2}(5) + 7b_{n-3}(5) - 2b_{n-4}(5) + k\,b_{n-5}(5)$$

with the initial conditions $b_n(r) = 2^n$ for $n = 0, 1, 2 \dots r-1$

2.3 Sums of the sequences $BF_k(r)$

In the sequel we will prove the formulas for the sums of the sequences $BF_k(2)$ and $BF_k(3)$ and show for $BF_k(4)$.

Let $b_n(2) = BF_{k,n}(2)$ be and we will indicate as b_n.

From (3) it is $b_n = 2b_{n-1} + k\,b_{n-2}$. Then:

$$\begin{aligned}
S_n(2) &= \sum_{j=0}^{n} b_j = b_0 + b_1 + \sum_{2}^{n} b_j = 1 + 2 + 2\sum_{2}^{n} b_{j-1} + k\sum_{2}^{n} b_{j-2} \\
&= 3 + 2(S_n - b_0 - b_n) + k(S_n - b_{n-1} - b_n) \\
&= 1 + 2S_n + k\,S_n - (2+k)b_n - k\,b_{n-1} \to
\end{aligned}$$

$$S_n(2) = \frac{1}{1+k}\Big((2+k)b_n(2) + k\,b_{n-1}(2) - 1\Big)$$

Let $b_n(3) = BF_{k,n}(3)$ be and we will idicate as b_n.

From (4) it is $b_n = 3b_{n-1} - 2b_{n-2} + k\,b_{n-3}$. Then

$$\begin{aligned}
S_n(3) &= \sum_{j=0}^{n} b_j = b_0 + b_1 + b_2 + \sum_{3}^{n} b_j \\
&= 1 + 2 + 4 + 3\sum_{3}^{n} b_{j-1} - 2\sum_{3}^{n} b_{j-2} + k\sum_{3}^{n} b_{j-3} \\
&= 7 + 3(S_n - b_0 - b_1 - b_n) - 2(S_n - b_0 - b_{n-1} - b_n) + k(S_n - b_{n-2} - b_{n-1} - b_n) \\
&= S_n + k\,S_n - (1+k)S_n - (k-2)b_{n-1} - k\,b_{n-2} \to \\
S_n(3) &= \frac{1}{k}((1+k)b_n + (k-2)b_{n-1} + k\,b_{n-2}) \\
&= \frac{1}{k}(b_n - 2b_{n-1} + k\,b_{n-2} + (b_n + b_{n-1} + b_{n-2}) \\
&= \frac{1}{k}(b_{n+1} - 2b_n) + S_n - S_{n-3} \to S_{n-3}(3) = \frac{1}{k}(b_{n+1} - 2b_n) \to \\
S_n(3) &= \frac{1}{k}\Big(b_{n+4}(3) - 2b_{n+3}(3)\Big)
\end{aligned}$$

and so,

$$S_n(4) = \frac{1}{k}\Big(b_{n+4}(4) - 3b_{n+3}(4) + 2b_{n+2}(4)\Big)$$

$$S_n(5) = \frac{1}{k}\Big(b_{n+4}(5) - 4b_{n+3}(5) + 5b_{n+2}(5) - 2b_{n+1}(5)\Big)$$

Finally, we see that the coefficients of the sum $S_n(r)$ are the constant coefficients of the polynomial $D(x)$ for $r = n-1$. So, the following sum must be

$$S_n(6) = \frac{1}{k}\Big(b_{n+4}(6) - 5B_{n+3}(6) + 9b_{n+2}(6) - 7b_{n+1}(6) + 2b_n(6)\Big)$$

and rightly so!

3. Conclusions

We have generalized the k-Fibonacci numbers according a distance $r \geq 1$ and obtained general formulas for these new sequences. We also indicate another way of finding the generalized k-Fibonacci sequences from the generating function.

We apply the Binomial Transform to these sequences and find its generating function. Later, we found both the recurrence relation for the sequences of the binomial transforms of the generalized k-Fibonacci numbers and the formula for the sum of the terms of these sequences.

Competing Interests

The author declares that he has no competing interests.

Authors' Contributions

The author wrote, read and approved the final manuscript.

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