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Research Article

Homogeneous Sagbi Bases Under Polynomial Composition

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Abstract. The process of replacing indeterminates in a Polynomial with other polynomials is the polynomial composition. Homogeneous Sagbi bases are the Sagbi bases generated by the subset of homogeneous polynomials. In this article, we present adequate and essential criterion on a set $\ominus$ of polynomials to guarantee that the composed set $S \circ \ominus$ is Homogeneous Sagbi basis whenever S is a Homogeneous Sagbi basis.

Keywords. Homogeneoous Sagbi basis; Polynomial composition

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1. Introduction

Our main interest in this field is inspirited by [6, 7], where the authors address the issue of the behaviour of *Gröbner* bases [1, 2] under polynomials composition. Further expressly, let $\ominus$ be a list of polynomials which may be as many in number as the indeterminates in our polynomial ring. The research investigation then is under which criterion on these polynomials the facts demonstrate that for any Homogeneous *Gröbner* basis G (under some term ordering), the composed set $G \circ \ominus$ would be Homogeneous *Gröbner* basis (possibly under different term ordering). If this happens, we state that the composition commutes with Homogeneous *Gröbner* basis computation. The result given in [6] is that, this is affirmative if and only if $\ominus$ is a list of permuted powering and composition by $\ominus$ is homogeneously compatible under the term ordering.

The behaviour of Sagbi (Subalgebra Anolog to *Gröbner* Bases for Ideals) bases under polynomials composition has extensively been discussed in research papers [5, 11]. Finally, composition of polynomial is an interesting and useful operation with an extensive number of uses in material sciences and arithmetic. Actually, we usually work with a list of polynomials where the indeterminates are given in the form of other indeterminates.

In this paper, we developed a slightly distinct case, that is, we examine the issue of the behaviour of HSB (homogeneous Sagbi bases) under composition of homogeneous polynomials of same degree. Let $K[x_1,\dots,x_n]$ denote the polynomial ring over the field K and $F \subset K[x_1,\dots,x_n]$ and let S be a HSB (under the term ordering $>$) and $\ominus = (\theta_1,\dots,\theta_n)$ such that, θ_i are homogeneous polynomials in $K[x_1,\dots,x_n]$ with the property that, degree of each θ_i is same and $S \circ \ominus$ is the composed set obtained by replacing each x_i by θ_i in S. Therefore, when does $S \circ \ominus$ is HSB under the same term ordering? We investigate this issue and provide a brief answer. We say that HSB computation commutes with composition by $\ominus$, if the composed set $S \circ \ominus$ is also HSB.

The paper is organised as follows. In Section 2, we give a brief review of Sagbi bases theory and composition of polynomials and also give the theory of HSB. In Section 3, we state the main result (Theorem 3.1) of this paper and also provide its proof.

2. Notation and Definitions

In this section, we recall some concepts and basic properties of Sagbi basis theory and composition of polynomials that will be used in the ensuing sections. The reader who knows about the theory is also additionally urge to skim through this area so as to get acquainted with the notation.

By mean of a monomial in $K[x_1,\dots,x_n]$, is an element of the form $x_1^{\alpha_1}\dots x_n^{\alpha_n}$ with $\alpha_1,\dots,\alpha_n \in \mathbb{N} = 0,1,2,\dots$, Mon_n represent the set of all monomials. Note that $1 = x_1^0\dots x_n^0 \in Mon_n$.

If $G \subset K[x_1,\dots,x_n]$ (may be infinite), then $K[G]$ represent the subalgebra of $K[x_1,\dots,x_n]$ generated by G. This notion is natural as the elements of $K[G]$ are specifically the polynomials in the set of given indeterminates G, observed as elements of $K[G]$.

Here, we list the notations which will be used throughout the paper.

Notation 2.1.

K	A field
σ, δ, η	A monomial, that is, $x_1^{\alpha_1},\dots,x_n^{\alpha_n}$ for some $\alpha_1,\dots,\alpha_n \in \mathbb{N}$
$>$	A global ordering (i.e. well ordering)
$LC_>(g)$	The leading coefficient of g under the term order $>$
$LM_>(g)$	The leading monomial of g under the term order $>$
$LT_>(g)$	The leading term of g under the term order $>$, and $LT_>(g) = LC_>(g)LM_>(g)$
$LM_>(G)$	The set $\{LM_>(g) \mid g \in G\}$
$\ominus$	A list $(\theta_1,\dots,\theta_n)$ of n non-zero homogeneous polynomials of same degree in $K[x_1,\dots,x_n]$
$LM(\ominus)$	The list $(LM(\theta_1),\dots,LT(\theta_n))$

2.1 Review of Sagbi Bases

In this section we collect the theory regarding Sagbi bases that we required (for a detailed annotation, see [9],[10]).

Definition 2.2 ([5, Definition 2.1])**.** A finite power product of the form $m(G) = g_1^{\alpha_1} \dots g_s^{\alpha_s}$ is called a G-monomial, where $g_i \in G$ for $i = 1,\dots,s$, and $\alpha_1,\dots,\alpha_s \in \mathbb{N}$.

Definition 2.3 ([5, Definition 2.2])**.** A subset S (may be infinite) of $K[G]$ is called Sagbi basis of $K[G]$ with respect to $>$ if

$$K[LM_>(K[G])] = K[LM_>(S)].$$

Thus a parallel criterion for Sagbi bases is that, if the leading monomial of each element in S can be written in the power products of leading monomials of elements in $K[G]$ then S is a Sagbi basis.

We can show that, S generates $K[G]$ whenever S is a Sagbi basis of $K[G]$, i.e. $K[G] = K[S]$. When we say that S is a Sagbi basis, we simply mean that S is a Sagbi basis of $K[S]$.

Definition 2.4 ([11, Definition 2])**.** If $LM_>(m(G)) = LM_>(\bar{m}(G))$, then we state that, the G-monomials $m(G)$ and $\bar{m}(G)$ form a critical pair $(m(G), \bar{m}(G))$ of G and if there exist a constant $a \in K$ such that leading coefficients of $m(G)$ and $a\bar{m}(G)$ becomes same, then corresponding T-polynomial of critical pair $(m(G), \bar{m}(G))$ is:

$$T(m(G), \bar{m}(G)) = m(G) - a\bar{m}(G).$$

Following theorem gives the criterion for a set to be a Sagbi basis of $K[G]$.

Theorem 2.5 ([11, Theorem 1])**.** *A subset S of $K[x_1,\dots,x_n]$ is a Sagbi basis with respect to $>$, if and only if, the T-polynomials of each critical pair $(m(S), \bar{m}(S))$ of S either equal to zero, or has a following representation*

$$T(m, \bar{m}) = \sum_{i=1}^{t} a_i m_i(S), \quad LM(\bar{m}(S)) = LM_>(m(S)) > LM_>(m_i(S)), \quad \textit{for all } i,$$

where the a_i are constant $\in K$ and m_i are monomials.

2.2 Composition of Polynomials

Now, we explain the procedure of composition of polynomials.

Definition 2.6 ([11, Definition 3])**.** Let $\ominus = (\theta_1,\dots,\theta_n)$ be a list of polynomials of $K[x_1,\dots,x_n]$, and let $h \in K[x_1,\dots,x_n]$. We describe the composition of h by $\ominus$, denoted by $h \circ \ominus$, is the polynomial obtained from h by replacing each occurrence of the x_i by θ_i. For a subset $H \in K[x_1,\dots,x_n]$, $H \circ \ominus = \{h \circ \ominus \mid h \in H\}$ is the composed set of H by $\ominus$.

Now, we state some elementary properties and facts about the composition and leading monomials. We will be used these throughout in the composition of Sagbi bases.

Proposition 2.7 (Hong [4, Proposition 4.1]).

(1): $(g_1 + g_2) \circ \ominus = g_1 \circ \ominus + g_2 \circ \ominus$.

(2): $(g_1 g_2) \circ \ominus = (g_1 \circ \ominus)(g_2 \circ \ominus)$.

(3): $LM_>(g_1 g_2) = LM_>(g_1) LM_>(g_2)$.

(4): $LM_>(\sigma \circ \ominus) = \sigma \circ (LM_>(\ominus))$.

Remark 2.8 ([5, Remark 2.7]). We have a natural correspondence between the set $G = \{g_1, g_2, \dots\}$ and $G \circ \ominus = \{g_1 \circ \ominus, g_2 \circ \ominus, \dots\}$, therefore for any G-monomial $m(G)$, its composition with $\ominus$ satisfies

$$m(G) \circ \ominus = m(G \circ \ominus).$$

Also, all the critical pairs of $G \circ \ominus$ are of the form $(m(G \circ \ominus), \bar{m}(G \circ \ominus))$, for some G-monomials $m(G), \bar{m}(G)$.

Definition 2.9 ([4, Definition 3.3]). We state that composition by $\ominus$ is compatible with term ordering $>$ if and only if all monomials σ and δ, we have

$$\sigma > \delta \Longrightarrow \sigma \circ LM(\ominus) > \delta \circ LM(\ominus).$$

Definition 2.10 ([5, Definition 2.11]). We state that composition by $\ominus$ is compatible with non-equality, if, for all monomials σ, δ we have

$$\sigma \neq \delta \Longrightarrow \sigma \circ LM(\ominus) \neq \delta \circ LM(\ominus).$$

2.3 Homogeneous Sagbi Bases

Homogeneous Sagbi bases are one of the important structures in commutative algebra. Here, we collect the theory related to HSB that we will required (for a detailed review, we refer to [8–10]).

Definition 2.11. A Sagbi basis S is said to be HSB if each element of S is a homogeneous polynomial.

Remark 2.12. Particularly it pursues that any subset S of $K[x_1, \dots, x_n]$ involving only of monomials (or coefficients times monomials) is HSB; and all T-polynomials are then surely equal to zero.

Following is an example of computing HSB of a given subalgebra.

Example 2.13. Here, we are computing HSB.
$\mathbb{Q}[G] \subset \mathbb{Q}[x, y]$, where $G = \{x, y^2 + xy, x^3 y\}$, we use lexicographical ordering with $y > x$.
We have $LM(G) = \{x, y^2, x^3 y\}$ and $deg(g_1) = 1, deg(g_2) = 2, deg(g_3) = 4$ and a critical pair $(m(G), m'(G)) = (g_1^6\, g_2, g_3^2)$ and $T(m(G), m'(G)) = x^7 y = g_1^4\, g_3$, thus $T(m(G), m'(G)) = 0$. On checking, we see that $(m(G), m'(G)) = (g_1^6\, g_2, g_3^2)$ is the only critical pair for G and their corresponding T-polynomial is zero. Therefore $G = \{x, y^2 + xy, x^3 y\}$ is HSB for the subalgebra $\mathbb{Q}[G]$ under lexicographical term ordering.

Now, we define the commutation of composition with HSB.

Definition 2.14 (Commutation of HSB)**.** Let S be a HSB under the term ordering $>$. If $S \circ \ominus$ is also HSB under the term ordering $>$, then we state that composition by $\ominus$ commutes with HSB computation.

Definition 2.15 ([6, Definition 2.6], Homogeneously Compatible with Term Ordering)**.** We state that composition by $\ominus$ is homogeneously compatible with the term ordering $>$ if and only if for all monomials σ and δ, such that

$$\sigma > \delta, \deg(\sigma) = \deg(\delta) \Longrightarrow \sigma \circ LM_{>}(\ominus) > \delta \circ LM(\ominus).$$

Definition 2.16 (Homogeneously Compatible with non-equality)**.** If for all monomials σ and δ, such that

$$\sigma \neq \delta, deg(\sigma) = deg(\delta) \Longrightarrow \sigma \circ LM(\ominus) \neq \delta \circ LM(\ominus)$$

then, we state that composition by $\ominus$ is homogeneously compatible with the non-equality.

3. Main Results

We state our main result.

Theorem 3.1 (Main Theorem)**.** *Let*

(A) *composition by $\ominus$ commutes with HSB computation.*

(B) *composition by $\ominus$ is homogeneously compatible with the term ordering $>$.*

then $(A) \Longleftrightarrow (B)$.

The proof of the main theorem is based on the following subsections.

3.1 Proof of Sufficiency

We give some basic results which can be found in [3].

Lemma 3.2. *Let composition by $\ominus$ be homogeneously compatible with the term ordering $>$ then for every polynomial g, we have*

(1): $LT_{>}(g \circ \ominus) = LT_{>}(g) \circ LT_{>}(\ominus)$; *and*

(2): $LM_{>}(g \circ \ominus) = LM_{>}(g) \circ LM_{>}(\ominus)$.

Proof. The proofs are similar as in [3]. □

Lemma 3.3. *Let*

(A): *composition by $\ominus$ is homogeneously compatible with the term ordering; and*

(B): *composition by $\ominus$ is homogeneously compatible with non-equality.*

then (A)$\Longrightarrow$(B).

Proof. The proof is similar as the proof (of Lemma 1) in [11]. □

Lemma 3.4. *Let $(m(S \circ \ominus), \bar{m}(S \circ \ominus))$ is a critical pair of $S \circ \ominus$, and composition by $\ominus$ is homogeneously compatible with the term ordering, then $(m(S), \bar{m}(S))$ is also a critical pair of S.*

Proof. Since S-monomials is a power products of polynomials of S and each polynomial is homogeneous this implies that $m(S)$ is homogeneous, also, $\ominus$ is the list of homogeneous polynomials of same degree therefore, $m(S \circ \ominus) = m(S) \circ \ominus$ is homogeneous (since $m(S)$ is arbitrary, this hold for $m_i(S)$ and $m_i(S \circ \ominus)$ for all i).

Let $(m(S \circ \ominus), \bar{m}(S \circ \ominus))$ be a critical pair of $S \circ \ominus$. We have $m(S \circ \ominus) = m(S) \circ \ominus$ and $\bar{m}(S \circ \ominus) = \bar{m}(S) \circ \ominus$, so by Lemma 3.2 we get $LM_{>} m(S) \circ LM_{>}(\ominus) = LM_{>} \bar{m}(S) \circ LM_{>}(\ominus)$. Since our composition is also homogeneously compatible with non-equality(by Lemma 3.3), $(m(S)), (\bar{m}(S))$ are homogeneous. We get $LM_{>} m(S) = LM_{>} \bar{m}(S)$.
Hence $(m(S), \bar{m}(S))$ is a critical pair of S. □

Now, the sufficiency side of the Theorem 3.1 is state as:

Proposition 3.5. (A): *the composition by $\ominus$ is homogeneously compatible with the term ordering $>$; and*

(B): *the composition by $\ominus$ commutes with HSB computation.*

then (A)$\Longrightarrow$(B).

Proof. Since composition by $\ominus$ is homogeneously compatible with the term ordering, therefore for all σ, δ with $\sigma > \delta$, $\deg(\sigma) = \deg(\delta) \Longrightarrow \sigma \circ LM(\ominus) > \delta \circ LM(\ominus)$, for an arbitrary HSB S. We claim that $S \circ \ominus$ is HSB.

By using Theorem 2.5, let $(m(S \circ \ominus), \bar{m}(S \circ \ominus))$ be any critical pair of $S \circ \ominus$. By Lemma 3.4, we realized that, $(m(S), \bar{m}(S))$ is also a critical pair of S. Since S is HSB therefore, by Theorem 2.5 we can write

$$m(S) - a\bar{m}(S) = \sum_i a_i m_i(S), \quad \text{(or zero) where } a, a_i \in K \tag{3.1}$$

and

$$LM(\bar{m}(S)) = LM_{>}(m(S)) > LM_{>}(m_i(S)) \quad \text{for all } i. \tag{3.2}$$

All S-monomials $(m_i(S))$ are homogeneous as they are the power products of homogeneous polynomials of S. Composing the equation (3.1) with $\ominus$ and using Proposition 2.7, we get

$$m(S \circ \ominus) - a\bar{m}(S \circ \ominus) = \sum_i a_i m_i(S \circ \ominus), \quad \text{(or zero).} \tag{3.3}$$

We know that, $m_i(S \circ \ominus)$ are homogeneous for all i. Composing the inequality in (3.2) by $LM_{>}(\ominus)$, we get

$$LM_{>}(\bar{m}(S)) \circ LM_{>}(\ominus) = LM_{>}(m(S)) \circ LM_{>}(\ominus) > LM_{>}(m_i(S)) \circ LM_{>}(\ominus) \quad \text{for all } i.$$

Using Lemma 3.2, this becomes

$$LM(\bar{m}(S \circ \ominus)) = LM_{>}(m(S \circ \ominus)) > LM_{>}(m_i(S \circ \ominus)) \quad \text{for all } i. \tag{3.4}$$

The leading terms of left-hands sides of (3.3) cancel. Thus (3.3) and (3.4) together give a representation of Theorem 2.5, also the each polynomial of $S \circ \ominus$ is homogeneous therefore, we conclude that $S \circ \ominus$ is a HSB with respect to $>$. □

3.2 Proof of Necessity

In this section, we give the converse with regard to the Theorem 3.1, that is, we will prove that commutativity implies homogeneously compatibility.

We first show the relation, that is, commutation of HSB implies homogeneously compatibility with non-equality, and by the use of this conclusion we will proof the compatibility with the term ordering. We begin with the following lemma.

Lemma 3.6. *Let δ,σ be two monomials of same degree, that is;* $\deg(\delta) = deg(\sigma), \delta \neq \sigma$ *but* $\delta \circ LM_>(\ominus) = \sigma \circ LM_>(\ominus)$. *Then for every $\eta < \delta$ with* $\deg(\eta) = \deg(\delta)$, *$S = \{\delta - \eta, \sigma\}$ is HSB.*

Proof. We must have both δ and σ are different from 1. Namely if $\sigma = 1$ then $\delta \neq 1$, and also we have θ_i are non-constant ([11, Remark 3]), $\delta \circ LM_>(\ominus) = \sigma \circ LM_>(\ominus) \neq 1$, a contradiction.

Since $deg(\eta) = deg(\delta)$ implies all polynomials of S are homogeneous. So, we just need to prove the more strong case, that is, S do not has any non-trivial critical pairs, that is, if $LM_> m(S) = LM_> \bar{m}(S)$, then m and $\bar{m}$ both are equal. Then it is obvious that S is HSB; and all T-polynomials surely equal to zero. Therefore, assume an arbitrary critical pair $(m(S), \bar{m}(S))$ of S and $LM_>(S) = \{\delta, \sigma\}$ we have $LM_> m(S)(= m(LM_>(S))) = \delta^k \sigma^l$ and $LM_> \bar{m}(S)(= \bar{m}(LM_>(S))) = \delta^s \sigma^t$, and $\delta^k \sigma^l = \delta^s \sigma^t$.

Suppose that, this is a non trivial critical pair. therefore, $k \neq s$ and $l \neq t$, further expressly $k > s$ and $l < t$ or with the order reversed. After the cancellation of common powers of δ and σ we thus finish up with $\delta^a = \sigma^b, a, b > 0$ also $\ominus$ is the list of homogeneous polynomials of same degree. Composing this last equality by $LM_> \ominus$ we get, $(\delta \circ LM_>(\ominus))^a = (\sigma \circ LM_>(\ominus))^b$.

Since $\delta \circ LM_>(\ominus) = \sigma \circ LM_>(\ominus) \neq 1$, at that point we obviously have $a = b$. From $\delta^a = \sigma^b$ it now lastly follows that $\delta = \sigma$ a contradiction.

Therefore, we conclude that $(m(S), \bar{m}(S))$ is trivial. Thus, S is HSB. □

Proposition 3.7. *Let*

(A): *composition by $\ominus$ commutes with HSB computation*; *and*

(B): *composition by $\ominus$ is homogeneously compatible with non-equality.*

then (A)$\Longrightarrow$(B).

Proof. Since composition by $\ominus$ commutes with HSB computation, i.e. each polynomial of $S \circ \ominus$ is homogeneous. On contrary suppose that, there exist two monomials δ, σ with $deg(\delta) = deg(\sigma), \delta \neq \sigma$ but $\delta \circ LM_>(\ominus) = \sigma \circ LM_>(\ominus)$; as in the proof of Lemma 3.6 we have $\delta, \sigma \neq 1$. By using Remark 2.12, we know that $S = \{\delta, \sigma\}$ is HSB, so $S \circ \ominus = \{\delta \circ \ominus, \sigma \circ \ominus\}$ is again HSB. If $g = \delta \circ \ominus - \sigma \circ \ominus \in K[S \circ \ominus]$ is different from 1 or zero, at that point we are finished; as $LM_> g < (\delta \circ LM_>(\ominus) = \sigma \circ LM_>(\ominus))$, can then not be written as product from $LM_> S \circ \ominus = \{\delta \circ LM_> \ominus, \sigma \circ LM_> \ominus\}$, so $S \circ \ominus$ cannot be HSB.

Now, let $\delta' = \delta x_i, \sigma' = \sigma x_i$ for some (arbitrary) $x_i \in K[x_1, \ldots, x_n]$ such that $x_i^{\lambda+1} < \delta'$, where $\lambda = deg(\delta)$. It is clear that for example $\delta' \circ \ominus = (\delta \circ \ominus)\theta_i$, and that

$$\delta' \neq \sigma' \quad \text{and} \quad LM_> \delta' \circ \ominus = LM_> \sigma' \circ \ominus \tag{3.5}$$

If $g = \delta \circ \ominus - \sigma \circ \ominus = 1$, at that point we use $S' = \{\delta', \sigma'\}$ which is HSB.

We then have

$$g' = \delta' \circ \ominus - \sigma' \circ \ominus = (\delta \circ \ominus)\theta_i - (\sigma \circ \ominus)\theta_i = g\theta_i = \theta_i \in K[S' \circ \ominus],$$

and as above it pursues that $S = \{\delta' \circ \ominus, \sigma' \circ \ominus\}$ can not be HSB

$$\left(LM_{>}\theta_i < LM_{>}\delta' \circ \ominus = \sigma' \circ \ominus\right).$$

(Also we notice that, it is impossible to use $g = 1$ as a counter example straightly; $p = x_1^0 \dots x_n^0$ is an allowed monomial.)

Now we just need to eliminate the case $g = \delta \circ \ominus - \sigma \circ \ominus = 0$ (for example: if $\theta = LM_{>}\theta$). We presently using the set $S' = \{\delta' + x_i^{\lambda+1}, \sigma'\}$. Since we have (3.5) and $x_i^{\lambda+1} < \delta' = \delta x_i$, and also $\deg(\delta') = \deg(x_i^{\lambda+1})$ therefore, S' is HSB by Lemma 3.6. We now get $g' = \delta' \circ \ominus - \sigma' \circ \ominus = \theta_i \in K[S' \circ \ominus]$, a contradiction.

Hence $\deg(\delta) = \deg(\sigma), \delta \neq \sigma \Longrightarrow \delta \circ LM_{>}(\ominus) \neq \sigma \circ LM_{>}(\ominus)$. □

We can now complete the proof of the necessity side of Theorem 3.1.

Proposition 3.8. *Let*

(A): *composition by* $\ominus$ *commutes with* HSB *computation*; *and*

(B): *composition by* $\ominus$ *is homogeneously compatible with the term ordering.*

then (A)$\Longrightarrow$(B).

Proof. Let composition by $\ominus$ commutes with HSB computation. Assume that δ, σ are two monomials such that $\deg(\delta) = \deg(\sigma)$, $\delta > \sigma$, then we claim that $\delta \circ LM_{>}(\ominus) > \sigma \circ LM_{>}(\ominus)$. Particularly we have $\delta \neq \sigma$, we know from Proposition 3.7 that it is not possible to have $\delta \circ LM_{>}(\ominus) = \sigma \circ LM_{>}(\ominus)$. So we only need to eliminate the case $\delta \circ LM_{>}(\ominus) < \sigma \circ LM_{>}(\ominus)$.

First, we claim that $S = \{\delta - \sigma, \sigma\}$ is HSB. Infact we know that $S' = \{\delta, \sigma\}$ is HSB so, our claim follows since $K[S] = K[S']$ and $LM_{>}S = LM_{>}S'$. We conclude that $S \circ \ominus = \{\delta \circ \ominus - \sigma \circ \ominus, \sigma \circ \ominus\}$ requisite to be HSB.

Now, we assume that $\delta \circ \ominus < \sigma \circ \ominus$. We then have $LM_{>}S \circ (\ominus) = \{\sigma \circ LM_{>}(\ominus)\}$, and $\delta \circ \ominus = (\delta \circ \ominus - \sigma \circ \ominus) + \sigma \circ \ominus \in K[S \circ \ominus]$. But (as in the proof of Proposition 3.7) since $\delta \circ LM_{>}(\ominus) < \sigma \circ LM_{>}(\ominus)$. $\delta \circ LM_{>}\ominus \neq 1$ impossible to be written in the powers of $\sigma \circ LM_{>}(\ominus)$, therefore $S \circ \ominus$ cannot be HSB.

Thus our supposition that, $\delta \circ LM_{>}\ominus < \sigma \circ LM_{>}\ominus$ was incorrect, so composition by $\ominus$ is homogeneously compatible with the term ordering. □

Therefore HSB computation commutes with composition by $\ominus$.

The following example emphasize the result of Theorem 3.1.

Example 3.9. Let $S = \{x, y^2 + xy, x^3 y\}$ and $\ominus = \left(x^2 + 2xy, 2y^2\right)$ with lexicographical ordering $y > x$. Clearly all polynomials of $\ominus$ are of same degree(=2) and $\ominus$ is homogeneously compatible with the term ordering. Thus by Theorem 3.1 we conclude that,

$$S \circ \ominus = \{(x^2 + 2xy), ((2y^2)^2 + (x^2 + 2xy)2y^2), ((x^2 + 2xy)^3 2y^2)\}$$

is HSB under the lexicographical ordering $y > x$.

4. Conclusion

We discussed the problem of the behaviour of homogeneous Sagbi bases under polynomial composition. One natural application is in the computation of the subalgebra generated by composed polynomials: In order to compute the HSB; $S\circ\ominus$ of subalgebra $F\circ\ominus$, we first compute the HSB; S of subalgebra F and carry out the composition on S to obtain the HSB of $F\circ\ominus$. One research direction is to investigate the behaviour of λ-Sagbi bases under polynomial composition, that is, when does composition commutes with λ-Sagbi bases computation.

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Competing Interests

The author declares that he has no competing interests.

Authors' Contributions

The author wrote, read and approved the final manuscript.

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