Research Article

Bicomplex Tetranacci and Tetranacci-Lucas Quaternions

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Abstract. In this paper, we introduce the bicomplex Tetranacci and Tetranacci-Lucas quaternions. Moreover, we present Binet's formulas, generating functions, and the summation formulas for those bicomplex quaternions.

Keywords. Bicomplex Tetranacci numbers; Bicomplex quaternions; Bicomplex Tetranacci quaternions; Bicomplex Tetranacci-Lucas quaternions

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1. Introduction

In this paper, we define bicomplex Tetranacci and bicomplex Tetranacci-Lucas quaternions by combining bicomplex numbers and Tetranacci, Tetranacci-Lucas numbers and give some properties of them. Before giving their definition, we present some information on bicomplex numbers and also on Tetranacci and Tetranacci-Lucas numbers.

The bicomplex numbers (quaternions) are defined by the four bases elements $1, i, j, ij$ where i, j and ij satisfy the following properties:

$$i^2 = -1, \quad j^2 = -1, \quad ij = ji.$$

A bicomplex number can be expressed as follows:

$$q = a_0 + ia_1 + ja_2 + ija_3 = (a_0 + ia_1) + j(a_2 + ia_3) = z_0 + jz_1,$$

where a_0, a_1, a_2, a_3 are real numbers and z_0, z_1 are complex numbers. So the set of bicomplex number is

$$\mathbb{BC} = \{z_0 + jz_1 : z_0, z_1 \in \mathbb{C}, j^2 = -1\}.$$

Moreover, for any bicomplex numbers $q = a_0 + ia_1 + ja_2 + ija_3$ and $p = b_0 + ib_1 + jb_2 + ijb_3$ and scaler $\lambda \in \mathbb{R}$, the addition, subtraction and multiplication with scalar are defined as componentwise, i.e

$$q + p = (a_0 + b_0) + i(a_1 + b_1) + j(a_2 + b_2) + ij(a_3 + b_3),$$
$$q - p = (a_0 - b_0) + i(a_1 - b_1) + j(a_2 - b_2) + ij(a_3 - b_3),$$
$$\lambda q = \lambda a_0 + i\lambda a_1 + j\lambda a_2 + ij\lambda a_3$$

respectively, and product (multiplication) is defined as follows:

$$q \times p = (a_0 b_0 - a_1 b_1 - a_2 b_2 + a_3 b_3) + i(a_0 b_1 + a_1 b_0 - a_2 b_3 - a_3 b_2)$$
$$+ j(a_0 b_2 - a_1 b_3 + a_2 b_0 - a_3 b_1) + ij(a_0 b_3 + a_1 b_2 + a_2 b_1 + a_3 b_0).$$

Multiplication of basis elements of bicomplex numbers can be done according to Table 1:

Table 1. Multiplication table

$\times$	1	i	j	ij
1	1	i	j	ij
i	i	-1	ij	$-j$
j	j	ij	-1	$-i$
ij	ij	$-j$	$-i$	-1

There are three different conjugations (involutions) for bicomplex numbers, namely

$$q_i^* = a_0 - ia_1 + ja_2 - ija_3 = \overline{z}_0 + j\overline{z}_1,$$
$$q_j^* = a_0 + ia_1 - ja_2 - ija_3 = z_0 - jz_1,$$
$$q_{ij}^* = a_0 - ia_1 - ja_2 + ija_3 = \overline{z}_0 - j\overline{z}_1,$$

for $q = a_0 + ia_1 + ja_2 + ija_3$. The squares of norms of the bicomplex numbers which arise from the definitions of conjugations are given by

$$N_i^2(q) = |q_i \times q_i^*| := |a_0^2 + a_1^2 - a_2^2 - a_3^2 + 2j(a_0 a_2 + a_1 a_3)|,$$
$$N_j^2(q) = |q_j \times q_j^*| := |a_0^2 + a_1^2 - a_2^2 - a_3^2 + 2i(a_0 a_1 + a_2 a_3)|,$$
$$N_{ij}^2(q) = |q_{ij} \times q_{ij}^*| := |a_0^2 + a_1^2 + a_2^2 + a_3^2 + 2ij(a_0 a_3 - a_2 a_1)|.$$

We now give some basic informations on quaternions. Quaternions were formally invented by Irish mathematician W. R. Hamilton (1805-1865) as an extension to the complex numbers and for some background about this type of hypercomplex numbers we refer the works, for

example, in [3], [6], [23]. The field $\mathbb{H}$ of quaternions is a four-dimensional non-commutative $\mathbb{R}$-field generated by four base elements $1, i, j$ and k that satisfy the following rules:

$$i^2 = j^2 = k^2 = ijk = -1 \tag{1.1}$$

and

$$ij = k = -ji, \ jk = i = -kj, \ ki = j = -ik. \tag{1.2}$$

Briefly $\mathbb{BC}$, the set of bicomplex numbers, has the following properties:

- Quaternions and bicomplex numbers are generalizations of complex numbers, but one difference between them is that quaternions are non-commutative, whereas bicomplex numbers are commutative.

- Real quaternions are non-commutative, and don't have zero divisors and non-trivial idempotent elements. But bicomplex numbers are commutative, have zero divisors and non-trivial idempotent elements:

$$ij = ji,$$
$$(i+j)(i-j) = i^2 - ij + ji - j^2 = 0,$$
$$\left(\frac{1+ij}{2}\right)^2 = \frac{1+ij}{2}.$$

- All above norms are isotropic. For example, for N_i, we calculate $N_i(q)$ for $q = 1 + ij$ as

$$N_i^2(1+ij) = (1+ij)(1-ij) = 1^2 - ij + ij - (ij)^2 = 0.$$

- $\mathbb{BC}$ is a real vector space with the addition of bicomplex numbers and the multiplication of a bicomplex number by a real scalar.

- $\mathbb{BC}$ forms a commutative ring with unity which contains $\mathbb{C}$.

- $\mathbb{BC}$ forms a two-dimensional algebra over $\mathbb{C}$, and since $\mathbb{C}$ is of dimension two over $\mathbb{R}$, the bicomplex numbers are an algebra over $\mathbb{R}$ of dimension four.

- $\mathbb{BC}$ is a real associative algebra with the bicomplex number product $\times$.

For more details about these type of numbers (quaternions), we refer to, for example, the works [13], [18], among others.

Tetranacci sequence $\{M_n\}_{n \geq 0}$ and Tetranacci-Lucas sequence $\{R_n\}_{n \geq 0}$ are defined by the fourth-order recurrence relations

$$M_n = M_{n-1} + M_{n-2} + M_{n-3} + M_{n-4}, \quad M_0 = 0, M_1 = 1, M_2 = 1, M_3 = 2 \tag{1.3}$$

and

$$R_n = R_{n-1} + R_{n-2} + R_{n-3} + R_{n-4}, \quad R_0 = 4, R_1 = 1, R_2 = 3, R_3 = 7 \tag{1.4}$$

respectively. More detail on these sequences can be found, for example, in [10], [14], [15], [19], [21] and [22].

The sequences $\{M_n\}_{n \geq 0}$ and $\{R_n\}_{n \geq 0}$ can be extended to negative subscripts by defining

$$M_{-n} = -M_{-(n-1)} - M_{-(n-2)} - M_{-(n-3)} + M_{-(n-4)}$$

and

$$R_{-n} = -R_{-(n-1)} - R_{-(n-2)} - R_{-(n-3)} + R_{-(n-4)}$$

for $n = 1, 2, 3, \ldots$, respectively. Therefore, recurrences (1.3) and (1.4) hold for all integer n.

Table 2 presents the first few values of the Tetranacci and Tetranacci-Lucas numbers with positive and negative subscripts:

Table 2. Tetranacci and Tetranacci-Lucas Numbers with non-negative and negative indices

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	$\cdots$
M_n	0	1	1	2	4	8	15	29	56	108	208	401	773	1490	$\cdots$
M_{-n}	0	0	0	1	-1	0	0	2	-3	1	0	4	-8	5	$\cdots$
R_n	4	1	3	7	15	26	51	99	191	367	708	1365	2631	5071	$\cdots$
R_{-n}	4	-1	-1	-1	7	-6	-1	-1	15	-19	4	-1	31	-53	$\cdots$

It is well known that for all integers n, usual Tetranacci and Tetranacci-Lucas numbers can be expressed using Binet's formulas

$$M_n = \frac{\alpha^{n+2}}{(\alpha - \beta)(\alpha - \gamma)(\alpha - \delta)} + \frac{\beta^{n+2}}{(\beta - \alpha)(\beta - \gamma)(\beta - \delta)} + \frac{\gamma^{n+2}}{(\gamma - \alpha)(\gamma - \beta)(\gamma - \delta)} + \frac{\delta^{n+2}}{(\delta - \alpha)(\delta - \beta)(\delta - \gamma)}$$

(see for example [24] or [10]), or

$$M_n = \frac{\alpha - 1}{5\alpha - 8}\alpha^{n-1} + \frac{\beta - 1}{5\beta - 8}\beta^{n-1} + \frac{\gamma - 1}{5\gamma - 8}\gamma^{n-1} + +\frac{\delta - 1}{5\delta - 8}\delta^{n-1} \tag{1.5}$$

(see for example [7]) and

$$R_n = \alpha^n + \beta^n + \gamma^n + \delta^n$$

respectively, where α, β, γ and δ are the roots of the equation $x^4 - x^3 - x^2 - x - 1 = 0$. Moreover,

$$\alpha = \frac{1}{4} + \frac{1}{2}\omega + \frac{1}{2}\sqrt{\frac{11}{4} - \omega^2 + \frac{13}{4}\omega^{-1}}, \qquad \beta = \frac{1}{4} + \frac{1}{2}\omega - \frac{1}{2}\sqrt{\frac{11}{4} - \omega^2 + \frac{13}{4}\omega^{-1}},$$

$$\gamma = \frac{1}{4} - \frac{1}{2}\omega + \frac{1}{2}\sqrt{\frac{11}{4} - \omega^2 - \frac{13}{4}\omega^{-1}}, \qquad \delta = \frac{1}{4} - \frac{1}{2}\omega - \frac{1}{2}\sqrt{\frac{11}{4} - \omega^2 - \frac{13}{4}\omega^{-1}},$$

where

$$\omega = \sqrt{\frac{11}{12} + \left(\frac{-65}{54} + \sqrt{\frac{563}{108}}\right)^{1/3} + \left(\frac{-65}{54} - \sqrt{\frac{563}{108}}\right)^{1/3}}.$$

Note that the Binet form of a sequence satisfying (1.3) and (1.4) for non-negative integers is valid for all integers n. This result of Howard and Saidak [11] is even true in the case of higher-order recurrence relations.

The generating functions for Tetranacci sequence $\{M_n\}_{n \geq 0}$ and Tetranacci-Lucas sequence $\{R_n\}_{n \geq 0}$ are

$$\sum_{n=0}^{\infty} M_n x^n = \frac{x}{1 - x - x^2 - x^3 - x^4} \quad \text{and} \quad \sum_{n=0}^{\infty} R_n x^n = \frac{4 - 3x - 2x^2 - x^3}{1 - x - x^2 - x^3 - x^4},$$

respectively.

2. The Bicomplex Tetranacci and Tetranacci-Lucas Quaternions and their Generating Functions, Binet's Formulas and Summations Formulas

In this section we define the bicomplex Tetranacci and Tetranacci-Lucas quaternions and give generating functions and Binet formulas for them. First, we give some information about bicomplex type quaternion sequences from the literature.

Nurkan and Güven [17] (see also [16]) introduced nth bicomplex Fibonacci and nth bicomplex Lucas numbers (quaternions) as

$$BF_n = F_n + F_{n+1}i + F_{n+2}j + F_{n+3}ij$$

and

$$BL_n = L_n + L_{n+1}i + L_{n+2}j + L_{n+3}ij$$

respectively, where F_n and L_n are the nth Fibonacci and Lucas numbers, respectively. Various families of bicomplex number (quaternion) sequences have been defined and studied by a number of authors. See, for example, [1], [2], [4], [8], [9] for second order bicomplex quaternion sequences and [5], [12]] for third order bicomplex quaternion sequences.

We now define bicomplex Tetranacci and Tetranacci-Lucas quaternions over the algebra $\mathbb{BC}$.

Definition 1. The nth bicomplex Tetranacci quaternion is

$$\mathbb{BC}M_n = M_n + iM_{n+1} + jM_{n+2} + ijM_{n+3} \tag{2.1}$$

and the nth Tetranacci-Lucas quaternion is

$$\mathbb{BC}R_n = R_n + iR_{n+1} + jR_{n+2} + ijR_{n+3}. \tag{2.2}$$

It can be easily shown that $\{\mathbb{BC}M_n\}_{n\geq 0}$ and $\{\mathbb{BC}R_n\}_{n\geq 0}$ can also be defined by the recurrence relations:

$$\mathbb{BC}M_n = \mathbb{BC}M_{n-1} + \mathbb{BC}M_{n-2} + \mathbb{BC}M_{n-3} + \mathbb{BC}M_{n-4} \tag{2.3}$$

and

$$\mathbb{BC}R_n = \mathbb{BC}R_{n-1} + \mathbb{BC}R_{n-2} + \mathbb{BC}R_{n-3} + \mathbb{BC}R_{n-4} \tag{2.4}$$

with the intial conditions

$$\mathbb{BC}M_0 = i + j + 2ij, \quad \mathbb{BC}M_1 = 1 + i + 2j + 4ij,$$
$$\mathbb{BC}M_2 = 1 + 2i + 4j + 8ij, \quad \mathbb{BC}M_3 = 2 + 4i + 8j + 15ij$$

and

$$\mathbb{BC}R_0 = 4 + i + 3j + 7ij, \quad \mathbb{BC}R_1 = 1 + 3i + 7 + 15ij,$$
$$\mathbb{BC}R_2 = 3 + 7i + 15j + 26ij, \quad \mathbb{BC}R_3 = 7 + 15i + 26j + 51ij.$$

The sequences $\{\mathbb{BC}M_n\}_{n\geq 0}$ and $\{\mathbb{BC}R_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$\mathbb{BC}M_{-n} = -\mathbb{BC}M_{-(n-1)} - \mathbb{BC}M_{-(n-2)} - \mathbb{BC}M_{-(n-3)} + \mathbb{BC}M_{-(n-4)}$$

and

$$\mathbb{BC}R_{-n} = -\mathbb{BC}R_{-(n-1)} - \mathbb{BC}R_{-(n-2)} - \mathbb{BC}R_{-(n-3)} + \mathbb{BC}R_{-(n-4)}$$

for $n = 1, 2, 3, \ldots$, respectively. Therefore, recurrences (2.3) and (2.4) hold for all integer n.

The first few bicomplex Tetranacci and Tetranacci-Lucas quaternions with positive subscript and negative subscript are given in Table 3 and Table 4:

Table 3. Bicomplex Tetranacci quaternions

n	$\mathbb{BC}M_n$	$\mathbb{BC}M_{-n}$
0	$i + j + 2ij$	$i + j + 2ij$
1	$1 + i + 2j + 4ij$	$j + ij$
2	$1 + 2i + 4j + 8ij$	ij
3	$2 + 4i + 8j + 15ij$	1
4	$4 + 8i + 15j + 29ij$	$-1 + i$
5	$8 + 15i + 29j + 56ij$	$-i + j$
6	$15 + 29i + 56j + 108ij$	$-j + ij$
7	$29 + 56i + 108j + 208ij$	$2 - ij$

Table 4. Bicomplex Tetranacci-Lucas quaternions

n	$\mathbb{BC}R_n$	$\mathbb{BC}R_{-n}$
0	$4 + i + 3j + 7ij$	$4 + i + 3j + 7ij$
1	$1 + 3i + 7 + 15ij$	$-1 + 4i + j + 3ij$
2	$3 + 7i + 15j + 26ij$	$-1 - i + 4j + ij$
3	$7 + 15i + 26j + 51ij$	$-1 - i - j + 4ij$
4	$15 + 26i + 51j + 99ij$	$7 - i - j - ij$
5	$26 + 51i + 99j + 191ij$	$-6 + 7i - j - ij$
6	$51 + 99i + 191j + 367ij$	$-1 - 6i + 7j - ij$
7	$99 + 191i + 367j + 708ij$	$-1 - i - 6j + 7ij$

For two bicomplex Tetranacci quaternions $\mathbb{BC}M_n$ and $\mathbb{BC}M_k$ and for skaler $\lambda \in \mathbb{R}$, the addition, substraction and multiplication with scalar are defined as componentwise, i.e.,

$$\mathbb{BC}M_n + \mathbb{BC}M_k = (M_n + M_k) + i(M_{n+1} + M_{k+1}) + j(M_{n+2} + M_{k+2}) + ij(M_{n+3} + M_{k+3}),$$

$$\mathbb{BC}M_n - \mathbb{BC}M_k = (M_n - M_k) + i(M_{n+1} - M_{k+1}) + j(M_{n+2} - M_{k+2}) + ij(M_{n+3} - M_{k+3}),$$

$$\lambda \mathbb{BC}M_n = \lambda M_n + i\lambda M_{n+1} + j\lambda M_{n+2} + ij\lambda M_{n+3}$$

respectively, and product (multiplication) is defined as follows:

$$\mathbb{BC}M_n \times \mathbb{BC}M_k = (M_n M_k - M_{n+1}M_{k+1} - M_{n+2}M_{k+2} + M_{n+3}M_{k+3})$$

$$+ i(M_n M_{k+1} + M_{n+1} M_k - M_{n+2} M_{k+3} - M_{n+3} M_{k+2})$$
$$+ j(M_n M_{k+2} - M_{n+1} M_{k+3} + M_{n+2} M_k - M_{n+3} M_{k+1})$$
$$+ ij(M_n M_{k+3} + M_{n+1} M_{k+2} + M_{n+2} M_{k+1} + M_{n+3} M_k)$$
$$= \mathbb{BC} M_k \times \mathbb{BC} M_n.$$

Similarly, for two bicomplex Tetranacci-Lucas quaternions $\mathbb{BC} R_n$ and $\mathbb{BC} R_k$ and for skaler $\lambda \in \mathbb{R}$, the addition, substraction and multiplication with scalar are defined as componentwise, i.e.,

$$\mathbb{BC} R_n \pm \mathbb{BC} R_k = (R_n \pm R_k) + i(R_{n+1} \pm R_{k+1}) + j(R_{n+2} \pm R_{k+2}) + ij(R_{n+3} \pm R_{k+3}),$$
$$\lambda \mathbb{BC} R_n = \lambda R_n + i\lambda R_{n+1} + j\lambda R_{n+2} + ij\lambda R_{n+3}$$

respectively, and product (multiplication) is defined as follows:

$$\mathbb{BC} R_n \times \mathbb{BC} R_k = (R_n R_k - R_{n+1} R_{k+1} - R_{n+2} R_{k+2} + R_{n+3} R_{k+3})$$
$$+ i(R_n R_{k+1} + R_{n+1} R_k - R_{n+2} R_{k+3} - R_{n+3} R_{k+2})$$
$$+ j(R_n R_{k+2} - R_{n+1} R_{k+3} + R_{n+2} R_k - R_{n+3} R_{k+1})$$
$$+ ij(R_n R_{k+3} + R_{n+1} R_{k+2} + R_{n+2} R_{k+1} + R_{n+3} R_k)$$
$$= \mathbb{BC} R_k \times \mathbb{BC} R_n.$$

Note that

$$\mathbb{BC} M_n \times \mathbb{BC} M_n = (M_n^2 - M_{n+1}^2 - M_{n+2}^2 + M_{n+3}^2) + 2i(M_n M_{n+1} - M_{n+2} M_{n+3})$$
$$+ 2j(M_n M_{n+2} - M_{n+1} M_{n+3}) + 2ij(M_n M_{n+3} + M_{n+1} M_{n+2})$$

and

$$\mathbb{BC} R_n \times \mathbb{BC} R_n = (R_n^2 - R_{n+1}^2 - R_{n+2}^2 + R_{n+3}^2) + 2i(R_n R_{n+1} - R_{n+2} R_{n+3})$$
$$+ 2j(R_n R_{n+2} - R_{n+1} R_{n+3}) + 2ij(R_n R_{n+3} + R_{n+1} R_{n+2}).$$

Moreover, three different conjugations for the bicomplex Tribonacci quaternion $\mathbb{BC} M_n = M_n + iM_{n+1} + jM_{n+2} + ijM_{n+3}$ are given as

$$(\mathbb{BC} M_n)_i^* = M_n - iM_{n+1} + jM_{n+2} - ijM_{n+3},$$
$$(\mathbb{BC} M_n)_j^* = M_n + iM_{n+1} - jM_{n+2} - ijM_{n+3},$$
$$(\mathbb{BC} M_n)_{ij}^* = M_n - iM_{n+1} - jM_{n+2} + ijM_{n+3},$$

and the squares of norms of the bicomplex Tribonacci quaternion are given by

$$N_i^2(\mathbb{BC} M_n) = |(\mathbb{BC} M_n)_i \times (\mathbb{BC} M_n)_i^*|$$
$$:= |M_n^2 + M_{n+1}^2 - M_{n+2}^2 - M_{n+3}^2 + 2j(M_n M_{n+2} + M_{n+1} M_{n+3})|,$$
$$N_j^2(\mathbb{BC} M_n) = |(\mathbb{BC} M_n)_j \times (\mathbb{BC} M_n)_j^*|$$
$$:= |M_n^2 + M_{n+1}^2 - M_{n+2}^2 - M_{n+3}^2 + 2i(M_n M_{n+1} + M_{n+2} M_{n+3})|,$$
$$N_{ij}^2(\mathbb{BC} M_n) = |(\mathbb{BC} M_n)_{ij} \times (\mathbb{BC} M_n)_{ij}^*|$$
$$:= |M_n^2 + M_{n+1}^2 + M_{n+2}^2 + M_{n+3}^2 + 2ij(M_n M_{n+3} - M_{n+2} M_{n+1})|.$$

Similarly, we can give three different conjugations and the squares of norms for the bicomplex Tribonacci-Lucas quaternion $\mathbb{BC}R_n = R_n + iR_{n+1} + jR_{n+2} + ijR_{n+3}$.

Now, we will state Binet's formula for the bicomplex Tetranacci and Tetranacci-Lucas quaternions and in the rest of the paper we fix the following notations.

$$\widehat{\alpha} = 1 + i\alpha + j\alpha^2 + ij\alpha^3,$$
$$\widehat{\beta} = 1 + i\beta + j\beta^2 + ij\beta^3,$$
$$\widehat{\gamma} = 1 + i\gamma + j\gamma^2 + ij\gamma^3,$$
$$\widehat{\delta} = 1 + i\delta + j\delta^2 + ij\delta^3.$$

Theorem 2. (Binet's Formulas) For any integer n, the nth bicomplex Tetranacci quaternion is

$$\mathbb{BC}M_n = \frac{\widehat{\alpha}\alpha^{n+2}}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)} + \frac{\widehat{\beta}\beta^{n+2}}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)}$$

$$+ \frac{\widehat{\gamma}\gamma^{n+2}}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)} + \frac{\widehat{\delta}\delta^{n+2}}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)} \tag{2.5}$$

$$= \frac{\alpha-1}{5\alpha-8}\widehat{\alpha}\alpha^{n-1} + \frac{\beta-1}{5\beta-8}\widehat{\beta}\beta^{n-1} + \frac{\gamma-1}{5\gamma-8}\widehat{\gamma}\gamma^{n-1} + \frac{\delta-1}{5\delta-8}\widehat{\delta}\delta^{n-1} \tag{2.6}$$

and the nth bicomplex Tetranacci-Lucas quaternion is

$$\mathbb{BC}R_n = \widehat{\alpha}\alpha^n + \widehat{\beta}\beta^n + \widehat{\gamma}\gamma^n + \widehat{\delta}\delta^n. \tag{2.7}$$

Proof. Using Binet's formula of the Tetranacci-Lucas numbers we have

$$\mathbb{BC}R_n = R_n + iR_{n+1} + jR_{n+2} + ijR_{n+3}$$
$$= (\alpha^n + \beta^n + \gamma^n + \delta^n) + i(\alpha^{n+1} + \beta^{n+1} + \gamma^{n+1} + \delta^{n+1})$$
$$+ j(\alpha^{n+2} + \beta^{n+2} + \gamma^{n+2} + \delta^{n+2}) + ij(\alpha^{n+3} + \beta^{n+3} + \gamma^{n+3} + \delta^{n+3})$$
$$= \widehat{\alpha}\alpha^n + \widehat{\beta}\beta^n + \widehat{\gamma}\gamma^n + \widehat{\delta}\delta^n.$$

Note that using Binet's formula (1.5) of the Tetranacci numbers we have

$$\mathbb{BC}M_n = M_n + iM_{n+1} + jM_{n+2} + ijM_{n+3}$$
$$= \left(\frac{\alpha-1}{5\alpha-8}\alpha^{n-1} + \frac{\beta-1}{5\beta-8}\beta^{n-1} + \frac{\gamma-1}{5\gamma-8}\gamma^{n-1} + \frac{\delta-1}{5\delta-8}\delta^{n-1}\right)$$
$$+ i\left(\frac{\alpha-1}{5\alpha-8}\alpha^n + \frac{\beta-1}{5\beta-8}\beta^n + \frac{\gamma-1}{5\gamma-8}\gamma^n + \frac{\delta-1}{5\delta-8}\delta^n\right)$$
$$+ j\left(\frac{\alpha-1}{5\alpha-8}\alpha^{n+1} + \frac{\beta-1}{5\beta-8}\beta^{n+1} + \frac{\gamma-1}{5\gamma-8}\gamma^{n+1} + \frac{\delta-1}{5\delta-8}\delta^{n+1}\right)$$
$$+ ij\left(\frac{\alpha-1}{5\alpha-8}\alpha^{n+2} + \frac{\beta-1}{5\beta-8}\beta^{n+2} + \frac{\gamma-1}{5\gamma-8}\gamma^{n+2} + \frac{\delta-1}{5\delta-8}\delta^{n+2}\right)$$
$$= \frac{\alpha-1}{5\alpha-8}\widehat{\alpha}\alpha^{n-1} + \frac{\beta-1}{5\beta-8}\widehat{\beta}\beta^{n-1} + \frac{\gamma-1}{5\gamma-8}\widehat{\gamma}\gamma^{n-1} + \frac{\delta-1}{5\delta-8}\widehat{\delta}\delta^{n-1}.$$

This proves (2.6). Similarly, we can obtain (2.5). $\qquad\square$

Next, we present generating functions.

Theorem 3. The generating functions for the bicomplex Tetranacci and Tetranacci-Lucas quaternions are

$$\sum_{n=0}^{\infty} \mathbb{BC}M_n x^n = \frac{(i+j+2ij)+(1+j+2ij)x+(j+2ij)x^2+(j+ij)x^3}{1-x-x^2-x^3-x^4} \tag{2.8}$$

and

$$\sum_{n=0}^{\infty} \mathbb{BC}R_n x^n = \frac{\left(\begin{array}{c}(4+i+3j+7ij)+(-3+2i+4j+8ij)x\\+(-2+3i+5j+4ij)x^2+(-1+4i+j+3ij)x^3\end{array}\right)}{1-x-x^2-x^3-x^4} \tag{2.9}$$

respectively.

Proof. Let

$$g(x) = \sum_{n=0}^{\infty} \mathbb{BC}M_n x^n$$

be generating function of the bicomplex Tetranacci quaternions. Then using the definition of the bicomplex Tetranacci quaternions, and substracting $xg(x)$, $x^2g(x)$, $x^3g(x)$ and $x^4g(x)$ from $g(x)$ and using the recurrence relation $\mathbb{BC}M_n = \mathbb{BC}M_{n-1} + \mathbb{BC}M_{n-2} + \mathbb{BC}M_{n-3} + \mathbb{BC}M_{n-4}$, we obtain

$$(1-x-x^2-x^3-x^4)g(x) = \mathbb{BC}M_0 + (\mathbb{BC}M_1 - \mathbb{BC}M_0)x + (\mathbb{BC}M_2 - \mathbb{BC}M_1 - \mathbb{BC}M_0)x^2$$
$$+ (\mathbb{BC}M_3 - \mathbb{BC}M_2 - \mathbb{BC}M_1 - \mathbb{BC}M_0)x^3.$$

Now using

$$\mathbb{BC}M_{-1} = j + ij$$
$$\mathbb{BC}M_0 = i + j + 2ij$$
$$\mathbb{BC}M_1 = 1 + i + 2j + 4ij$$
$$\mathbb{BC}M_2 = 1 + 2i + 4j + 8ij$$
$$\mathbb{BC}M_3 = 2 + 4i + 8j + 15ij$$

we obtain

$$g(x) = \frac{(i+j+2ij)+(1+j+2ij)x+(j+2ij)x^2+(j+ij)x^3}{1-x-x^2-x^3-x^4}.$$

Similarly, we can obtain (2.9). □

Next we present some summation formulas of Tetranacci numbers.

Lemma 4. For $n \geq 1$ we have the following formulas:

(a) $\sum_{p=1}^{n} M_p = \frac{1}{3}(M_{n+2} + 2M_n + M_{n-1} - 1)$,

(b) $\sum_{p=1}^{n} M_{2p+1} = \frac{1}{3}(2M_{2n+2} + M_{2n} - M_{2n-1} - 2)$,

(c) $\sum_{p=1}^{n} M_{2p} = \frac{1}{3}(2M_{2n+1} + M_{2n-1} - M_{2n-2} - 2)$.

The above Lemma is given in Soykan [20, Corollary 2.7]. It now follows that for every integer $n \geq 0$,

$$\sum_{p=0}^{n} M_p = M_0 + \sum_{p=1}^{n} M_p = \frac{1}{3}(M_{n+2} + 2M_n + M_{n-1} - 1), \tag{2.10}$$

$$\sum_{p=0}^{n} M_{2p+1} = M_1 + \sum_{p=1}^{n} M_{2p+1} = \frac{1}{3}(2M_{2n+2} + M_{2n} - M_{2n-1} + 1), \tag{2.11}$$

$$\sum_{p=0}^{n} M_{2p} = M_0 + \sum_{p=1}^{n} M_{2p} = \frac{1}{3}(2M_{2n+1} + M_{2n-1} - M_{2n-2} - 2). \tag{2.12}$$

In the following Lemma we present some summation formulas of Tetranacci-Lucas numbers.

Lemma 5. For $n \geq 1$ we have the following formulas:

(a) $\sum\limits_{p=1}^{n} R_p = \frac{1}{3}(R_{n+2} + 2R_n + R_{n-1} - 10)$,

(b) $\sum\limits_{p=1}^{n} R_{2p+1} = \frac{1}{3}(2R_{2n+2} + R_{2n} - R_{2n-1} - 11)$,

(c) $\sum\limits_{p=1}^{n} R_{2p} = \frac{1}{3}(2R_{2n+1} + R_{2n-1} - R_{2n-2} - 2)$.

The above Lemma is given in Soykan [20, Corollary 2.8]. It now follows that for every integer $n \geq 0$,

$$\sum_{p=0}^{n} R_p = R_0 + \sum_{p=1}^{n} R_p = \frac{1}{3}(R_{n+2} + 2R_n + R_{n-1} + 2), \tag{2.13}$$

$$\sum_{p=0}^{n} R_{2p+1} = R_1 + \sum_{p=1}^{n} R_{2p+1} = +\frac{1}{3}(2R_{2n+2} + R_{2n} - R_{2n-1} - 8), \tag{2.14}$$

$$\sum_{p=0}^{n} R_{2p} = R_0 + \sum_{p=1}^{n} R_{2p} = \frac{1}{3}(2R_{2n+1} + R_{2n-1} - R_{2n-2} + 10). \tag{2.15}$$

Next we present some summation formulas of bicomplex Tetranacci quaternions.

Theorem 6. For $n \geq 0$ we have the following formulas:

(a) $\sum\limits_{p=0}^{n} \mathbb{BC}M_p = \frac{1}{3}(\mathbb{BC}M_{n+2} + 2\mathbb{BC}M_n + \mathbb{BC}M_{n-1} - (1 + i + 4j + 7ij))$ \hfill (2.16)

(b) $\sum\limits_{p=0}^{n} \mathbb{BC}M_{2p+1} = \frac{1}{3}(2\mathbb{BC}M_{2n+2} + \mathbb{BC}M_{2n} - \mathbb{BC}M_{2n-1} + (1 - 2i - 2j - 5ij))$

(c) $\sum\limits_{p=0}^{n} \mathbb{BC}M_{2p} = \frac{1}{3}(2\mathbb{BC}M_{2n+1} + \mathbb{BC}M_{2n-1} - \mathbb{BC}M_{2n-2} - (2 - i + 2j + 2ij))$.

Proof. (a) Using (2.1) and (2.10), we obtain

$$\sum_{p=0}^{n} \mathbb{BC}M_p = \sum_{p=0}^{n} M_p + i \sum_{p=0}^{n} M_{p+1} + j \sum_{p=0}^{n} M_{p+2} + ij \sum_{p=0}^{n} M_{p+3}$$

$$= (M_0 + \ldots + M_n) + i(M_1 + \ldots + M_{n+1}) + j(M_2 + \ldots + M_{n+2}) + ij(M_3 + \ldots + M_{n+2})$$

and so

$$3\sum_{p=0}^{n}\mathbb{BC}M_p = (M_{n+2} + 2M_n + M_{n-1} - 1) + i(M_{n+3} + 2M_{n+1} + M_n - 1 - 3M_0)$$

$$+ j(M_{n+4} + 2M_{n+2} + M_{n+1} - 1 - 3(M_0 + M_1))$$

$$+ ij(M_{n+5} + 2M_{n+3} + M_{n+2} - 1 - 3(M_0 + M_1 + M_2))$$

$$= \mathbb{BC}M_{n+2} + 2\mathbb{BC}M_n + \mathbb{BC}M_{n-1} + c,$$

where

$$c = -1 + i(-1 - 3M_0) + j(-1 - 3(M_0 + M_1)) + ij(-1 - 3(M_0 + M_1 + M_2))$$

$$= -1 - i - 4j - 7ij.$$

Hence

$$\sum_{p=0}^{n}\mathbb{BC}M_p = \frac{1}{3}(\mathbb{BC}M_{n+2} + 2\mathbb{BC}M_n + \mathbb{BC}M_{n-1} - (1 + i + 4j + 7ij)).$$

This proves (2.16).

(b) and (c) follows from the identities (2.11) and (2.12). □

In the following Theorem, we give some summation formulas of bicomplex Tetranacci-Lucas quaternions.

Theorem 7. For $n \geq 0$ we have the following formulas:

(a) $\displaystyle\sum_{p=0}^{n}\mathbb{BC}R_p = \frac{1}{3}(\mathbb{BC}R_{n+2} + 2\mathbb{BC}R_n + \mathbb{BC}R_{n-1} + (2 - 10i - 13j - 22ij)).$ (2.17)

(b) $\displaystyle\sum_{p=0}^{n}\mathbb{BC}R_{2p+1} = \frac{1}{3}(2\mathbb{BC}R_{2n+2} + \mathbb{BC}R_{2n} - \mathbb{BC}R_{2n-1} - (8 + 2i + 11j + 11ij))$

(c) $\displaystyle\sum_{p=0}^{n}\mathbb{BC}R_{2p} = \frac{1}{3}(2\mathbb{BC}R_{2n+1} + \mathbb{BC}R_{2n-1} - \mathbb{BC}R_{2n-2} + (10 - 8i - 2j - 11ij)).$

Proof. (a) Using (2.3) and (2.13), we obtain (2.17).

(b) and (c) follows from the identities (2.14) and (2.15).

3. Matrices and Determinants Related with Tetranacci and Tetranacci-Lucas Quaternions

We define the square matrix B of order 4 as:

$$B = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

such that $\det B = -1$.

Induction proof may be used to establish

$$
B^n = \begin{pmatrix} M_{n+1} & M_n + M_{n-1} + M_{n-2} & M_n + M_{n-1} & M_n \\ M_n & M_{n-1} + M_{n-2} + M_{n-3} & M_{n-1} + M_{n-2} & M_{n-1} \\ M_{n-1} & M_{n-2} + M_{n-3} + M_{n-4} & M_{n-2} + M_{n-3} & M_{n-2} \\ M_{n-2} & M_{n-3} + M_{n-4} + M_{n-5} & M_{n-3} + M_{n-4} & M_{n-3} \end{pmatrix}. \tag{3.1}
$$

Matrix formulation of M_n and R_n can be given as

$$
\begin{pmatrix} M_{n+3} \\ M_{n+2} \\ M_{n+1} \\ M_n \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}^n \begin{pmatrix} M_3 \\ M_2 \\ M_1 \\ M_0 \end{pmatrix} \tag{3.2}
$$

and

$$
\begin{pmatrix} R_{n+3} \\ R_{n+2} \\ R_{n+1} \\ R_n \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}^n \begin{pmatrix} R_3 \\ R_2 \\ R_1 \\ R_0 \end{pmatrix}. \tag{3.3}
$$

Induction proofs may be used to establish the matrix formulations M_n and R_n.

Now, we define the matrices B_M and B_R as

$$
B_M = \begin{pmatrix} \mathbb{BC}M_5 & \mathbb{BC}M_4 + \mathbb{BC}M_3 + \mathbb{BC}M_2 & \mathbb{BC}M_4 + \mathbb{BC}M_3 & \mathbb{BC}M_4 \\ \mathbb{BC}M_4 & \mathbb{BC}M_3 + \mathbb{BC}M_2 + \mathbb{BC}M_1 & \mathbb{BC}M_3 + \mathbb{BC}M_2 & \mathbb{BC}M_3 \\ \mathbb{BC}M_3 & \mathbb{BC}M_2 + \mathbb{BC}M_1 + \mathbb{BC}M_0 & \mathbb{BC}M_2 + \mathbb{BC}M_1 & \mathbb{BC}M_2 \\ \mathbb{BC}M_2 & \mathbb{BC}M_1 + \mathbb{BC}M_0 + \mathbb{BC}M_{-1} & \mathbb{BC}M_1 + \mathbb{BC}M_0 & \mathbb{BC}M_1 \end{pmatrix}
$$

and

$$
B_R = \begin{pmatrix} \mathbb{BC}R_5 & \mathbb{BC}R_4 + \mathbb{BC}R_3 + \mathbb{BC}R_2 & \mathbb{BC}R_4 + \mathbb{BC}R_3 & \mathbb{BC}R_4 \\ \mathbb{BC}R_4 & \mathbb{BC}R_3 + \mathbb{BC}R_2 + \mathbb{BC}R_1 & \mathbb{BC}R_3 + \mathbb{BC}R_2 & \mathbb{BC}R_3 \\ \mathbb{BC}R_3 & \mathbb{BC}R_2 + \mathbb{BC}R_1 + \mathbb{BC}R_0 & \mathbb{BC}R_2 + \mathbb{BC}R_1 & \mathbb{BC}R_2 \\ \mathbb{BC}R_2 & \mathbb{BC}R_1 + \mathbb{BC}R_0 + \mathbb{BC}R_{-1} & \mathbb{BC}R_1 + \mathbb{BC}R_0 & \mathbb{BC}R_1 \end{pmatrix}.
$$

These matrices B_M and B_R can be called bicomplex Tetranacci quaternion matrix and bicomplex Tetranacci-Lucas quaternion matrix, respectively.

Theorem 8. For $n \geq 0$, the followings are valid:

(a) $B_M \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}^n = \begin{pmatrix} \mathbb{BC}M_{n+5} & \mathbb{BC}M_{n+4} + \mathbb{BC}M_{n+3} + \mathbb{BC}M_{n+2} & \mathbb{BC}M_{n+4} + \mathbb{BC}M_{n+3} & \mathbb{BC}M_{n+4} \\ \mathbb{BC}M_{n+4} & \mathbb{BC}M_{n+3} + \mathbb{BC}M_{n+2} + \mathbb{BC}M_{n+1} & \mathbb{BC}M_{n+3} + \mathbb{BC}M_{n+2} & \mathbb{BC}M_{n+3} \\ \mathbb{BC}M_{n+3} & \mathbb{BC}M_{n+2} + \mathbb{BC}M_{n+1} + \mathbb{BC}M_n & \mathbb{BC}M_{n+2} + \mathbb{BC}M_{n+1} & \mathbb{BC}M_{n+2} \\ \mathbb{BC}M_{n+2} & \mathbb{BC}M_{n+1} + \mathbb{BC}M_n + \mathbb{BC}M_{n-1} & \mathbb{BC}M_{n+1} + \mathbb{BC}M_n & \mathbb{BC}M_{n+1} \end{pmatrix}$ (3.4)

(b) $B_R \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}^n = \begin{pmatrix} \mathbb{BC}R_{n+5} & \mathbb{BC}R_{n+4} + \mathbb{BC}R_{n+3} + \mathbb{BC}R_{n+2} & \mathbb{BC}R_{n+4} + \mathbb{BC}R_{n+3} & \mathbb{BC}R_{n+4} \\ \mathbb{BC}R_{n+4} & \mathbb{BC}R_{n+3} + \mathbb{BC}R_{n+2} + \mathbb{BC}R_{n+1} & \mathbb{BC}R_{n+3} + \mathbb{BC}R_{n+2} & \mathbb{BC}R_{n+3} \\ \mathbb{BC}R_{n+3} & \mathbb{BC}R_{n+2} + \mathbb{BC}R_{n+1} + \mathbb{BC}R_n & \mathbb{BC}R_{n+2} + \mathbb{BC}R_{n+1} & \mathbb{BC}R_{n+2} \\ \mathbb{BC}R_{n+2} & \mathbb{BC}R_{n+1} + \mathbb{BC}R_n + \mathbb{BC}R_{n-1} & \mathbb{BC}R_{n+1} + \mathbb{BC}R_n & \mathbb{BC}R_{n+1} \end{pmatrix}$ (3.5)

Proof. We prove (a) by mathematical induction on n. If $n = 0$ then the result is clear. Now, we assume it is true for $n = k$, that is

$$
B_M B^k = \begin{pmatrix} \mathbb{BC}M_{k+5} & \mathbb{BC}M_{k+4} + \mathbb{BC}M_{k+3} + \mathbb{BC}M_{k+2} & \mathbb{BC}M_{k+4} + \mathbb{BC}M_{k+3} & \mathbb{BC}M_{k+4} \\ \mathbb{BC}M_{k+4} & \mathbb{BC}M_{k+3} + \mathbb{BC}M_{k+2} + \mathbb{BC}M_{k+1} & \mathbb{BC}M_{k+3} + \mathbb{BC}M_{k+2} & \mathbb{BC}M_{k+3} \\ \mathbb{BC}M_{k+3} & \mathbb{BC}M_{k+2} + \mathbb{BC}M_{k+1} + \mathbb{BC}M_k & \mathbb{BC}M_{k+2} + \mathbb{BC}M_{k+1} & \mathbb{BC}M_{k+2} \\ \mathbb{BC}M_{k+2} & \mathbb{BC}M_{k+1} + \mathbb{BC}M_k + \mathbb{BC}M_{k-1} & \mathbb{BC}M_{k+1} + \mathbb{BC}M_k & \mathbb{BC}M_{k+1} \end{pmatrix}.
$$

If we use (2.3), then we have $\mathbb{BC}M_{k+4} = \mathbb{BC}M_{k+3} + \mathbb{BC}M_{k+2} + \mathbb{BC}M_{k+1} + \mathbb{BC}M_k$. Then by induction hypothesis, we obtain

$$B_M B^{k+1} = (B_M B^k)B$$

$$= \begin{pmatrix} \mathbb{BC}M_{k+5} & \mathbb{BC}M_{k+4} + \mathbb{BC}M_{k+3} + \mathbb{BC}M_{k+2} & \mathbb{BC}M_{k+4} + \mathbb{BC}M_{k+3} & \mathbb{BC}M_{k+4} \\ \mathbb{BC}M_{k+4} & \mathbb{BC}M_{k+3} + \mathbb{BC}M_{k+2} + \mathbb{BC}M_{k+1} & \mathbb{BC}M_{k+3} + \mathbb{BC}M_{k+2} & \mathbb{BC}M_{k+3} \\ \mathbb{BC}M_{k+3} & \mathbb{BC}M_{k+2} + \mathbb{BC}M_{k+1} + \mathbb{BC}M_k & \mathbb{BC}M_{k+2} + \mathbb{BC}M_{k+1} & \mathbb{BC}M_{k+2} \\ \mathbb{BC}M_{k+2} & \mathbb{BC}M_{k+1} + \mathbb{BC}M_k + \mathbb{BC}M_{k-1} & \mathbb{BC}M_{k+1} + \mathbb{BC}M_k & \mathbb{BC}M_{k+1} \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \mathbb{BC}M_{k+6} & \mathbb{BC}M_{k+5} + \mathbb{BC}M_{k+4} + \mathbb{BC}M_{k+3} & \mathbb{BC}M_{k+5} + \mathbb{BC}M_{k+4} & \mathbb{BC}M_{k+5} \\ \mathbb{BC}M_{k+5} & \mathbb{BC}M_{k+4} + \mathbb{BC}M_{k+3} + \mathbb{BC}M_{k+2} & \mathbb{BC}M_{k+4} + \mathbb{BC}M_{k+3} & \mathbb{BC}M_{k+4} \\ \mathbb{BC}M_{k+4} & \mathbb{BC}M_{k+3} + \mathbb{BC}M_{k+2} + \mathbb{BC}M_{k+1} & \mathbb{BC}M_{k+3} + \mathbb{BC}M_{k+2} & \mathbb{BC}M_{k+3} \\ \mathbb{BC}M_{k+3} & \mathbb{BC}M_{k+2} + \mathbb{BC}M_{k+1} + \mathbb{BC}M_k & \mathbb{BC}M_{k+2} + \mathbb{BC}M_{k+1} & \mathbb{BC}M_{k+2} \end{pmatrix}$$

Thus, (3.4) holds for all non-negative integers n.

Eq. (3.5) can be similarly proved. $\qquad\square$

Corollary 9. For $n \geq 0$, the followings hold:

(a) $\mathbb{BC}M_{n+3} = \mathbb{BC}M_3 M_{n+1} + (\mathbb{BC}M_2 + \mathbb{BC}M_1 + \mathbb{BC}M_0)M_n + (\mathbb{BC}M_1 + \mathbb{BC}M_2)M_{n-1} + \mathbb{BC}M_2 M_{n-2}$

(b) $\mathbb{BC}R_{n+3} = \mathbb{BC}R_3 M_{n+1} + (\mathbb{BC}R_2 + \mathbb{BC}R_1 + \mathbb{BC}R_0)M_n + (\mathbb{BC}R_1 + \mathbb{BC}R_2)M_{n-1} + \mathbb{BC}R_2 M_{n-2}$.

Proof. The proof of (a) can be seen by the coefficient of the matrix B_M and (3.1). The proof of (b) can be seen by the coefficient of the matrix B_R and (3.1). $\qquad\square$

4. Five-Diagonal Matrix with Fourth Order Sequences and Applications

In this section we give another way to obtain nth term of the bicomplex Tetranacci and Tetranacci-Lucas quaternions. For this we need the following theorem.

Theorem 10. Let $\{x_n\}$ be any fourth-order linear sequence defined recursively as follows:

$$x_n = rx_{n-1} + sx_{n-2} + tx_{n-3} + ux_{n-4}, \qquad n \geq 4$$

with the initial conditions $x_0 = a$, $x_1 = b$, $x_2 = c$, $x_3 = d$. Then for all $n \geq 0$, we have

$$x_n = \begin{vmatrix} a & -1 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ b & 0 & -1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ c & 0 & 0 & -1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ d & 0 & 0 & 0 & -1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & u & t & s & r & -1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & u & t & s & r & -1 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ddots & s & r & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ddots & t & s & r & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ddots & u & t & s & r \end{vmatrix}_{(n+1)\times(n+1)}.$$

Proof. We proceed by induction on n. Since

$$x_0 = |a|_{1 \times 1} = a, \; x_1 = \begin{vmatrix} a & -1 \\ b & 0 \end{vmatrix}_{2 \times 2} = b, \; x_2 = \begin{vmatrix} a & -1 & 0 \\ b & 0 & -1 \\ c & 0 & 0 \end{vmatrix}_{3 \times 3} = c, \; x_3 = \begin{vmatrix} a & -1 & 0 & 0 \\ b & 0 & -1 & 0 \\ c & 0 & 0 & -1 \\ d & 0 & 0 & 0 \end{vmatrix}_{4 \times 4} = d,$$

the equality holds for $n = 0, 1, 2, 3$. Now we assume that the equality is true for $4 \leq k \leq n$. Then we will complete the inductive step $n + 1$ as follows: Note that

$$x_{n+1} = \begin{vmatrix} a & -1 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ b & 0 & -1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ c & 0 & 0 & -1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ d & 0 & 0 & 0 & -1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & u & t & s & r & -1 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ddots & r & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ddots & s & r & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ddots & t & s & r \end{vmatrix}_{(n+2) \times (n+2)}$$

and

$$x_{n+1} = r(-1)^{(n+2)+(n+2)} \begin{vmatrix} a & -1 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ b & 0 & -1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ c & 0 & 0 & -1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ d & 0 & 0 & 0 & -1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & u & t & s & r & -1 & 0 & \cdots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ddots & r & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ddots & s & r \end{vmatrix}_{(n+1) \times (n+1)}$$

$$+ (-1)(-1)^{(n+2)+(n+1)} \begin{vmatrix} a & -1 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ b & 0 & -1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ c & 0 & 0 & -1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ d & 0 & 0 & 0 & -1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & u & t & s & r & -1 & 0 & \cdots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ddots & r & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ddots & t & s \end{vmatrix}_{(n+1) \times (n+1)}$$

and so

$$x_{n+1} = r x_n + s \begin{vmatrix} a & -1 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ b & 0 & -1 & 0 & 0 & 0 & 0 & \cdots & 0 \\ c & 0 & 0 & -1 & 0 & 0 & 0 & \cdots & 0 \\ d & 0 & 0 & 0 & -1 & 0 & 0 & \cdots & 0 \\ 0 & u & t & s & r & -1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ddots & r \end{vmatrix}_{n \times n}$$

$$+\begin{vmatrix} a & -1 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ b & 0 & -1 & 0 & 0 & 0 & 0 & \cdots & 0 \\ c & 0 & 0 & -1 & 0 & 0 & 0 & \cdots & 0 \\ d & 0 & 0 & 0 & -1 & 0 & 0 & \cdots & 0 \\ 0 & u & t & s & r & -1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ddots & t \end{vmatrix}_{n\times n}.$$

Now it follows that

$$x_n = rx_n + sx_{n-1} + t\begin{vmatrix} a & -1 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ b & 0 & -1 & 0 & 0 & 0 & 0 & \cdots & 0 \\ c & 0 & 0 & -1 & 0 & 0 & 0 & \cdots & 0 \\ d & 0 & 0 & 0 & -1 & 0 & 0 & \cdots & 0 \\ 0 & u & t & s & r & -1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ddots & t \end{vmatrix}_{(n-1)\times(n-1)}$$

$$+\begin{vmatrix} a & -1 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ b & 0 & -1 & 0 & 0 & 0 & 0 & \cdots & 0 \\ c & 0 & 0 & -1 & 0 & 0 & 0 & \cdots & 0 \\ d & 0 & 0 & 0 & -1 & 0 & 0 & \cdots & 0 \\ 0 & u & t & s & r & -1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ddots & u \end{vmatrix}_{(n-1)\times(n-1)}$$

$$= rx_n + sx_{n-1} + tx_{n-2} + \begin{vmatrix} a & -1 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ b & 0 & -1 & 0 & 0 & 0 & 0 & \cdots & 0 \\ c & 0 & 0 & -1 & 0 & 0 & 0 & \cdots & 0 \\ d & 0 & 0 & 0 & -1 & 0 & 0 & \cdots & 0 \\ 0 & u & t & s & r & -1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ddots & u \end{vmatrix}_{(n-1)\times(n-1)}$$

$$= rx_n + sx_{n-1} + tx_{n-2} + u(-1)^{(n-1)+(n-1)}\begin{vmatrix} a & -1 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ b & 0 & -1 & 0 & 0 & 0 & 0 & \cdots & 0 \\ c & 0 & 0 & -1 & 0 & 0 & 0 & \cdots & 0 \\ d & 0 & 0 & 0 & -1 & 0 & 0 & \cdots & 0 \\ 0 & u & t & s & r & -1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ddots & -1 \end{vmatrix}_{(n-2)\times(n-2)}$$

$$= rx_n + sx_{n-1} + tx_{n-2} + ux_{n-3}.$$

This completes the inductive step and the proof of the theorem. □

Note that in our cases $r = s = t = u = 1$. As a corallary of the above theorem, in the following we present another way to obtain nth term of the bicomplex Tetranacci and Tetranacci-Lucas quaternions.

Corollary 11. For all $n \geq 0$, we have

(a) $\mathbb{BC}M_n =$

$$\begin{vmatrix} \mathbb{BC}M_0 & -1 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \mathbb{BC}M_1 & 0 & -1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \mathbb{BC}M_2 & 0 & 0 & -1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \mathbb{BC}M_3 & 0 & 0 & 0 & -1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & -1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & -1 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ddots & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ddots & 1 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ddots & 1 & 1 & 1 & 1 \end{vmatrix}_{(n+1)\times(n+1)}.$$

(b) $\mathbb{BC}R_n =$

$$\begin{vmatrix} \mathbb{BC}R_0 & -1 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \mathbb{BC}R_1 & 0 & -1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \mathbb{BC}R_2 & 0 & 0 & -1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \mathbb{BC}R_3 & 0 & 0 & 0 & -1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & -1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & -1 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ddots & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ddots & 1 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ddots & 1 & 1 & 1 & 1 \end{vmatrix}_{(n+1)\times(n+1)}.$$

Proof. (a) follows from (2.3) and Theorem 10. (b) follows from (2.4) and Theorem 10. ◻

5. Conclusion

Recently, there have been so many studies of the sequences of numbers in the literature and the sequences of numbers were widely used in many research areas, such as physics, engineering, architecture, nature and art. $\mathbb{BC}$, the set of bicomplex numbers, is a complex Clifford Algebra which is the simplest example, and the only commutative one, see [13, p. 65]. If we use together sequences of integers defined recursively and bicomplex numbers, we obtain a new sequences such as bicomplex Fibonacci quaternions, bicomplex Pell quaternions, bicomplex Padovan quaternions.

Various families of bicomplex number (quaternion) sequences have been defined and studied by a number of authors. Some authors studied on second order bicomplex quaternion sequences such as [1], [2], [4], [8], [9] and some authors studied on third order bicomplex quaternion sequences such as [5], [12]. In this work, we introduce (fourth order) bicomplex Tetranacci and bicomplex Tetranacci-Lucas quaternions and prove some important properties of them. The methods used in this paper can be used for the other linear recurrence sequences, too. It is our intention to continue the study and explore properties of some type of bicomplex number sequences, such as bicomplex generalized Tetranacci quaternions and other fourth and fifth order bicomplex quaternions.

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Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

References

[1] F. T. Aydın, Bicomplex k-Fibonacci quaternions, arXiv 1810.05003 [math.NT] (2018), `https://arxiv.org/abs/1810.05003`.

[2] F. T. Aydın, On bicomplex Pell and Pell-Lucas numbers, *Communications in Advanced Mathematical Sciences* **1**(2) (2018), 142 − 155, DOI: 10.33434/cams.439752.

[3] J. Baez, The octonions, *Bull. Amer. Math. Soc.* **39**(2) (2002), 145 − 205, DOI: 10.1090/S0273-0979-01-00934-X.

[4] P. Catarino, Bicomplex k-Pell quaternions, *Computational Methods and Function Theory* **19**(1) (2018), 65 − 76, DOI: 10.1007/s40315-018-0251-5.

[5] G. Cerda, Bicomplex third-order Jacobsthal quaternions, arXiv 1809.06979 [math.AC] (2018), `https://arxiv.org/abs/1809.06979`.

[6] J. H. Conway and D. A. Smith, *On Quaternions and Octonions: Their Geometry, Arithmetic and Symmetry*, A. K. Peters/CRC Press (2003), DOI: 10.1017/S002555720017545X.

[7] G. P. Dresden and Z. Du, A simplified Binet formula for k-generalized Fibonacci numbers, *Journal of Integer Sequences* **17** (2014), article 14.4.7, pages 9, `https://cs.uwaterloo.ca/journals/JIS/VOL17/Dresden/dresden6.pdf`.

[8] S. Halıcı and A. Karataş, Bicomplex Lucas and Horadam numbers, arXiv 1806.05038v2 [math.RA] (2018), `https://arxiv.org/abs/1806.05038`.

[9] S. Halici, On bicomplex Fibonacci numbers and their generalization, in *Models and Theories in Social Systems. Studies in Systems, Decision and Control*, C. Flaut, Š. Hošková-Mayerová and D. Flaut (eds.), Vol. **179**, 509 − 524, Springer (2019), DOI: 10.1007/978-3-030-00084-4_26.

[10] G. S. Hathiwala and D. V. Shah, Binet-type formula for the sequence of Tetranacci numbers by alternate methods, *Mathematical Journal of Interdisciplinary Sciences* **6**(1) (2017), 37 − 48, DOI: 10.15415/mjis.2017.61004.

[11] F. T. Howard and F. Saidak, Zhou's theory of constructing identities, *Congress Numer.* **200** (2010), 225 − 237.

[12] C. Kızılateş, P. Catarino and N. Tuğlu, On the bicomplex generalized Tribonacci quaternions, *Mathematics* **7**(1) (2019), 80, DOI: 10.3390/math7010080.

[13] M. E. Luna-Elizarrarás, M. Shapiro, D. C. Struppa and A. Vajiac, Bicomplex numbers and their elementary functions, *CUBO A Mathematical Journal* **14**(2) (2012), 61 – 80, DOI: 10.4067/S0719-06462012000200004.

[14] R. S. Melham, Some analogs of the identity $F_n^2 + F_{n+1}^2 = F_{2n+1}^2$, *Fibonacci Quarterly* (1999), 305 – 311.

[15] L. R. Natividad, On solving Fibonacci-like sequences of fourth, fifth and sixth order, *International Journal of Mathematics and Computing* **3**(2) (2013), 38 – 40.

[16] S. K. Nurkan and İ. A. Güven, A note on bicomplex Fibonacci and Lucas numbers, *International Journal of Pure and Applied Mathematics* **120**(3) (2018), 365 – 377, DOI: 10.12732/ijpam.v120i3.7.

[17] S. K. Nurkan and İ. A. Güven, A note on bicomplex Fibonacci and Lucas numbers, arXiv 1508.03972 [math.NT] (2015), https://arxiv.org/abs/1508.03972.

[18] D. Rochon and M. Shapiro, On algebraic properties of Bicomplex and Hyperbolic numbers, *Anal. Univ. Oradea Fascicola. Matematica.* **11** (2004), 1 – 28.

[19] B. Singh, P. Bhadouria, O. Sikhwal and K. Sisodiya, A formula for Tetranacci-like sequence, *General Mathematics Notes* **20**(2) (2014), 136 – 141.

[20] Y. Soykan, Gaussian generalized tetranacci numbers, arXiv 1902.03936 [math.NT] (2019), https://arxiv.org/abs/1902.03936.

[21] M. E. Waddill, Another generalized Fibonacci sequence, *Fibonacci Quarterly* **5**(3) (1967), 209 – 227.

[22] M. E. Waddill, The tetranacci sequence and generalizations, *Fibonacci Quarterly* (1992), 9 – 20.

[23] J. P. Ward, *Quaternions and Cayley Numbers: Algebra and Applications*, Kluwer Academic Publishers, London (1997), DOI: 0.1007/978-94-011-5768-1.

[24] M. N. Zaveri and J. K. Patel, Binet's formula for the tetranacci sequence, *International Journal of Science and Research (IJSR)*, **5**(12) (2016), 1911 – 1914.