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Research Article

Generalization of Favard's and Berwald's Inequalities for Strongly Convex Functions

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Abstract. In this paper, we give generalization of discrete weighted Favard's and Berwald's inequalities for strongly convex functions. The obtained results are the improvement and generalization of the earlier results.

Keywords. Majorization theorem; Strongly convex function; Favard's inequality; Berwald's inequality

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1. Introduction

The theory of mathematical inequalities is fairly linked with the concept of convex functions and many crucial inequalities have been got and structured for this family of functions. The unearthing of this idea has opened up a very productive, advanced and comprehensive discipline of mathematics, namely, convex analysis. Convexity is the most fundamental, important, natural notion in mathematics and presented over 100 years ago. In the last few years, several extensions and generalizations have been made for convexity. These generalizations and extensions in the theory of inequalities have made precious contributions in different areas of mathematics. In this point of view, the new generalized concepts are quasi-convex [17], φ-convex [19], λ-convex [20], approximately convex [23], midconvex functions [24], pseudo-convex [29], strongly convex [31], logarithmically convex [35], h-convex [39], delta-convex

[36], Schur convexity [14–16] and others [1–3, 10–13, 26, 34, 38, 40–44, 46].

The main aim of this article is on the inequalities related to strongly convex function.

Definition 1 (see [31]). Let Ψ be a real-valued function defined on the interval $[\lambda_1,\xi_1]$ and c be positive real number. Then, the function Ψ is said to be strongly convex with modulus c if the inequality

$$\Psi(\eta u_1+(1-\eta)v_1)\le \eta\Psi(u_1)+(1-\eta)\Psi(v_1)-c\eta(1-\eta)(u_1-v_1)^2 \tag{1}$$

holds for all $u_1,v_1\in[\lambda_1,\xi_1]$ and $\eta\in[0,1]$. From (1) we clearly see that

$$\Psi(u_1)-\Psi(v_1)\ge \Psi'_+(v_1)(u_1-v_1)+c(u_1-v_1)^2. \tag{2}$$

Now, we are going to present some basic theory of majorization.

For fixed $n\ge 2$, let $\boldsymbol{\beta}=(\beta_1,\beta_2,\dots,\beta_n)$ and $\boldsymbol{\zeta}=(\zeta_1,\zeta_2,\dots,\zeta_n)$ be two n-tuples of real numbers such that

$$\beta_1^{\downarrow}\ge\beta_2^{\downarrow}\ge\dots\ge\beta_n^{\downarrow},\qquad \zeta_1^{\downarrow}\ge\zeta_2^{\downarrow}\ge\dots\ge\zeta_n^{\downarrow},$$
$$\beta_{(1)}\ge\beta_{(2)}\ge\dots\ge\beta_{(n)},\quad \zeta_{(1)}\ge\zeta_{(2)}\ge\dots\ge\zeta_{(n)},$$

be their ordered components.

Definition 2 ([35, p.319]). The n-tuple $\boldsymbol{\beta}$ is said to majorizes $\boldsymbol{\zeta}$ (or $\boldsymbol{\zeta}$ is to be majorized by $\boldsymbol{\beta}$), i.e., $\boldsymbol{\beta}\succ\boldsymbol{\zeta}$, if

$$\sum_{i=1}^{m}\beta_i^{\downarrow}\ge\sum_{i=1}^{m}\zeta_i^{\downarrow}$$

holds for $m=1,2,\dots,n-1$, and

$$\sum_{i=1}^{n}\beta_i=\sum_{i=1}^{n}\zeta_i.$$

In literature the following theorem is well-known as majorization theorem and a suitable reference for its proof is Olkin and Marshall (see [32, p. 11], see also [35, p. 320]). The result is due to Littlewood, Hardy and Polya (see [22, p. 75]) and can also be found in Karamata [27]. For a detailed regarding the matter of priority can be found in [30, p. 169].

Theorem 1. *Let $[\lambda_1,\xi_1]$ be an interval in $\mathbf{R}$ and suppose $\boldsymbol{\beta}=(\beta_1,\beta_2,\dots,\beta_n)$, $\boldsymbol{\zeta}=(\zeta_1,\zeta_2,\dots,\zeta_n)$ are n-tuples such that $\beta_i,\zeta_i\in[\lambda_1,\xi_1]$, for $i=1,2,\dots,n$, then*

$$\sum_{i=1}^{n}\Psi(\beta_i)\ge\sum_{i=1}^{n}\Psi(\zeta_i) \tag{3}$$

holds for every continuous convex function $\Psi:[\lambda_1,\xi_1]\to\mathbf{R}$ if and only if $\boldsymbol{\beta}\succ\boldsymbol{\zeta}$ holds.

A weighted version of Theorem 1 was proved by Fuchs in [21] (see also [35]):

Theorem 2. *Let $\boldsymbol{\beta}=(\beta_1,\beta_2,\dots,\beta_n)$, $\boldsymbol{\zeta}=(\zeta_1,\zeta_2,\dots,\zeta_n)$ be two decreasing n-tuples and $\mathbf{p}=(p_1,p_2,\dots,p_n)$ be a real n-tuple such that*

$$\sum_{i=1}^{m}p_i\beta_i\ge\sum_{i=1}^{m}p_i\zeta_i,\quad \text{for } m=1,2,\dots,n-1$$

and

$$\sum_{i=1}^{n} p_i\beta_i = \sum_{i=1}^{n} p_i\zeta_i.$$

Then for every continuous convex function $\Psi : [\lambda_1, \xi_1] \to \mathbf{R}$, the following inequality holds

$$\sum_{i=1}^{n} p_i\Psi(\beta_i) \geq \sum_{i=1}^{n} p_i\Psi(\zeta_i). \tag{4}$$

The following majorization theorem for positive weight and monotonic condition on single tuple can be found in the literature [45].

Theorem 3. *Let $\Psi : [\lambda_1, \xi_1] \to \mathbf{R}$ be strongly convex function with modulus c. Suppose $\boldsymbol{\beta} = (\beta_1, \beta_2, \ldots, \beta_n)$, $\boldsymbol{\zeta} = (\zeta_1, \zeta_2, \ldots, \zeta_n)$ and $\mathbf{p} = (p_1, p_2, \ldots, p_n)$ are positive n-tuples such that $\beta_i, \zeta_i \in [\lambda_1, \xi_1]$, $p_i \geq 0$ for $i = 1, 2, \ldots, n$ and satisfying*

$$\sum_{i=1}^{m} p_i\beta_i \geq \sum_{i=1}^{m} p_i\zeta_i, \quad \text{for } m = 1, 2, \ldots, n-1$$

and

$$\sum_{i=1}^{n} p_i\beta_i = \sum_{i=1}^{n} p_i\zeta_i.$$

(a): *If the n-tuple $\boldsymbol{\zeta}$ is decreasing, then the following inequality holds*

$$\sum_{i=1}^{n} p_i\Psi(\beta_i) \geq \sum_{i=1}^{n} p_i\Psi(\zeta_i) + c\sum_{i=1}^{n} p_i(\beta_i - \zeta_i)^2. \tag{5}$$

(b): *If the n-tuple $\boldsymbol{\beta}$ is increasing, then the following inequality holds*

$$\sum_{i=1}^{n} p_i\Psi(\zeta_i) \geq \sum_{i=1}^{n} p_i\Psi(\beta_i) + c\sum_{i=1}^{n} p_i(\zeta_i - \beta_i)^2. \tag{6}$$

The following theorem is in fact the generalization of majorization theorem for positive weight and monotonic condition on single tuple [45].

Theorem 4. *Let φ be a strictly increasing function from (λ_1, ξ_1) onto (λ_2, ξ_2) and $\Psi \circ \varphi^{-1}$ be a strongly convex functions on $[\lambda_2, \xi_2]$ with respect to c. Suppose $\boldsymbol{\beta} = (\beta_1, \beta_2, \ldots, \beta_n)$, $\boldsymbol{\zeta} = (\zeta_1, \zeta_2, \ldots, \zeta_n)$ and $\mathbf{p} = (p_1, p_2, \ldots, p_n)$ are positive n-tuples such that $\beta_i, \zeta_i \in (\lambda_1, \xi_1)$ for $i = 1, 2, \ldots, n$ and satisfying*

$$\sum_{i=1}^{m} p_i\varphi(\beta_i) \geq \sum_{i=1}^{m} p_i\varphi(\zeta_i), \quad \text{for } m = 1, 2, \ldots, n-1$$

and

$$\sum_{i=1}^{n} p_i\varphi(\beta_i) = \sum_{i=1}^{n} p_i\varphi(\zeta_i).$$

Then the following statements are true:

(a): *If the n-tuple $\boldsymbol{\zeta}$ is decreasing, then the following inequality holds*

$$\sum_{i=1}^{n} p_i\Psi(\beta_i) \geq \sum_{i=1}^{n} p_i\Psi(\zeta_i) + c\sum_{i=1}^{n} p_i(\varphi(\beta_i) - \varphi(\zeta_i))^2. \tag{7}$$

(b): *If the n-tuple $\boldsymbol{\beta}$ is increasing, then the following inequality holds*

$$\sum_{i=1}^{n} p_i \Psi(\zeta_i) \geq \sum_{i=1}^{n} p_i \Psi\left(\beta_i\right) + c\sum_{i=1}^{n} p_i \left(\varphi(\zeta_i) - \varphi(\beta_i)\right)^2. \tag{8}$$

The following theorem is a generalization of discrete weighted Favard's inequality [28]:

Theorem 5. *Let $\Psi : (0,1) \to \mathbf{R}$ be convex function and $\boldsymbol{\beta} = (\beta_1, \beta_2, \dots, \beta_n)$, $\boldsymbol{\zeta} = (\zeta_1, \zeta_2, \dots, \zeta_n)$ and $\mathbf{p} = (p_1, p_2, \dots, p_n)$ be positive n-tuples.*

(i): *If $\frac{\boldsymbol{\beta}}{\boldsymbol{\zeta}}$ is decreasing n-tuple and $\boldsymbol{\beta}$ is an increasing n-tuple, then the following inequality holds*

$$\sum_{i=1}^{n} p_i \Psi\left(\frac{\zeta_i}{\sum_{i=1}^{n} p_i \zeta_i}\right) \geq \sum_{i=1}^{n} p_i \Psi\left(\frac{\beta_i}{\sum_{i=1}^{n} p_i \beta_i}\right). \tag{9}$$

If $\boldsymbol{\zeta}$ is decreasing n-tuple, then the reverse inequality holds in (9)*.*

(ii): *If $\frac{\boldsymbol{\beta}}{\boldsymbol{\zeta}}$ is an increasing n-tuple and $\boldsymbol{\zeta}$ is an increasing n-tuple, then the following inequality holds*

$$\sum_{i=1}^{n} p_i \Psi\left(\frac{\beta_i}{\sum_{i=1}^{n} p_i \beta_i}\right) \geq \sum_{i=1}^{n} p_i \Psi\left(\frac{\zeta_i}{\sum_{i=1}^{n} p_i \zeta_i}\right). \tag{10}$$

If $\boldsymbol{\beta}$ is decreasing n-tuple, then the reverse inequality holds in (10)*.*

If Ψ is strictly convex function and $\boldsymbol{\beta} \neq \boldsymbol{\zeta}$, then the strict inequalities hold in (9) *and* (10) *and their reverse cases.*

The following theorem is a generalization of discrete weighted Berwald's inequality [28]:

Theorem 6. *Let $\Psi, \varphi : [0,\infty) \to \mathbf{R}$ be such that φ is continues and strictly increasing function and Ψ be a convex function with respect to φ that is $\Psi \circ \varphi^{-1}$ be convex. If $\boldsymbol{\beta} = (\beta_1, \beta_2, \dots, \beta_n)$, $\boldsymbol{\zeta} = (\zeta_1, \zeta_2, \dots, \zeta_n)$ and $\mathbf{p} = (p_1, p_2, \dots, p_n)$ are positive n-tuples and κ is such that*

$$\sum_{i=1}^{n} p_i \varphi(\beta_i) = \sum_{i=1}^{n} p_i \varphi(\kappa \zeta_i)$$

(i): *If $\frac{\boldsymbol{\beta}}{\boldsymbol{\zeta}}$ is decreasing n-tuple and $\boldsymbol{\beta}$ is an increasing n-tuple, then the following inequality holds*

$$\sum_{i=1}^{n} p_i \Psi(\kappa \zeta_i) \geq \sum_{i=1}^{n} p_i \Psi\left(\beta_i\right). \tag{11}$$

If $\boldsymbol{\zeta}$ is decreasing n-tuple, then (11) *holds in the reverse direction.*

(ii): *If $\frac{\boldsymbol{\beta}}{\boldsymbol{\zeta}}$ is an increasing n-tuple and $\boldsymbol{\zeta}$ is an increasing n-tuple, then the following inequality holds*

$$\sum_{i=1}^{n} p_i \Psi\left(\beta_i\right) \geq \sum_{i=1}^{n} p_i \Psi(\kappa \zeta_i). \tag{12}$$

If $\boldsymbol{\beta}$ is decreasing n-tuple, then (12) *holds in the reverse direction.*

If $\Psi \circ \varphi^{-1}$ is strictly convex function and $\boldsymbol{\beta} \neq \kappa \boldsymbol{\zeta}$, then the strict inequalities hold in (11) *and* (12) *and their reverse cases.*

The following lemma is given in [28]:

Lemma 1. *Let* $\mathbf{u}=(u_1,u_2,\dots,u_n)$ *be a positive* n*-tuple and* $\mathbf{b}=(b_1,b_2,\dots,b_n)$ *be increasing real* n*-tuple, then*

$$\sum_{i=1}^{m} b_i u_i \sum_{i=1}^{n} u_i \le \sum_{i=1}^{n} b_i u_i \sum_{i=1}^{m} u_i, \quad \text{for } m=1,2,\dots,n. \tag{13}$$

If $\mathbf{b}$ *is decreasing real* n*-tuple, then* (13) *holds in the reverse direction.*

If $\boldsymbol{\beta}=(\beta_1,\beta_2,\dots,\beta_n)$ and $\boldsymbol{\zeta}=(\zeta_1,\zeta_2,\dots,\zeta_n)$ are n-tuples with $\zeta_i \neq 0$, for $i=1,2,\dots,n$, then we can define the n-tuple $\frac{\boldsymbol{\beta}}{\boldsymbol{\zeta}}$ by $\left(\frac{\beta_1}{\zeta_1},\frac{\beta_1}{\zeta_2},\dots,\frac{\beta_n}{\zeta_n}\right)$.

For more recent results related to strongly convex functions, majorization inequality for convex functions, Favard's and Berwald's inequalities we recommend [4–9, 18, 25, 28, 33, 37, 45].

The main purpose of the article is to establish the extension of Favard's and Berwald's inequalities for strongly convex functions. The given results improve the previously results. Our approach may have further applications in the theory of majorization.

2. Main Results

In the following theorem we present a generalization of discrete weighted Favard's inequality for strongly convex functions.

Theorem 7. *Let* $\Psi:(0,1)\to\mathbf{R}$ *be strongly convex function with respect to* c*,* $\boldsymbol{\beta}=(\beta_1,\beta_2,\dots,\beta_n)$*,* $\boldsymbol{\zeta}=(\zeta_1,\zeta_2,\dots,\zeta_n)$ *and* $\mathbf{p}=(p_1,p_2,\dots,p_n)$ *be positive* n*-tuples. Also, let* $\overline{\beta}=\frac{\beta_i}{\sum_{i=1}^n p_i\beta_i}$ *and* $\overline{\zeta}=\frac{\zeta_i}{\sum_{i=1}^n p_i\zeta_i}$.

(a): *If* $\frac{\boldsymbol{\beta}}{\boldsymbol{\zeta}}$ *is decreasing* n*-tuple and* $\boldsymbol{\beta}$ *is an increasing* n*-tuple, then the following inequality holds*

$$\sum_{i=1}^{n} p_i \Psi\left(\overline{\zeta}\right) \ge \sum_{i=1}^{n} p_i \Psi\left(\overline{\beta}\right) + c\sum_{i=1}^{n} p_i \left(\overline{\zeta}-\overline{\beta}\right)^2. \tag{14}$$

If $\boldsymbol{\zeta}$ *is decreasing* n*-tuple, then* (14) *holds in the reverse direction.*

(b): *If* $\frac{\boldsymbol{\beta}}{\boldsymbol{\zeta}}$ *is an increasing* n*-tuple and* $\boldsymbol{\zeta}$ *is an increasing* n*-tuple, then the following inequality holds*

$$\sum_{i=1}^{n} p_i \Psi\left(\overline{\beta}\right) \ge \sum_{i=1}^{n} p_i \Psi\left(\overline{\zeta}\right) + c\sum_{i=1}^{n} p_i \left(\overline{\beta}-\overline{\zeta}\right)^2. \tag{15}$$

If $\boldsymbol{\beta}$ *is decreasing* n*-tuple, then* (15) *holds in the reverse direction.*

If Ψ *is strictly strongly convex function and* $\boldsymbol{\beta}\neq\boldsymbol{\zeta}$*, then the strict inequalities hold in* (14) *and* (15) *and their reverse cases.*

Proof. Using Lemma 1 for $\mathbf{u}=\mathbf{p}\boldsymbol{\zeta}$ and $\mathbf{b}=\frac{\boldsymbol{\beta}}{\boldsymbol{\zeta}}$, we obtain

$$\sum_{i=1}^{n} p_i\beta_i \sum_{i=1}^{m} p_i\zeta_i \le \sum_{i=1}^{m} p_i\beta_i \sum_{i=1}^{n} p_i\zeta_i, \quad \text{for } m=1,2,\dots,n,$$

i.e.,

$$\sum_{i=1}^{m} p_i \left(\frac{\zeta_i}{\sum_{i=1}^n p_i\zeta_i}\right) \le \sum_{i=1}^{m} p_i \left(\frac{\beta_i}{\sum_{i=1}^n p_i\beta_i}\right).$$

If $\boldsymbol{\beta}$ is increasing, then using Theorem 3(b), we obtain

$$\sum_{i=1}^{n} p_i \Psi\left(\overline{\zeta}\right) \geq \sum_{i=1}^{n} p_i \Psi\left(\overline{\beta}\right) + c\sum_{i=1}^{n} p_i \left(\overline{\zeta} - \overline{\beta}\right)^2.$$

In similar fashion we can prove the remaining cases. □

Theorem 8. *Under the assumptions of Theorem* 7.

(a): *If* $\frac{\boldsymbol{\beta}}{\boldsymbol{\zeta}}$ *is decreasing* n*-tuple and* $\boldsymbol{\beta}$ *is an increasing* n*-tuple, then the following inequality holds*

$$\sum_{i=1}^{n} p_i \Psi\left(\overline{\zeta}\right) \geq \sum_{i=1}^{n} p_i \Psi\left(\overline{\beta}\right) + c\sum_{i=1}^{n} p_i \left(\overline{\zeta}^2 - \overline{\beta}^2\right). \tag{16}$$

If $\boldsymbol{\zeta}$ *is decreasing* n*-tuple, then* (16) *holds in the reverse direction.*

(b): *If* $\frac{\boldsymbol{\beta}}{\boldsymbol{\zeta}}$ *is an increasing* n*-tuple and* $\boldsymbol{\zeta}$ *is an increasing* n*-tuple, then the following inequality holds*

$$\sum_{i=1}^{n} p_i \Psi\left(\overline{\beta}\right) \geq \sum_{i=1}^{n} p_i \Psi\left(\overline{\zeta}\right) + c\sum_{i=1}^{n} p_i \left(\overline{\beta}^2 - \overline{\zeta}^2\right). \tag{17}$$

If $\boldsymbol{\beta}$ *is decreasing* n*-tuple, then* (17) *holds in the reverse direction.*

Proof. Since Ψ is strongly convex function with respect to c, therefore $\Psi(x) - cx^2$ is convex function. Applying this convex function on (9), we have

$$\sum_{i=1}^{n} p_i \Psi\left(\overline{\zeta}\right) - c\sum_{i=1}^{n} p_i \overline{\zeta}^2 \geq \sum_{i=1}^{n} p_i \Psi\left(\overline{\beta}\right) - c\sum_{i=1}^{n} p_i \overline{\beta}^2,$$

which is equivalent to (16). Similarly, from (10) we can deduce (17). □

Remark 1. From (16) we can easily obtain Theorem 5 (9), because for convex function $\Psi(x) = x^2$ we obtain $\sum_{i=1}^{n} p_i \left(\overline{\zeta}^2 - \overline{\beta}^2\right) \geq 0$. Similarly from (17) we can easily deduce Theorem 5 (10).

Corollary 1. *Let* $\Psi : [0,\infty) \to \mathbf{R}$ *be strongly convex function with respect to* c, $\mathbf{p} = (p_1, p_2, \dots, p_n)$ *be positive* n*-tuple. Assume that* $\overline{\beta} = \frac{\beta_i}{\sum_{i=1}^{n} p_i \beta_i}$, $\overline{z}_1 = \frac{i-1}{\sum_{i=1}^{n} p_i (i-1)}$ *and* $\overline{z}_2 = \frac{n-i}{\sum_{i=1}^{n} p_i (n-i)}$.

(a): *If* $\boldsymbol{\beta}$ *is positive increasing concave* n*-tuple, then*

$$\sum_{i=1}^{n} p_i \Psi(\overline{z}_1) \geq \sum_{i=1}^{n} p_i \Psi\left(\overline{\beta}\right) + c\sum_{i=1}^{n} p_i \left(\overline{z}_1 - \overline{\beta}\right)^2. \tag{18}$$

(b): *If* $\boldsymbol{\beta}$ *is an increasing convex real* n*-tuple and* $\beta_1 = 0$, *then*

$$\sum_{i=1}^{n} p_i \Psi\left(\overline{\beta}\right) \geq \sum_{i=1}^{n} p_i \Psi(\overline{z}_1) + c\sum_{i=1}^{n} p_i \left(\overline{z}_1 - \overline{\beta}\right)^2. \tag{19}$$

(c): *If* $\boldsymbol{\beta}$ *is positive decreasing concave* n*-tuple, then*

$$\sum_{i=1}^{n} p_i \Psi(\overline{z}_2) \geq \sum_{i=1}^{n} p_i \Psi\left(\overline{\beta}\right) + c\sum_{i=1}^{n} p_i \left(\overline{z}_2 - \overline{\beta}\right)^2. \tag{20}$$

(d): *If* $\boldsymbol{\beta}$ *is decreasing convex real* n*-tuple and* $\beta_n = 0$, *then*

$$\sum_{i=1}^{n} p_i \Psi\left(\overline{\beta}\right) \geq \sum_{i=1}^{n} p_i \Psi(\overline{z}_2) + c\sum_{i=1}^{n} p_i \left(\overline{z}_2 - \overline{\beta}\right)^2. \tag{21}$$

Proof. (a): Taking $\zeta_1 = \varepsilon < \frac{\beta_1}{\beta_2}$, $\zeta_i = i-1$ for $2 \leq i \leq n$ and by concavity of $\boldsymbol{\beta}$, we have $\frac{\beta_i}{\zeta_i}$ for

$1 \le i \le n$ is decreasing n-tuple. Thus, by using Theorem 7 (14), we have

$$p_1\Psi\left(\frac{\varepsilon}{p_1\varepsilon+\sum_{i=2}^n p_i(i-1)}\right)+\sum_{i=2}^n p_i\Psi\left(\frac{i-1}{p_1\varepsilon+\sum_{i=2}^n p_i(i-1)}\right)$$
$$\ge \sum_{i=1}^n p_i\Psi\left(\overline{\beta}\right)+c\sum_{i=1}^n p_i\left(\frac{\varepsilon+(i-1)}{p_1\varepsilon+\sum_{i=2}^n p_i(i-1)}-\overline{\beta}\right)^2.$$

By taking $\varepsilon \to 0$, we obtain

$$p_1\Psi(0)+\sum_{i=2}^n p_i\Psi\left(\frac{i-1}{\sum_{i=2}^n p_i(i-1)}\right)\ge \sum_{i=1}^n p_i\Psi\left(\overline{\beta}\right)+c\sum_{i=1}^n p_i\left(\frac{(i-1)}{\sum_{i=2}^n p_i(i-1)}-\overline{\beta}\right)^2,$$

which is equivalent to (18).

(b): If $\boldsymbol{\beta}$ is an increasing convex real n-tuple and $\beta_1 = 0$, then $\frac{\beta_i}{i-1}$ for $2 \le i \le n$ is increasing. Since $\frac{\beta_i}{i-1}$ for $2 \le i \le n$ is increasing and also $\zeta_i = i-1$ for $2 \le i \le n$ is increasing, thus by using Theorem 7 (15), we get

$$\sum_{i=2}^n p_i\Psi\left(\frac{\beta_i}{\sum_{i=2}^n p_i\beta_i}\right)\ge \sum_{i=2}^n p_i\Psi\left(\frac{i-1}{\sum_{i=2}^n p_i(i-1)}\right)+c\sum_{i=2}^n p_i\left(\frac{\beta_i}{\sum_{i=2}^n p_i\beta_i}-\frac{i-1}{\sum_{i=2}^n p_i(i-1)}\right)^2,$$

which can be written as

$$p_1\Psi\left(\frac{0}{\sum_{i=1}^n p_i\beta_i}\right)+\sum_{i=2}^n p_i\Psi\left(\frac{\beta_i}{\sum_{i=1}^n p_i\beta_i}\right)$$
$$\ge p_1\Psi\left(\frac{0}{\sum_{i=1}^n p_i(i-1)}\right)+\sum_{i=2}^n p_i\Psi\left(\frac{i-1}{\sum_{i=1}^n p_i(i-1)}\right)+c\sum_{i=1}^n p_i\left(\frac{\beta_i}{\sum_{i=1}^n p_i\beta_i}-\frac{i-1}{\sum_{i=1}^n p_i(i-1)}\right)^2,$$

which is equivalent to (19).

In similar fashion we can prove the remaining cases. □

Corollary 2. *Under the assumptions of Corollary* 1.

(a): *If $\boldsymbol{\beta}$ is positive increasing concave n-tuple, then*

$$\sum_{i=1}^n p_i\Psi(\overline{z}_1)\ge \sum_{i=1}^n p_i\Psi\left(\overline{\beta}\right)+c\sum_{i=1}^n p_i\left(\overline{z}_1^2-\overline{\beta}^2\right). \tag{22}$$

(b): *If $\boldsymbol{\beta}$ is an increasing convex real n-tuple and $\beta_1 = 0$, then*

$$\sum_{i=1}^n p_i\Psi\left(\overline{\beta}\right)\ge \sum_{i=1}^n p_i\Psi(\overline{z}_1)+c\sum_{i=1}^n p_i\left(\overline{\beta}^2-\overline{z}_1^2\right). \tag{23}$$

(c): *If $\boldsymbol{\beta}$ is positive decreasing concave n-tuple, then*

$$\sum_{i=1}^n p_i\Psi(\overline{z}_2)\ge \sum_{i=1}^n p_i\Psi\left(\overline{\beta}\right)+c\sum_{i=1}^n p_i\left(\overline{z}_2^2-\overline{\beta}^2\right). \tag{24}$$

(d): *If $\boldsymbol{\beta}$ is decreasing convex real n-tuple and $\beta_n = 0$, then*

$$\sum_{i=1}^n p_i\Psi\left(\overline{\beta}\right)\ge \sum_{i=1}^n p_i\Psi(\overline{z}_2)+c\sum_{i=1}^n p_i\left(\overline{\beta}^2-\overline{z}_2^2\right). \tag{25}$$

Proof. Since Ψ is strongly convex function with respect to c, therefore $\Psi(x)-cx^2$ is convex function. Applying this convex function on [28, Corollary 2.4], we deduce (22), (23), (24) and (25). □

Remark 2. From (22) we can easily obtain Corollary 2.4 (2.9) in [28] because for convex function $\Psi(x)=x^2$ we obtain $\sum\limits_{i=1}^{n} p_i\left(\overline{z}_1^2-\overline{\beta}^2\right)\geq 0$. Similarly for the remaining cases.

The following theorem is discrete weighted Berwald's inequality for strongly convex function.

Theorem 9. *Let $\varphi,\Psi:[0,\infty)\to\mathbf{R}$ be functions such that φ is a strictly increasing function and Ψ is a strongly convex function with respect to φ that is $\Psi\circ\varphi^{-1}$ is strongly convex function with respect to c. Suppose $\boldsymbol{\beta}$, $\boldsymbol{\zeta}$ and $\mathbf{p}$ are positive n-tuples and κ is such that*

$$\sum_{i=1}^{n} p_i\varphi(\beta_i)=\sum_{i=1}^{n} p_i\varphi(\kappa\zeta_i) \tag{26}$$

(a): *If $\frac{\boldsymbol{\beta}}{\boldsymbol{\zeta}}$ is decreasing n-tuple and $\boldsymbol{\beta}$ is an increasing n-tuple, then the following inequality holds*

$$\sum_{i=1}^{n} p_i\Psi(\kappa\zeta_i)\geq\sum_{i=1}^{n} p_i\Psi\left(\beta_i\right)+c\sum_{i=1}^{n} p_i\left(\varphi(\kappa\zeta_i)-\varphi(\beta_i)\right)^2. \tag{27}$$

If $\boldsymbol{\zeta}$ is decreasing n-tuple, then (27) *holds in the reverse direction.*

(b): *If $\frac{\boldsymbol{\beta}}{\boldsymbol{\zeta}}$ is an increasing n-tuple and $\boldsymbol{\zeta}$ is an increasing n-tuple, then the following inequality holds*

$$\sum_{i=1}^{n} p_i\Psi\left(\beta_i\right)\geq\sum_{i=1}^{n} p_i\Psi(\kappa\zeta_i)+c\sum_{i=1}^{n} p_i\left(\varphi(\beta_i)-\varphi(\kappa\zeta_i)\right)^2. \tag{28}$$

If $\boldsymbol{\beta}$ is decreasing n-tuple, then (28) *holds in the reverse direction.*

Proof. Under the given condition in [28] it has been shown that

$$\sum_{i=1}^{m} p_i\varphi(\beta_i)\geq\sum_{i=1}^{m} p_i\varphi(\kappa\zeta_i),\quad \text{for } m=1,2,\dots,n-1. \tag{29}$$

Since $\Psi\circ\varphi^{-1}$ is strongly convex function and $\boldsymbol{\beta}$ is an increasing n-tuple. So, by using (29), (26) and Theorem 4, we have

$$\sum_{i=1}^{n} p_i\Psi(\kappa\zeta_i)\geq\sum_{i=1}^{n} p_i\Psi\left(\beta_i\right)+c\sum_{i=1}^{n} p_i\left(\varphi(\kappa\zeta_i)-\varphi(\beta_i)\right)^2. \tag{30}$$

Since $\Psi\circ\varphi^{-1}$ is strongly convex function and $\boldsymbol{\zeta}$ is an decreasing n-tuple. So, by using (29), (26) and Theorem 4, we have

$$\sum_{i=1}^{n} p_i\Psi\left(\beta_i\right)\geq\sum_{i=1}^{n} p_i\Psi(\kappa\zeta_i)+c\sum_{i=1}^{n} p_i\left(\varphi(\beta_i)-\varphi(\kappa\zeta_i)\right)^2. \tag{31}$$

In similar fashion we can prove the remaining cases. □

Theorem 10. *Under the assumptions of Theorem* 9.

(a): *If $\frac{\boldsymbol{\beta}}{\boldsymbol{\zeta}}$ is a decreasing n-tuple and $\boldsymbol{\beta}$ is an increasing n-tuple, then the following inequality holds*

$$\sum_{i=1}^{n} p_i\Psi(\kappa\zeta_i)\geq\sum_{i=1}^{n} p_i\Psi\left(\beta_i\right)+c\sum_{i=1}^{n} p_i\left\{\left(\varphi(\kappa\zeta_i)\right)^2-\left(\varphi(\beta_i)\right)^2\right\}. \tag{32}$$

If $\boldsymbol{\zeta}$ is decreasing n-tuple, then reverse inequality holds in (32).

(b): *If $\frac{\boldsymbol{\beta}}{\boldsymbol{\zeta}}$ is an increasing n-tuple and $\boldsymbol{\zeta}$ is an increasing n-tuple, then the following inequality holds*

$$\sum_{i=1}^{n} p_i \Psi\left(\beta_i\right) \geq \sum_{i=1}^{n} p_i \Psi(\kappa\zeta_i) + c\sum_{i=1}^{n} p_i \left\{\left(\varphi(\beta_i)\right)^2 - \left(\varphi(\kappa\zeta_i)\right)^2\right\}. \tag{33}$$

If $\boldsymbol{\beta}$ is decreasing n-tuple, then reverse inequality holds in (33).

Proof. Since $\Psi\circ\varphi^{-1}$ is strongly convex function with respect to c, therefore $\Psi(x) - c(\varphi(x))^2$ is convex function. Applying this convex function on (11), we have

$$\sum_{i=1}^{n} p_i \Psi(\kappa\zeta_i) - c\sum_{i=1}^{n} p_i \left(\varphi(\kappa\zeta_i)\right)^2 \geq \sum_{i=1}^{n} p_i \Psi\left(\beta_i\right) - c\sum_{i=1}^{n} p_i \left(\varphi(\beta_i)\right)^2$$

which is equivalent to (32). Similarly, from (12) we can deduce (33). □

Remark 3. From (32) we can easily obtain Theorem 6 (11), because for convex function $\Psi(x) = x^2$ we obtain $\sum\limits_{i=1}^{n} p_i \left\{\left(\varphi(\kappa\zeta_i)\right)^2 - \left(\varphi(\beta_i)\right)^2\right\} \geq 0$. Similarly, from (33) we can easily deduce Theorem 6 (12).

Corollary 3. *Let $\varphi, \Psi : [0,\infty) \to \mathbf{R}$ be such that φ is strictly increasing function and Ψ be a strongly convex function with respect to φ that is $\Psi\circ\varphi^{-1}$ is strongly convex function with respect to c. Assume that $\mathbf{p} = (p_1, p_2, \dots, p_n)$ is positive n-tuple and κ_1 and κ_2 are such that*

$$\sum_{i=1}^{n} p_i \varphi(\beta_i) = \sum_{i=1}^{n} p_i \varphi\{(i-1)\kappa_1\} \tag{34}$$

and

$$\sum_{i=1}^{n} p_i \varphi(\beta_i) = \sum_{i=1}^{n} p_i \varphi\{(n-i)\kappa_2\} \tag{35}$$

(a): *If $\boldsymbol{\beta}$ is positive increasing concave n-tuple, then*

$$\sum_{i=1}^{n} p_i \Psi\{(i-1)\kappa_1\} \geq \sum_{i=1}^{n} p_i \Psi\left(\beta_i\right) + c\sum_{i=1}^{n} p_i \left\{\varphi\{(i-1)\kappa_1\} - \varphi(\beta_i)\right\}^2. \tag{36}$$

(b): *If $\boldsymbol{\beta}$ is an increasing convex n-tuple and $\beta_1 = 0$, then*

$$\sum_{i=1}^{n} p_i \Psi\left(\beta_i\right) \geq \sum_{i=1}^{n} p_i \Psi\{(i-1)\kappa_1\} + c\sum_{i=1}^{n} p_i \left\{\varphi(\beta_i) - \varphi\{(i-1)\kappa_1\}\right\}^2. \tag{37}$$

(c): *If $\boldsymbol{\beta}$ is positive decreasing concave n-tuple, then*

$$\sum_{i=1}^{n} p_i \Psi\{(n-i)\kappa_2\} \geq \sum_{i=1}^{n} p_i \Psi\left(\beta_i\right) + c\sum_{i=1}^{n} p_i \left\{\varphi\{(n-i)\kappa_2\} - \varphi(\beta_i)\right\}^2. \tag{38}$$

(d): *If $\boldsymbol{\beta}$ is an decreasing convex n-tuple and $\beta_n = 0$, then*

$$\sum_{i=1}^{n} p_i \Psi\left(\beta_i\right) \geq \sum_{i=1}^{n} p_i \Psi\{(n-i)\kappa_2\} + c\sum_{i=1}^{n} p_i \left\{\varphi(\beta_i) - \varphi\{(n-i)\kappa_2\}\right\}^2. \tag{39}$$

Proof. (a): Taking $\zeta_1 = \varepsilon < \frac{\beta_1}{\beta_2}$, $\zeta_i = i-1$ for $2 \leq i \leq n$ and by concavity of $\boldsymbol{\beta}$, we have $\frac{\beta_i}{\zeta_i}$ for $1 \leq i \leq n$ is decreasing n-tuple. Thus, from (34), we have

$$\sum_{i=1}^{n} p_i \varphi(\beta_i) = p_1\varphi(\varepsilon\kappa_1) + \sum_{i=2}^{n} p_i \varphi\{(i-1)\kappa_1\}.$$

By using Theorem 9 (27), we get

$$p_1\Psi(\varepsilon\kappa_1)+\sum_{i=2}^{n}p_i\Psi(\kappa_1(i-1))\geq\sum_{i=1}^{n}p_i\Psi\left(\beta_i\right)+c\sum_{i=1}^{n}p_i\left\{\left(\varphi(\kappa_1\varepsilon)+\varphi(\kappa_1(i-1))\right)-\left(\varphi(\beta_i)\right)\right\}^2.$$

and taking $\varepsilon\to 0$, we obtain (36).

(b): If $\boldsymbol{\beta}$ is an increasing convex n-tuple and $\beta_1=0$, then $\frac{\beta_i}{i-1}$ for $2\leq i\leq n$ is increasing. Since $\frac{\beta_i}{i-1}$ for $2\leq i\leq n$ is increasing and also $\zeta_i=i-1$ for $2\leq i\leq n$ is increasing, thus by using Theorem 9 (28), we have

$$\sum_{i=2}^{n}p_i\Psi\left(\beta_i\right)\geq\sum_{i=2}^{n}p_i\Psi\{(i-1)\kappa_1\}+c\sum_{i=2}^{n}p_i\left\{\varphi(\beta_i)-\varphi\{(i-1)\kappa_1\}\right\}^2$$

$$p_1\Psi(0)+\sum_{i=2}^{n}p_i\Psi\left(\beta_i\right)\geq p_1\Psi(\kappa_1 0)+\sum_{i=2}^{n}p_i\Psi\{(i-1)\kappa_1\}+c\sum_{i=1}^{n}p_i\left\{\varphi(\beta_i)-\varphi\{(i-1)\kappa_1\}\right\}^2,$$

which is equivalent to (37).

In similar fashion we can prove the remaining cases. □

Corollary 4. *Under the assumptions of Corollary* 3.

(a): *If $\boldsymbol{\beta}$ is positive increasing concave n-tuple, then*

$$\sum_{i=1}^{n}p_i\Psi\{(i-1)\kappa_1\}\geq\sum_{i=1}^{n}p_i\Psi\left(\beta_i\right)+c\sum_{i=1}^{n}p_i\left\{\left(\varphi\{(i-1)\kappa_1\}\right)^2-\left(\varphi(\beta_i)\right)^2\right\}.\tag{40}$$

(b): *If $\boldsymbol{\beta}$ is an increasing convex n-tuple and $\beta_1=0$, then*

$$\sum_{i=1}^{n}p_i\Psi\left(\beta_i\right)\geq\sum_{i=1}^{n}p_i\Psi\{(i-1)\kappa_1\}+c\sum_{i=1}^{n}p_i\left\{\left(\varphi(\beta_i)\right)^2-\left(\varphi\{(i-1)\kappa_1\}\right)^2\right\}.\tag{41}$$

(c): *If $\boldsymbol{\beta}$ is positive decreasing concave n-tuple, then*

$$\sum_{i=1}^{n}p_i\Psi\{(n-i)\kappa_2\}\geq\sum_{i=1}^{n}p_i\Psi\left(\beta_i\right)+c\sum_{i=1}^{n}p_i\left\{\left(\varphi\{(n-i)\kappa_2\}\right)^2-\left(\varphi(\beta_i)\right)^2\right\}.\tag{42}$$

(d): *If $\boldsymbol{\beta}$ is decreasing convex n-tuple and $\beta_n=0$, then*

$$\sum_{i=1}^{n}p_i\Psi\left(\beta_i\right)\geq\sum_{i=1}^{n}p_i\Psi\{(n-i)\kappa_2\}+c\sum_{i=1}^{n}p_i\left\{\left(\varphi(\beta_i)\right)^2-\left(\varphi\{(n-i)\kappa_2\}\right)^2\right\}.\tag{43}$$

Proof. Since $\Psi\circ\varphi^{-1}$ is strongly convex function with respect to c, therefore $\Psi(x)-c(\varphi(x))^2$ is convex function. Applying this convex function on [28, Corollary 2.9], we deduce (40), (41), (42) and (43). □

Remark 4. From (40) we can easily obtain Corollary 2.9(2.26) in [28] because for convex function $\Psi(x)=x^2$ we obtain $\sum_{i=1}^{n}p_i\left\{\left(\varphi\{(i-1)\kappa_1\}\right)^2-\left(\varphi(\beta_i)\right)^2\right\}\geq 0$.

Similarly, we can obtain the remaining cases.

3. Conclusion

In the article, we generalize the discrete weighted version of the well known Favard's and Berwald's inequalities to the strongly convex function by use of majorization theory. Our obtained results are the improvements and generalizations of some previously results. The given ideas and methods may lead to a large number of follow-up research.

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Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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