Communications in Mathematics and Applications
Vol. 10, No. 3, pp. 369–377, 2019
ISSN 0975-8607 (online); 0976-5905 (print)
Published by RGN Publications
DOI: 10.26713/cma.v10i3.1199

http://www.rgnpublications.com

Research Article

Existence and Convergence Theorems For Best Proximity Points of Proximal Multi-Valued Nonexpansive Mappings

Panitarn Sarnmeta[1] and Suthep Suantai[2,*]

[1] *Ph.D. Degree Program in Mathematics, Faculty of Science, Chiang Mai University, Chiang Mai, 50200, Thailand*
[2] *Center of Excellence in Mathematics and Applied Mathematics, Department of Mathematics, Faculty of Science, Chiang Mai University, Chiang Mai 50200, Thailand*
[*] **Corresponding author:** suthep.s@cmu.ac.th

Abstract. The concepts of proximal contraction and proximal nonexpansive mapping have been investigated and extended in many direction. However, most of these works concern only single-valued mappings. So, in this paper, we introduce a concept of proximal nonexpansive for non-self set-valued mappings and prove the existence of best proximity point for such mappings under appropriate conditions. We also provide an algorithm to approximate a best proximity point of such mappings, and prove its convergence theorem. Moreover, a numerical example supporting our main results is also given.

Keywords. Proximal multi-valued nonexpansive; Best proximity point

MSC. 47H04; 47H09; 47H10

Received: February 22, 2019 **Accepted:** March 26, 2019

1. Introduction

Fixed point theory focuses on solving nonlinear equation in the form $Tx = x$ where T is a self-mapping defined on a subset of metric spaces or normed spaces. A solution of said equation is called fixed point of T. The study of fixed point theory can be classified into two problems, one is an existence problem and the other one is an approximation problem. The well-known

Banach contraction principle [1] assures that every contraction self-mapping has a unique fixed point, this theorem also provides the iteration for finding the fixed point of the mapping. This work was extended in many directions, one of that is to extend from single-valued contraction mappings to multi-valued contraction mappings. The first well-known result was established by Nadler [16]. There are many interesting extensions of this kind of mappings (for more information see [4–6, 15, 20]). However, all of these works concern on fixed point problem.

In the case that T is non-self mappings, it is possible for T to not having a fixed point. So, it is natural to ask for a point in domain which lies closet to its image. This idea was first investigated by Fan [8]. He established the best approximation theorem which asserts that a continuous mapping $T : A \to X$, where A is nonempty compact and convex subset of a Hausdorff locally convex topological vector space X, has a point $x \in A$ such that $d(x,Tx) = D(Tx,A)$. This idea led to the definition of best proximity point which states as follows:

Let A and B be two nonempty subsets of a metric space (X,d) and T be non-self mapping of A into B. A point x in A is said to be a best proximity point of T if the distance between x and Tx is equal to the distance from A to B or dist(A,B).

Following this concept, it can be said that x is actually an optimal approximate solution of $x = Tx$. The study of best proximity point theorem has been growing in recent years. Many researchers established very interesting results, for example, Kim and Lee [14] combined the optimal form of Fan's best approximation theorem and equilibrium existence theorem into a single existence theorem which can be use to answer some economic problems. For more works related to best proximity point theorem see [7, 13, 18, 19]. In 2011, Basha [2] introduced a mapping called proximal contraction mapping. He proved the existence of best proximity point of such mapping under appropriate conditions. However, this mapping is a single-valued mapping. Motivated by this concept, we aim to introduce a proximal multi-valued nonexpansive mapping, and prove the existence of best proximity point for such mappings under certain conditions. We also provide a convergence theorem for this kind of mappings.

2. Preliminaries

In this section, we give definitions, notations and lemmas which will be used in our main results.

Definition 2.1. Let A and B be nonempty subsets of a metric space (X,d), a distance between A and B is as follows

$$\operatorname{dist}(A,B) = \inf\{d(x,y) : (x,y) \in A \times B\}.$$

Let x be any element in X, a distance between x and a set B is defined by

$$D(x,B) = \inf\{d(x,y) : y \in B\}.$$

Let $T : A \to 2^B$ be a multi-valued non-self mapping, a point $x \in A$ is a called best proximity point of T if

$$D(x,Tx) = \operatorname{dist}(A,B).$$

The subsets A_0 and B_0 of A and B, respectively, are given as follows:

$$A_0 = \{x \in A : d(x,y) = \operatorname{dist}(A,B), \text{ for some } y \in B\};$$

$$B_0 = \{y \in B : d(x,y) = \text{dist}(A,B), \text{ for some } x \in A\}.$$

We denote $CB(X)$ a set of all closed and bound subsets of X. For $A,B \in CB(X)$, define

$$H(A,B) = \max\{\sup_{a\in A} D(a,B), \sup_{b\in B} D(b,A)\}.$$

This mapping H is called Pompeiu-Hausdorff distance from A to B.

We note that if A and B are nonempty, weakly compact and convex subsets of a Banach space X, then A_0 and B_0 are nonempty.

The following notions and lemma play crucial roles in our main results.

Definition 2.2. Let A and B be nonempty subsets of a metric space (X,d). B is said to be approximatively compact with respect to A if for any sequence $\{y_n\} \in B$ satisfying the condition that $d(x,y_n) \to D(x,B)$, for some $x \in A$ has a convergent subsequence. We note that B is approximatively compact with respect to itself.

Definition 2.3 ([17])**.** Let $(X,\|\cdot\|)$ be normed space. X satisfies *Opial's condition* if for any sequence $\{x_n\}$ in X with $x_n \rightharpoonup x$, the inequality

$$\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|$$

holds for any $y \neq x$.

Definition 2.4. Let A and B be nonempty subsets of a metric space (X,d) such that both A_0 and B_0 are nonempty. A mapping $T : A \to CB(B)$ is said to have the best proximity property if for any best proximity point x of T the following holds:

$$D(x,Tx) < D(y,Tx), \quad \text{for all } y \in A, \text{ with } y \neq x.$$

The following lemma can be found in [12].

Lemma 2.5. *Let $(X,\|\cdot\|)$ be a normed space and $\alpha \in (0,1)$. Suppose sequences $\{x_n\}$ and $\{y_n\}$ in X satisfy for all $n \in \mathbb{N}$,*

(i) $x_{n+1} = (1-\alpha)x_n + \alpha y_n$,

(ii) $\|y_{n+1} - y_n\| \le \|x_{n+1} - x_n\|$. *Then for all $i,n \in \mathbb{N}$,*

$$(1+n\alpha)\|x_i - y_i\| \le \|y_{i+n} - x_i\| + (1-\alpha)^{-n}(\|y_i - x_i\| - \|y_{i+n} - x_{i+n}\|).$$

The notion of proximal contraction was introduced by Basha [2] as follows.

Definition 2.6. Let A and B be nonempty subsets of metric space (X,d). A mapping $T : A \to B$ is said to be proximal contraction if there exists non-negative real number $\alpha < 1$ such that, for all u_1,u_2,x_1,x_2 in A,

$$d(u_1,Tx_1) = \text{dist}(A,B) = d(u_2,Tx_2) \quad \Rightarrow \quad d(u_1,u_2) \le \alpha d(x_1,x_2).$$

The following theorem is the main result of [2].

Theorem 2.7. *Let A and B be nonempty, closed subsets of a complete metric space (X,d) such that B is approximatively compact with respect to A. Moreover, assume that A_0 and B_0 are* *nonempty. Let $T : A \to B$ and $g : A \to A$ satisfy the following conditions:*

(i) *T is a proximal contraction,*

(ii) *$T(A_0)$ is contained in B_0,*

(iii) *g is an isometry,*

(iv) *$g(A_0)$ contains A_0.*

Then, there exists an element $x \in A$ such that $d(gx, Tx) = \operatorname{dist}(A, B.)$.

This result was extended in many directions, for more information (see [**?**, 3, 9–11]).

3. Main Results

We first begin our main results by giving the notion of our new mapping. In order to do this, we need the following lemma:

Lemma 3.1. *Let A and B be nonempty subsets of a metric space (X,d), $T : A \to CB(B)$ a mapping. For each $x \in A$, set $U_x = \{y \in A : D(y,Tx) = \operatorname{dist}(A,B)\}$. If U_x is nonempty then U_x is closed and bounded.*

Proof. To show the closedness of U_x, let $\{y_n\}$ be a sequence in U_x such that $y_n \to y$. So, $D(y_n, Tx) = \operatorname{dist}(A,B)$ for all $n \in \mathbb{N}$. Moreover $D(y_n, Tx) \to D(y, Tx)$ as $n \to \infty$. Hence $D(y, Tx) = \operatorname{dist}(A,B)$, so $y \in U_x$ which implies U_x is closed.

In order to show U_x is bounded, we suppose, by the contrary, that U_x is unbounded. So, for each $n \in \mathbb{N}$, there exist $x_n, y_n \in U_x$ such that $d(x_n, y_n) \geq n$. Since $x_n, y_n \in U_x$, we can find $x'_n, y'_n \in Tx$ such that $d(x_n, x'_n) \leq \operatorname{dist}(A,B) + \frac{1}{n}$ and $d(y_n, y'_n) \leq d(A,B) + \frac{1}{n}$, for all $n \in \mathbb{N}$. So,

$$n \leq d(x_n, y_n) \leq d(x_n, x'_n) + d(x'_n, y'_n) + d(y_n, y'_n) \leq d(x'_n, y'_n) + 2\operatorname{dist}(A,B) + \frac{2}{n}, \quad \text{for all } n \in \mathbb{N},$$

which implies $n - 2\operatorname{dist}(A,B) + \frac{2}{n} \leq d(x'_n, y'_n)$, for all $n \in \mathbb{N}$. Hence Tx is unbounded, a contradiction. Therefore U_x is bounded. □

Definition 3.2. Let A, B be subsets of a normed space $(X, \|\cdot\|)$. A mapping $T : A \to CB(B)$ is said to be a proximal multi-valued nonexpansive if for any $x_1, x_2 \in A$ with U_{x_1} and U_{x_2} both are nonempty, we have

$$H(U_{x_1}, U_{x_2}) \leq \|x_1 - x_2\|.$$

As we can see, this mapping reduces to a multi-valued nonexpansive if $d(A,B) = 0$.

Theorem 3.3. *Let A, B be subsets of a normed space $(X, \|\cdot\|)$ satisfying the Opial's condition. Assume that A_0 is nonempty weakly compact convex subset of A and B_0 is nonempty. If $T : A \to CB(B)$ satisfies the following conditions:*

(i) *T is proximal multi-valued nonexpansive mapping,*

(ii) *for any $x \in A_0$, $Tx \cap B_0$ is nonempty,*

(iii) *U_x is compact, for any $x \in A_0$,*

then T has a best proximity point.

Proof. Let x_0 be fixed element in A_0. From (b), there exists $x_0' \in Tx_0 \cap B_0$. So, there exists $y_0 \in A_0$ such that $\|x_0 - y_0\| = \text{dist}(A,B)$. Set $x_1 = (1-\alpha)x_0 + \alpha y_0$, where $\alpha \in [0,1)$, so $x_1 \in A_0$. From (b), U_{x_1} is nonempty. So, there exists $y_1 \in U_{x_1}$ such that

$$\|y_1 - y_0\| \le H(U_{x_1}, U_{x_0}) \le \|x_1 - x_0\|.$$

Since $y_1 \in U_{x_1}$, there exists $y_n^1 \in Tx_1$ such that

$$\text{dist}(A,B) \le D(y_1,B) \le \|y_1 - y_n^1\| \le D(y_1,Tx_1) + \frac{1}{n} = \text{dist}(A,B) + \frac{1}{n} \le D(y_1,B) + \frac{1}{n},$$

for all $n \in \mathbb{N}$. Taking $n \to \infty$, we have $\|y_1 - y_n^1\| \to D(y_1,B) = \text{dist}(A,B)$. By the approximatively compactness with respect to A of B, there exists a subsequence $\{y_{n_k}^1\}$ of $\{y_n^1\}$ such that $y_{n_k}^1 \to y^1$, for some $y^1 \in B$. Hence $d(y_1,y^1) = \text{dist}(A,B)$ which implies $y_1 \in A_0$. Again set $x_2 = (1-\alpha)x_1 + \alpha y_1$, we have $x_2 \in A_0$. So, we can find $y_2 \in U_{x_2}$ such that

$$\|y_2 - y_1\| \le H(U_{x_2}, U_{x_1}) \le \|x_2 - x_1\|.$$

By the same argument, we can show that $y_2 \in A_0$. Inductively, we can construct sequences $\{x_n\}$ and $\{y_n\}$ in A_0 satisfying the following:

(1) $\|y_{n+1} - y_n\| \le \|x_{n+1} - x_n\|$,

(2) $\|x_{n+1} - x_n\| = \|(1-\alpha)x_n + \alpha y_n - x_n\| = \alpha\|x_n - y_n\|$,

(3) $$\begin{aligned}\|x_{n+1} - x_n\| &= \|(1-\alpha)x_n + \alpha y_n - (1-\alpha)x_{n-1} + \alpha y_{n-1}\| \\ &\le (1-\alpha)\|x_n - x_{n-1}\| + \alpha\|y_n - y_{n-1}\| \\ &\le \|x_n - x_{n-1}\|.\end{aligned}$$

The above inequalities imply that $\lim_{n\to\infty} \|x_n - y_n\|$ exists. Suppose that $\lim_{n\to\infty} \|x_n - y_n\| = r$, we claim that $r = 0$.

To show this, from Lemma 2.5 we have, for all $i,n \in \mathbb{N}$,

$$(1+n\alpha)\|x_i - y_i\| \le \|y_{i+n} - x_i\| + (1-\alpha)^{-n}(\|y_i - x_i\| - \|y_{i+n} - x_{i+n}\|).$$

Then,

$$\begin{aligned}\limsup_{i\to\infty}(1+n\alpha)\|x_i - y_i\| &\le \limsup_{i\to\infty}(\|x_i - y_{i+n}\| + (1-\alpha)^{-n}[\|y_i - x_i\| - \|y_{i+n} - x_{i+n}\|]), \\ (1+n\alpha)r &\le \limsup_{i\to\infty}\|x_i - y_{i+n}\| + (1-\alpha)^{-n}(r-r) \\ &= \limsup_{i\to\infty}\|x_i - y_{i+n}\| \\ &\le \text{diam}(C), \quad \text{for all } n \in \mathbb{N}.\end{aligned}$$

It follows that $r = 0$. Since A_0 is weakly compact, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightharpoonup x$, for some $x \in A_0$. So, U_x is nonempty. For each $k \in \mathbb{N}$, there exists $z_k \in U_x$ such that

$$\|y_{n_k} - z_k\| = H(U_{x_{n_k}}, U_x) \le \|x_{n_k} - x\|.$$

From the compactness of U_x, without loss of generality, we can say that $z_k \to z$, for some $z \in U_x$. So,

$$\begin{aligned}\|x_{n_k} - z\| &\le \|x_{n_k} - y_{n_k}\| + \|y_{n_k} - z_k\| + \|z_k - z\| \\ &\le \|x_{n_k} - y_{n_k}\| + \|x_{n_k} - x\| + \|z_k - z\|.\end{aligned}$$

So, $\limsup_{k\to\infty}\|x_{n_k} - z\| \le \limsup_{k\to\infty}\|x_{n_k} - x\|$. From Opial's condition, we have $z = x$ and hence $x \in U_x$. Therefore $D(x,Tx) = \text{dist}(A,B)$, and the proof is complete. □

From the proof of Theorem 3.3, we can generated a sequence $\{x_n\}$ by using the following iterative method:

Algorithm 3.4. Denote $P_K(x)$ a metric projection of x onto a subset K of X.
Choose an initial arbitrary point $x_0 \in A_0$, and $\alpha \in (0,1)$.
Step 1. Pick $x_0' \in Tx_0 \cap B_0$,
set $y_0 = P_{U_{x_0}}(x_0')$, and
set $x_1 = (1-\alpha)x_0 + \alpha y_0$.
Step 2. For $n \geq 1$, we know that U_{x_n} is nonempty,
set $y_n = P_{U_{x_n}}(y_{n-1})$, and
set $x_{n+1} = (1-\alpha)x_n + \alpha y_n$.
Go to *Step* 2 until we obtain the appropriate error.

The following theorem shows that a sequence $\{x_n\}$ generated by Algorithm 3.4 converges to a best proximity point of T under certain conditions.

Theorem 3.5. *Let A, B, A_0, X, T be the same as in Theorem* 3.3*, if T has the best proximity property then the sequence $\{x_n\}$, generated by Algorithm* 3.4*, converges weakly to a best proximity point of T.*

Proof. It suffices to show that $\{x_n\}$ has a unique weak subsequential limit which is a best proximity point of T. In order to show this, let z_1, z_2 be weak limits of subsequence $\{x_{n_k}\}$ and $\{x_{m_k}\}$ of $\{x_n\}$, respectively. From the proof of Theorem 3.3, z_1 and z_2 are best proximity points of T. Since T has the best proximity property, we obtain that U_{z_1} and U_{z_2} are singleton. So,

$$\begin{aligned}\|x_{n+1} - z\| &= \|(1-\alpha)x_n + \alpha y_n - z_1\| \\ &\leq (1-\alpha)\|x_n - z_1\| + \alpha\|y_n - z_1\| \\ &\leq (1-\alpha)\|x_n - z_1\| + \alpha H(U_{x_n}, U_{z_1}) \\ &\leq (1-\alpha)\|x_n - z_1\| + \alpha\|x_n - z_1\| \\ &= \|x_n - z_1\|.\end{aligned}$$

Hence, $\lim_{n\to\infty}\|x_n - z_1\|$ exists. By the same argument, we also have $\lim_{n\to\infty}\|x_n - z_2\|$ exists. Suppose, by the contrary, that $z_1 \neq z_2$. Using Opial's condition, we obtain

$$\limsup_{k\to\infty}\|x_{n_k} - z_1\| < \limsup_{k\to\infty}\|x_{n_k} - z_2\| = \limsup_{k\to\infty}\|x_{m_k} - z_2\|$$

and

$$\limsup_{k\to\infty}\|x_{m_k} - z_2\| < \limsup_{k\to\infty}\|x_{m_k} - z_1\| = \limsup_{k\to\infty}\|x_{n_k} - z_1\|$$

which is a contradiction. Hence $z_1 = z_2$, and the proof is complete. □

Example. Let $A = [0,1] \times [0,1]$ and $B = [2,3] \times [0,1]$. Both are subsets of $\mathbb{R}^2$ under Euclidean norm.

Define a mapping $T: A \to CB(B)$ by

$$T((a,b)) = \begin{cases} \bigcup\limits_{k\in[2,\frac{5}{2}]} \left(\{k\} \times \left[(2x_2-1)(k-2)+\dfrac{2b+1}{2}, \dfrac{1}{2}\right]\right), & \text{if } (a,b) \in \{1\} \times \left[0, \dfrac{1}{2}\right], \\ \bigcup\limits_{k\in[2,\frac{5}{2}]} \left(\{k\} \times \left[\dfrac{1}{2}, \left(\dfrac{3-2b}{2}\right)(a-2)+\dfrac{2x_2+1}{4}\right]\right), & \text{if } (a,b) \in \{1\} \times \left[\dfrac{1}{2}, 1\right], \\ \{a\} \times [b,1] & \text{otherwise.} \end{cases}$$

As we can see, $\operatorname{dist}(A,B) = 1$, $A_0 = \{1\} \times [0,1]$ and $B_0 = \{2\} \times [0,1]$. Moreover, T satisfies all hypothesis of Theorem 3.3. Hence T has a best proximity point. We also provide the numerical result showing that a sequence with the initial point $(1,1)$, generated by Algorithm 3.4 converges to $\left(1, \frac{1}{2}\right)$ which is a best proximity point for T.

Table 1. The value at each step of iteration with the initial point $(1,1)$ and the estimate error

n	(a_n, b_n)	Estimate error
1	(1, 1)	0.5
2	(1, 0.775)	0.275
3	(1, 0.65125)	0.15125
4	(1, 0.58318750)	0.08318750
⋮	⋮	⋮
21	(1, 0.50000320)	0.00000320
22	(1, 0.50000176)	0.00000176
23	(1, 0.50000097)	0.00000097
24	(1, 0.50000053)	0.00000053
25	(1, 0.50000029)	0.00000029

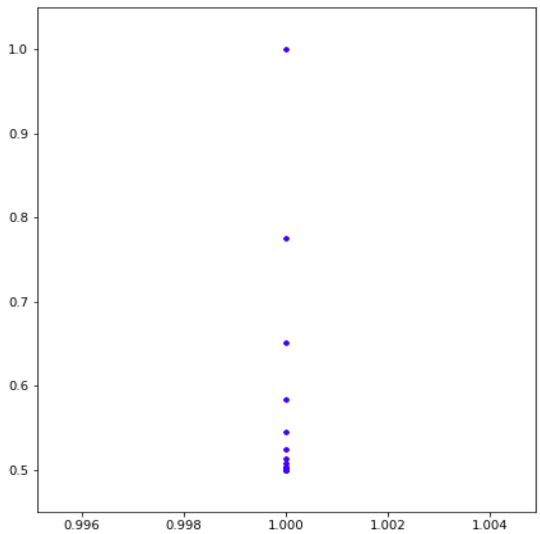
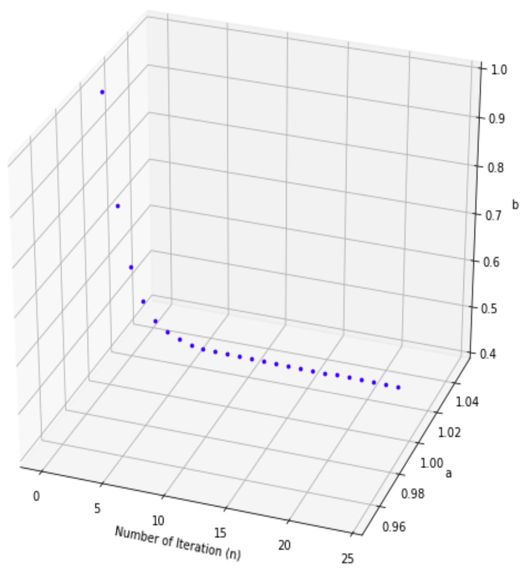


Figure of $x_n = (a_n, b_n)$



Value at each step

Figure 1. Illustrate that the sequence generated by Algorithm 3.4 converges to the best proximity point of T

From Table 1, we observe that $(1, 0.50000029)$ is an approximate solution of best proximity point of T with the iterative number 24 and accuracy of 6 D.P., we also see from Figure 1 that the iterative sequence $\{x_n\}$ converges to the best proximity point $\left(1, \frac{1}{2}\right)$.

4. Conclusion

In this paper, the concept of proximal multi-valued nonexpansive mapping in a Banach space is introduced and the existence of best proximity point of such mapping is also discussed. We also introduced an algorithm for finding such point under some appropriate conditions. Moreover, some numerical experiment of the proposed algorithm is also given. Finally, the readers who are interested in studying this particular concept may consider modifying the Algorithm 3.4 to improve its convergence behavior.

Acknowledgment

This research was supported by Chiang Mai University, Chiang Mai 50200, Thailand.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

References

[1] S. Banach, Sur les oprations dans les ensembles abstraits et leur application aux quations intgrales, *Fund. Math.* **3** (1922), 133 – 181, `http://eudml.org/doc/213289`.

[2] S. S. Basha, Best proximity points: optimal solutions, *J. Optim. Theory Appl.* **151** (2011), 210 – 216, DOI: 10.1007/s10957-011-9869-4.

[3] S. S. Basha and N. Shahzad, Best proximity point theorems for generalized proximal contractions, *Fixed Point Theory Appl.* **42** (2012), DOI: 10.1186/1687-1812-2012-42.

[4] P. Z. Daffer and H. Kaneko, Fixed points of generalized contractive multi-valued mappings, *J. Math. Anal. Appl.* **192** (1995), 655 – 666, DOI: 10.1006/jmaa.1995.1194.

[5] T. D. Benavides and P. L. Ramírez, Fixed point theorems for multivalued nonexpansive mappings without uniform convexity, *Abstr. Appl. Anal.* **2003** (2003), 375 – 386, DOI: 10.1155/S1085337503203080.

[6] E. L. Dozo, Multivalued nonexpansive mappings and Opial's condition, *Proc. Amer. Math. Soc.* **38** (1973), 286 – 292, DOI: 10.1090/S0002-9939-1973-0310718-0.

[7] A. A. Eldred and P. Veeramani, Existence and convergence of best proximity points, *J. Math. Anal. Appl.* **323** (2006), 1001 – 1006, DOI: 10.1016/j.jmaa.2005.10.081.

[8] K. Fan, Extensions of two fixed point theorems of F. E. Browder, *Math. Z.* **122** (1969), 234 – 240, DOI: 10.1007/BF01110225.

[9] A. Fernández-León, Best proximity points for proximal contractions, *J. Nonlinear Convex Anal.* **15** (2014), 313 – 324, `http://hdl.handle.net/11441/43082`.

[10] M. Gabeleh, Best proximity points for weak proximal contractions, *Bull. Malays. Math. Sci. Soc.* **38**(1) (2015), 143 – 154, DOI: 10.1007/s40840-014-0009-9.

[11] M. Gabeleh, Best proximity point theorems via proximal non-self mappings, *J. Optim. Theory Appl.* **164** (2015), 565 – 576, DOI: 10.1007/s10957-014-0585-8.

[12] K. Goebel and W. A. Kirk, Iteration processes for nonexpansive mappings, *Contemp. Math.* **21** (1983), 115 – 123, DOI: 10.2307/2047831.

[13] E. Karapinar, Best proximity points of cyclic mappings, *Appl. Math. Lett.* **25** (2012), 1761 – 1766, DOI: 10.1016/j.aml.2012.02.008.

[14] W. K. Kim and K. H. Lee, Existence of best proximity pairs and equilibrium pairs, *J. Math. Anal. Appl.* **316** (2006), 433 – 446, DOI: 10.1016/j.jmaa.2005.04.053.

[15] N. Mizoguchi and W. Takahashi, Fixed point theorems for multivalued mappings on complete metric spaces, *J. Math. Anal. Appl.* **141** (1989), 177 – 188, DOI: 10.1016/0022-247X(89)90214-X.

[16] S. B. Nadler Jr., Multi-valued contraction mappings, *Pacific J. Math.* **30** (1969), 475 – 487, DOI: 10.1007/BF02771543.

[17] Z. Opial, Weak convergence of the sequence of successive approximations for nonexpansive mappings, *Bull. Amer. Math. Soc.* **73** (1967), 591 – 597, DOI: 10.1090/S0002-9904-1967-11761-0.

[18] S. Pirvbavafa and S. M. Vaezpour, Equilibria of free abstract economies via best proximity point theorems, *Boletín de la Sociedad Matemática Mexicana* **24**(2) (2018), 471 – 481, DOI: 10.1007/s40590-017-0175-5.

[19] Sh. Rezapour, M. Derafshpour and N. Shahzad, Best proximity points of cyclic ϕ-contractions on reflexive Banach spaces, *Fixed Point Theory Appl.* **2010** (2010), Article ID 946178, DOI: 10.1155/2010/946178.

[20] C.-K. Zhong, J. Zhu and P.-H. Zhao, An extension of multi-valued contraction mappings and fixed points, *Proc. Amer. Math. Soc.* **128** (2000), 2439 – 2444, DOI: 10.1090/S0002-9939-99-05318-6.