Communications in Mathematics and Applications
Vol. 10, No. 3, pp. 599–606, 2019
ISSN 0975-8607 (online); 0976-5905 (print)
Published by RGN Publications

http://www.rgnpublications.com

DOI: 10.26713/cma.v10i3.1190

Research Article

Two Classes of Integrals Involving Extended Wright Type Generalized Hypergeometric Function

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Abstract. In this article, our main purpose is to investigate generalized integral formulas containing the extended Wright type generalized hypergeometric function. Moreover, certain special cases of the main results given here have also been pointed out for the Wright type hypergeometric function.

Keywords. Wright type hypergeometric function; Wright type generalized hypergeometric function; hypergeometric function; generalized hypergeometric function

MSC. 33B20; 33C20; 33C45; 33C60; 33B15; 33C05

Received: February 8, 2019 **Accepted:** May 9, 2019

1. Introduction

The hypergeometric function performs an essential role in mathematical analysis and its applications. Many special functions encountered in physics, engineering and probability theory are special cases of hypergeometric functions. Due to various implementations of generalized hypergeometric and Mittag-Leffler functions, many researchers have made their contribution to extend it in various forms. Recently, many authors [1–7, 14–16] introduced several extensions of the well-known special functions due to their importance in mathematical and functional analysis.

Throughout this article, we denote by $\mathbb{Z}_0^-$, $\mathbb{R}^+$ and $\mathbb{C}$ the sets of non positive integers, positive real numbers and complex numbers, respectively.

The classical generalized hypergeometric function [10] has been defined by

$$ {}_r\mathbf{F}_s[\gamma_1,\gamma_2,\dots,\gamma_r;\delta_1,\delta_2,\dots,\delta_s;z] = \sum_{n=0}^{\infty} \frac{(\gamma_1)_n(\gamma_2)_n\dots(\gamma_r)_n}{(\delta_1)_n(\delta_2)_n\dots(\delta_s)_n}\frac{z^n}{n!}, \tag{1.1}$$

where ($|z|<1, r=s+1$) and δ_k does not belong to $\mathbb{Z}_0^-$ for $k=1,2,\dots,s$.

Wright [17] has further extended the generalized hypergeometric function in the following form

$$ {}_r\Psi_s(z) = \sum_{n=0}^{\infty} \frac{\Gamma(\alpha_1+\tau_1 n)\dots\Gamma(\alpha_r+\tau_r n)}{\Gamma(\beta_1+\mu_1 n)\dots\Gamma(\beta_s+\mu_s n)}\frac{z^n}{n!}, \tag{1.2}$$

where τ_i and $\mu_k \in \mathbb{R}^+$ such that

$$1+\sum_{k=1}^{s}\mu_k - \sum_{i=1}^{r}\tau_i > 0$$

when τ_i and μ_k are equal to 1, then equation (1.2) differs from the classical generalized hypergeometric function ${}_r\mathbf{F}_s$ by a constant multiplier only.

Malovichko [12] has investigated the generalization of hypergeometric function, but Dotsenko [9] considered one of the special cases which has the following form

$$ {}_2\mathbf{R}_1^{\omega,\rho}(z) = {}_2\mathbf{R}_1(\delta_1,\delta_2;\delta_3;\omega,\rho;z) = \frac{\Gamma(\delta_3)}{\Gamma(\delta_1)\Gamma(\delta_2)}\sum_{n=0}^{\infty}\frac{\Gamma(\delta_1+n)\Gamma(\delta_2+\frac{\omega}{\rho}n)}{\Gamma(\delta_3+\frac{\omega}{\rho}n)}\frac{z^n}{n!} \tag{1.3}$$

and expressed its integral representation in the following form

$$\begin{aligned} {}_2\mathbf{R}_1^{\omega,\rho}(z) &= {}_2\mathbf{R}_1(\delta_1,\delta_2;\delta_3;\omega,\rho;z) \\ &= \frac{\Gamma(\delta_3)\rho}{\Gamma(\delta_3-\delta_2)\Gamma(\delta_2)}\int_0^1 t^{\rho\delta_2-1}(1-t^{\rho})^{\delta_3-\delta_2-1}(1-zt^{\omega})^{-\delta_1}dt, \end{aligned} \tag{1.4}$$

where $\mathbf{Re}(\delta_3) > \mathbf{Re}(\delta_2) > 0$.

In 2001, Virchenko et al. [8] introduced the said Wright type hypergeometric function by taking $\frac{\omega}{\rho} = \upsilon > 0$ in the following form

$$ {}_2\mathbf{R}_1^{\upsilon}(z) = \frac{\Gamma(\delta_3)}{\Gamma(\delta_2)}\sum_{n=0}^{\infty}\frac{(\delta_1)_n\Gamma(\delta_2+\upsilon n)}{\Gamma(\delta_3+\upsilon n)}\frac{z^n}{n!}, \quad \upsilon>0,\ |z|<1. \tag{1.5}$$

In this article, we establish new integral formulas involving the extended Wright type generalized hypergeometric function (1.2). Further, we give some special cases for the Wright type hypergeometric function (1.5). For the recent consideration we require the following results of Oberhettinger [13], and Lavoie and Trottier [11], respectively

$$\int_0^{\infty} t^{\gamma-1}(t+a+\sqrt{t^2+2at})^{-\delta}dt = 2\delta a^{-\delta}\left(\frac{a}{2}\right)^{\gamma}\frac{\Gamma(2\gamma)\Gamma(\delta-\gamma)}{\Gamma(1+\gamma+\delta)}, \tag{1.6}$$

where $\mathbf{Re}(\delta) > \mathbf{Re}(\gamma) > 0$.

$$\int_0^1 t^{\gamma-1}(1-t)^{2\delta-1}\left(1-\frac{t}{3}\right)^{2\gamma-1}\left(1-\frac{t}{4}\right)^{\delta-1}dt = \left(\frac{2}{3}\right)^{2\gamma}\frac{\Gamma(\gamma)\Gamma(\delta)}{\Gamma(\gamma+\delta)} \tag{1.7}$$

($\mathbf{Re}(\gamma) > 0$, $\mathbf{Re}(\delta) > 0$).

2. Main results

Here, four different formulas involving the extended Wright type generalized hypergeometric function are derived by inserting suitable argument in the integrand of (1.6) and (1.7).

Theorem 2.1. *Let $\delta,\lambda,\alpha_m \in \mathbb{C}$; $m = 1,2,\dots,r$ and $\beta_n \in \mathbb{C}/\mathbb{Z}_0^-$; $n = 1,2,\dots,s$ with* ***Re***$(\delta) > 0$, ***Re***$(\lambda)>0$, *$z>0$ and ρ_m; $m=1,2,\dots,r$, μ_j; $n=1,2,\dots,s$ belong to $\mathbb{R}^+$ such $1+\sum_{n=1}^{s}\mu_n-\sum_{m=1}^{r}\rho_m>0$. Then the following result holds true:*

$$\int_0^\infty z^{\delta-1}(z+a+\sqrt{z^2+2az})^{-\lambda}{}_r\Psi_s\left[\begin{matrix}(\alpha_1,\rho_1),\dots,(\alpha_r,\rho_r);\\(\beta_1,\mu_1),\dots,(\beta_s,\mu_s);\end{matrix}\Big|\frac{x}{z+a+\sqrt{z^2+2az}}\right]dz$$
$$=\frac{\Gamma(2\delta)a^{\delta-\lambda}}{2^{\delta-1}}{}_{r+2}\Psi_{s+2}\left[\begin{matrix}(\alpha_1,\rho_1),\dots,(\alpha_r,\rho_r),(\lambda+1,1),(\lambda-\delta,1);\\(\beta_1,\mu_1),\dots,(\beta_s,\mu_s),(\lambda,1),(1+\lambda+\delta,1);\end{matrix}\Big|\frac{x}{a}\right]. \tag{2.1}$$

Proof. Let I_1 be the left hand side of (2.1). Now applying the series representation of extended Wright type generalized hypergeometric function (1.2) to the integrand of (2.1), we get

$$I_1=\int_0^\infty z^{\delta-1}(z+a+\sqrt{z^2+2az})^{-\lambda}\sum_{k=0}^{\infty}\frac{\Gamma(\alpha_1+\rho_1k)\dots\Gamma(\alpha_r+\rho_rk)}{\Gamma(\beta_1+\mu_1k)\dots\Gamma(\beta_s+\mu_sk)}\frac{x^k}{k!}\frac{1}{(z+a+\sqrt{z^2+2az})^k}dz.$$

By interchanging the order of integration and summation under the assumption of Theorem 2.1, we have

$$I_1=\sum_{k=0}^{\infty}\frac{\Gamma(\alpha_1+\rho_1k)\dots\Gamma(\alpha_r+\rho_rk)}{\Gamma(\beta_1+\mu_1k)\dots\Gamma(\beta_s+\mu_sk)}\frac{x^k}{k!}\int_0^\infty z^{\delta-1}(z+a+\sqrt{z^2+2az})^{-(\lambda+k)}dz.$$

By using the result (1.6), we have

$$I_1=\sum_{k=0}^{\infty}\frac{\Gamma(\alpha_1+\rho_1k)\dots\Gamma(\alpha_r+\rho_rk)}{\Gamma(\beta_1+\mu_1k)\dots\Gamma(\beta_s+\mu_sk)}\frac{x^k}{k!}2(\lambda+k)a^{-(\lambda+k)}\left(\frac{a}{2}\right)^{\delta}\frac{\Gamma(2\delta)\Gamma(\lambda+k-\delta)}{\Gamma(1+\delta+\lambda+k)}$$
$$=\frac{\Gamma(2\delta)a^{\delta-\lambda}}{2^{\delta-1}}\sum_{k=0}^{\infty}\frac{\Gamma(\alpha_1+\rho_1k)\dots\Gamma(\alpha_r+\rho_rk)}{\Gamma(\beta_1+\mu_1k)\dots\Gamma(\beta_s+\mu_sk)}\frac{1}{k!}\left(\frac{x}{a}\right)^k\frac{\Gamma(\lambda+1+k)}{\Gamma(\lambda+k)}\frac{\Gamma(\lambda-\delta+k)}{\Gamma(1+\delta+\lambda+k)}.$$

Upon using the result (1.2), we obtain the required result. □

Theorem 2.2. *Let $\delta,\lambda,\alpha_m \in \mathbb{C}$; $m = 1,2,\dots,r$ and $\beta_n \in \mathbb{C}/\mathbb{Z}_0^-$; $n = 1,2,\dots,s$ with* ***Re***$(\delta) > 0$, ***Re***$(\lambda) > 0$, *$z > 0$ and ρ_m; $m = 1,2,\dots,r$, μ_n; $n = 1,2,\dots,s$ belong to $\mathbb{R}^+$ real positive numbers such $1+\sum_{n=1}^{s}\mu_n-\sum_{m=1}^{r}\rho_m > 0$. Then the following formula holds true:*

$$\int_0^\infty z^{\delta-1}(z+a+\sqrt{z^2+2az})^{-\lambda}{}_r\Psi_s\left[\begin{matrix}(\alpha_1,\rho_1),\dots,(\alpha_r,\rho_r);\\(\beta_1,\mu_1),\dots,(\beta_s,\mu_s);\end{matrix}\Big|\frac{xz}{z+a+\sqrt{z^2+2az}}\right]dz$$
$$=\frac{\Gamma(\lambda-\delta)a^{\delta-\lambda}}{2^{\lambda}}{}_{r+3}\Psi_{s+3}\left[\begin{matrix}(\alpha_1,\rho_1),\dots,(\alpha_r,\rho_r),(\delta,1),(\lambda+1,1),(\delta+\frac{1}{2},1);\\(\beta_1,\mu_1),\dots,(\beta_s,\mu_s),(\lambda,1),(\frac{\lambda+\delta+1}{2},1),(\frac{\lambda+\delta+2}{2},1);\end{matrix}\Big|\frac{x}{2}\right]. \tag{2.2}$$

Proof. Let I_2 be the left hand side of (2.2). Now applying the series representation of extended Wright type generalized hypergeometric function (1.2) to the integrand of (2.2), we get

$$I_2=\int_0^\infty z^{\delta-1}(z+a+\sqrt{z^2+2az})^{-\lambda}\sum_{k=0}^{\infty}\frac{\Gamma(\alpha_1+\rho_1k)\dots\Gamma(\alpha_r+\rho_rk)}{\Gamma(\beta_1+\mu_1k)\dots\Gamma(\beta_s+\mu_sk)}\frac{(xz)^k}{k!}\frac{1}{(z+a+\sqrt{z^2+2az})^k}dz.$$

By interchanging the order of integration and summation under the assumption of Theorem 2.2, we have

$$I_2=\sum_{k=0}^{\infty}\frac{\Gamma(\alpha_1+\rho_1k)\dots\Gamma(\alpha_r+\rho_rk)}{\Gamma(\beta_1+\mu_1k)\dots\Gamma(\beta_s+\mu_sk)}\frac{x^k}{k!}\int_0^{\infty}z^{(\delta+k)-1}(z+a+\sqrt{z^2+2az})^{-(\lambda+k)}dz.$$

By using the result (1.6), we have

$$\begin{aligned}I_2&=\sum_{k=0}^{\infty}\frac{\Gamma(\alpha_1+\rho_1k)\dots\Gamma(\alpha_r+\rho_rk)}{\Gamma(\beta_1+\mu_1k)\dots\Gamma(\beta_s+\mu_sk)}\frac{x^k}{k!}2(\lambda+k)a^{-(\lambda+k)}\left(\frac{a}{2}\right)^{\delta+k}\frac{\Gamma(2\delta+2k)\Gamma(\lambda-\delta)}{\Gamma(1+\delta+\lambda+2k)}\\
&=\frac{\Gamma(\lambda-\delta)a^{\delta-\lambda}}{2^{\delta-1}}\sum_{k=0}^{\infty}\frac{\Gamma(\alpha_1+\rho_1k)\dots\Gamma(\alpha_r+\rho_rk)}{\Gamma(\beta_1+\mu_1k)\dots\Gamma(\beta_s+\mu_sk)}\frac{1}{k!}\left(\frac{x}{2}\right)^k\frac{(\lambda+k)\Gamma(2\delta+2k)}{\Gamma(1+\delta+\lambda+2k)}\\
&=\frac{\Gamma(\lambda-\delta)a^{\delta-\lambda}}{2^{\lambda}}\sum_{k=0}^{\infty}\frac{\Gamma(\alpha_1+\rho_1k)\dots\Gamma(\alpha_r+\rho_rk)\Gamma(\delta+k)\Gamma(\lambda+1+k)\Gamma(\delta+\frac{1}{2}+k)}{\Gamma(\beta_1+\mu_1k)\dots\Gamma(\beta_s+\mu_sk)\Gamma(\lambda+k)\Gamma\left(\frac{\lambda+\delta+1}{2}+k\right)\Gamma(\frac{\lambda+\delta+2}{2}+k)}\left(\frac{x}{2}\right)^k\frac{1}{k!}.\end{aligned}$$

Upon using the result (1.2), we obtain the required result. □

Theorem 2.3. *Let $i,\delta,\alpha_m\in\mathbb{C}$; $m=1,2,\dots,r$ and $\beta_n\in\mathbb{C}/\mathbb{Z}_0^-$; $n=1,2,\dots,s$ with $\boldsymbol{Re}(\delta)>0$, $\boldsymbol{Re}(\delta+i)>0$, $\boldsymbol{Re}(2\delta+i)>0$, $z>0$ and ρ_m; $m=1,2,\dots,r$, μ_n; $n=1,2,\dots,s$ belong to $\mathbb{R}^+$ such $1+\sum_{n=1}^{s}\mu_n-\sum_{m=1}^{r}\rho_m>0$. Then the following formula holds true:*

$$\begin{aligned}&\int_0^1 z^{\delta+i-1}(1-z)^{2\delta-1}\left(1-\frac{z}{3}\right)^{2(\delta+i)-1}\left(1-\frac{z}{4}\right)^{\delta-1}{}_r\Psi_s\left[\begin{matrix}(\alpha_1,\rho_1),\dots,(\alpha_r,\rho_r);\\(\beta_1,\mu_1),\dots,(\beta_s,\mu_s);\end{matrix}\middle|x\left(1-\frac{z}{4}\right)(1-z)^2\right]dz\\
&=\left(\frac{2}{3}\right)^{2(\delta+i)}\Gamma(\delta+i){}_{r+1}\Psi_{s+1}\left[\begin{matrix}(\alpha_1,\rho_1),\dots,(\alpha_r,\rho_r),(\delta,1);\\(\beta_1,\mu_1),\dots,(\beta_s,\mu_s),(2\delta+i,1);\end{matrix}\middle|x\right].\end{aligned}\tag{2.3}$$

Proof. Let I_3 be the left hand side of (2.3). Now applying the series representation of extended Wright type generalized hypergeometric function (1.2) to the integrand of (2.3), we get

$$I_3=\int_0^1 z^{\delta+i-1}(1-z)^{2\delta-1}\left(1-\frac{z}{3}\right)^{2(\delta+i)-1}\left(1-\frac{z}{4}\right)^{\delta-1}\sum_{k=0}^{\infty}\frac{\Gamma(\alpha_1+\rho_1k)\dots\Gamma(\alpha_r+\rho_rk)}{\Gamma(\beta_1+\mu_1k)\dots\Gamma(\beta_s+\mu_sk)}\frac{x^k}{k!}\left(1-\frac{z}{4}\right)^k(1-z)^{2k}dz.$$

By interchanging the order of integration and summation under the assumption of Theorem 2.3, we have

$$I_3=\sum_{k=0}^{\infty}\frac{\Gamma(\alpha_1+\rho_1k)\dots\Gamma(\alpha_r+\rho_rk)}{\Gamma(\beta_1+\mu_1k)\dots\Gamma(\beta_s+\mu_sk)}\frac{x^k}{k!}\int_0^1 z^{\delta+i-1}(1-z)^{2(\delta+k)-1}\left(1-\frac{z}{3}\right)^{2(\delta+i)-1}\left(1-\frac{z}{4}\right)^{(\delta+k)-1}dz.$$

By using the result (1.7), we get

$$\begin{aligned}I_3&=\sum_{k=0}^{\infty}\frac{\Gamma(\alpha_1+\rho_1k)\dots\Gamma(\alpha_r+\rho_rk)}{\Gamma(\beta_1+\mu_1k)\dots\Gamma(\beta_s+\mu_sk)}\frac{x^k}{k!}\left(\frac{2}{3}\right)^{2(\delta+i)}\frac{\Gamma(\delta+i)\Gamma(\delta+k)}{\Gamma(2\delta+i+k)}\\
&=\left(\frac{2}{3}\right)^{2(\delta+i)}\Gamma(\delta+i)\sum_{k=0}^{\infty}\frac{\Gamma(\alpha_1+\rho_1k)\dots\Gamma(\alpha_r+\rho_rk)\Gamma(\delta+k)}{\Gamma(\beta_1+\mu_1k)\dots\Gamma(\beta_s+\mu_sk)\Gamma(2\delta+j+k)}\frac{x^k}{k!}.\end{aligned}$$

Upon using the result (1.2), we obtain the required result. □

Theorem 2.4. *Let $i,\delta,\alpha_m\in\mathbb{C}$; $m=1,2,\dots,r$ and $\beta_n\in\mathbb{C}/\mathbb{Z}_0^-$; $n=1,2,\dots,s$ with $\boldsymbol{Re}(\delta)>0$, $\boldsymbol{Re}(\delta+i)>0$, $\boldsymbol{Re}(2\delta+i)>0$, $z>0$ and ρ_m; $m=1,2,\dots,r$, μ_n; $n=1,2,\dots,s$ belong to $\mathbb{R}^+$ such $1+\sum_{n=1}^{s}\mu_n-\sum_{m=1}^{r}\rho_m>0$. Then the following formula holds true:*

$$\int_0^1 z^{\delta-1}(1-z)^{2(\delta+i)-1}\left(1-\frac{z}{3}\right)^{2\delta-1}\left(1-\frac{z}{4}\right)^{\delta+i-1}{}_r\Psi_s\left[\begin{matrix}(\alpha_1,\rho_1),\dots,(\alpha_r,\rho_r);\\(\beta_1,\mu_1),\dots,(\beta_s,\mu_s);\end{matrix}\middle|xz\left(1-\frac{z}{3}\right)^2\right]dz$$

$$= \left(\frac{2}{3}\right)^{2\delta} \Gamma(\delta+i)_{r+1}\Psi_{s+1}\left[\begin{array}{l}(\alpha_1,\rho_1),\ldots,(\alpha_r,\rho_r),(\delta,1);\\ (\beta_1,\mu_1),\ldots,(\beta_s,\mu_s),(2\delta+i,1);\end{array}\middle|\frac{4x}{9}\right]. \tag{2.4}$$

Proof. Let I_4 be the left hand side of (2.4). Now applying the series representation of extended Wright type generalized hypergeometric function (1.2) to the integrand of (2.4), we get

$$I_4 = \int_0^1 z^{\delta-1}(1-z)^{2(\delta+i)-1}\left(1-\frac{z}{3}\right)^{2\delta-1}\left(1-\frac{z}{4}\right)^{(\delta+i)-1}\sum_{k=0}^{\infty}\frac{\Gamma(\alpha_1+\rho_1 k)\ldots\Gamma(\alpha_r+\rho_r k)}{\Gamma(\beta_1+\mu_1 k)\ldots\Gamma(\beta_s+\mu_s k)}\frac{x^k z^k}{k!}\left(1-\frac{z}{3}\right)^{2k} dz.$$

By interchanging the order of integration and summation under the assumption of Theorem 2.4, we have

$$I_4 = \sum_{k=0}^{\infty}\frac{\Gamma(\alpha_1+\rho_1 k)\ldots\Gamma(\alpha_r+\rho_r k)}{\Gamma(\beta_1+\mu_1 k)\ldots\Gamma(\beta_s+\mu_s k)}\frac{x^k}{k!}\int_0^1 z^{\delta+k-1}(1-z)^{2(\delta+i)-1}\left(1-\frac{z}{3}\right)^{2(\delta+k)-1}\left(1-\frac{z}{4}\right)^{(\delta+i)-1} dz.$$

By using the result (1.7), we get

$$I_4 = \sum_{k=0}^{\infty}\frac{\Gamma(\alpha_1+\rho_1 k)\ldots\Gamma(\alpha_r+\rho_r k)}{\Gamma(\beta_1+\mu_1 k)\ldots\Gamma(\beta_s+\mu_s k)}\frac{x^k}{k!}\left(\frac{2}{3}\right)^{2(\delta+k)}\frac{\Gamma(\delta+i)\Gamma(\delta+k)}{\Gamma(2\delta+i+k)}$$

$$= \left(\frac{2}{3}\right)^{2\delta}\Gamma(\delta+i)\sum_{k=0}^{\infty}\frac{\Gamma(\alpha_1+\rho_1 k)\ldots\Gamma(\alpha_r+\rho_r k)\Gamma(\delta+k)}{\Gamma(\beta_1+\mu_1 k)\ldots\Gamma(\beta_s+\mu_s k)\Gamma(2\delta+i+k)}\left(\frac{4x}{9}\right)^k\frac{1}{k!}.$$

Upon using the result (1.2), we obtain the required result. □

3. Special Cases

In this portion, we present some particular cases of (2.1), (2.2), (2.3) and (2.4) as corollaries given below for Wright type hypergeometric function (1.5).

Corollary 3.1. *Let $\alpha,\beta,\delta\in\mathbb{C}$ and $\gamma\in\mathbb{C}\ \mathbb{Z}_0^-$ with $v>0$, $\boldsymbol{Re}(\lambda)>\boldsymbol{Re}(\delta)>0$, $z>0$ and $|z|<1$. Then the following formula holds true.*

$$\int_0^{\infty} z^{\delta-1}(z+a+\sqrt{z^2+2az})^{-\lambda}\,{}_2\mathbf{R}_1^{v}\left(\frac{x}{z+a+\sqrt{z^2+2az}}\right)dz$$

$$= \frac{\Gamma(\gamma)\Gamma(2\delta)a^{\delta-\lambda}}{\Gamma(\beta)\Gamma(\alpha)2^{\delta-1}}\,{}_4\Psi_3\left[\begin{array}{l}(\alpha,1),(\beta,v),(\lambda+1,1),(\lambda-\delta,1);\\ (\gamma,v),(\lambda,1),(1+\delta+\lambda,1);\end{array}\middle|\frac{x}{a}\right]. \tag{3.1}$$

Proof. Let I_5 be the left hand side of (3.1). Now applying the series representation of Wright type hypergeometric function (1.5) to the integrand of (3.1), we get

$$I_5 = \int_0^{\infty} z^{\delta-1}(z+a+\sqrt{z^2+2az})^{-\lambda}\frac{\Gamma(\gamma)}{\Gamma(\beta)}\sum_{k=0}^{\infty}\frac{(\alpha)_k\Gamma(\beta+vk)}{\Gamma(\gamma+vk)k!}\left(\frac{x}{z+a+\sqrt{z^2+2az}}\right)^k dz.$$

By interchanging the order of integration and summation under the assumption of Corollary 3.1, we have

$$I_5 = \frac{\Gamma(\gamma)}{\Gamma(\beta)}\sum_{k=0}^{\infty}\frac{(\alpha)_k\Gamma(\beta+vk)}{\Gamma(\gamma+vk)}\frac{x^k}{k!}\int_0^{\infty} z^{\delta-1}(z+a+\sqrt{z^2+2az})^{-(\lambda+k)}dz$$

By using the result (1.6), we have

$$I_5 = \frac{\Gamma(\gamma)}{\Gamma(\beta)}\sum_{k=0}^{\infty}\frac{(\alpha)_k\Gamma(\beta+vk)}{\Gamma(\gamma+vk)}\frac{x^k}{k!}2(\lambda+k)a^{-(\lambda+k)}\left(\frac{a}{2}\right)^{\delta}\frac{\Gamma(2\delta)\Gamma(\lambda-\delta+k)}{\Gamma(1+\delta+\lambda+k)}$$

$$= \frac{\Gamma(\gamma)\Gamma(2\delta)a^{\delta-\lambda}}{\Gamma(\alpha)\Gamma(\beta)2^{\delta-1}} \sum_{k=0}^{\infty} \frac{\Gamma(\alpha+k)\Gamma(\beta+vk)\Gamma(\lambda+1+k)\Gamma(\lambda-\delta+k)}{\Gamma(\gamma+vk)\Gamma(\lambda+k)\Gamma(1+\delta+\lambda+k)} \left(\frac{x}{a}\right)^k \frac{1}{k!}.$$

Upon using the result (1.2), we obtain the required result. □

Corollary 3.2. *Let $\alpha,\beta,\delta \in \mathbb{C}$ and $\gamma \in \mathbb{C}\ \mathbb{Z}_0^-$ with $v>0$, $\boldsymbol{Re}(\lambda) > \boldsymbol{Re}(\delta) > 0$, $z>0$ and $|z|<1$. Then the following formula holds true.*

$$\int_0^{\infty} z^{\delta-1}(z+a+\sqrt{z^2+2az})^{-\lambda}{}_2\mathbf{R}_1^v\left(\frac{xz}{z+a+\sqrt{z^2+2az}}\right)dz$$
$$= \frac{\Gamma(\gamma)\Gamma(\lambda-\delta)a^{\delta-\lambda}}{\Gamma(\alpha)\Gamma(\beta)2^{\lambda}}{}_5\Psi_4\left[\begin{matrix}(\alpha,1),(\beta,v),(\lambda+1,1),(\delta,1),(\delta+\frac{1}{2},1);\\ (\gamma,v),(\lambda,1),(\frac{\lambda+\delta+1}{2},1),(\frac{\lambda+\delta+2}{2},1);\end{matrix}\ \Bigg|\frac{x}{2}\right] \tag{3.2}$$

Proof. Let I_6 be the left hand side of (3.2). Now applying the series representation of Wright type hypergeometric function (1.5) to the integrand of (3.2), we get

$$I_6 = \int_0^{\infty} z^{\delta-1}(z+a+\sqrt{z^2+2az})^{-\lambda}\frac{\Gamma(\gamma)}{\Gamma(\beta)}\sum_{k=0}^{\infty}\frac{(\alpha)_k\Gamma(\beta+vk)}{\Gamma(\gamma+vk)k!}\left(\frac{xz}{z+a+\sqrt{z^2+2az}}\right)^k dz.$$

By interchanging the order of integration and summation under the assumption of Corollary 3.2, we have

$$I_6 = \frac{\Gamma(\gamma)}{\Gamma(\beta)}\sum_{k=0}^{\infty}\frac{(\alpha)_k\Gamma(\beta+vk)}{\Gamma(\gamma+vk)}\frac{x^k}{k!}\int_0^{\infty} z^{(\delta+k)-1}(z+a+\sqrt{z^2+2az})^{-(\lambda+k)}dz.$$

By using the result (1.6), we have

$$I_6 = \frac{\Gamma(\gamma)}{\Gamma(\beta)}\sum_{k=0}^{\infty}\frac{(\alpha)_k\Gamma(\beta+vk)}{\Gamma(\gamma+vk)}\frac{x^k}{k!}2(\lambda+k)a^{-(\lambda+k)}\left(\frac{a}{2}\right)^{\delta+k}\frac{\Gamma(2\delta+2k)\Gamma(\lambda-\delta)}{\Gamma(1+\delta+\lambda+2k)}$$
$$= \frac{\Gamma(\gamma)\Gamma(\lambda-\delta)a^{\delta-\lambda}}{\Gamma(\alpha)\Gamma(\beta)2^{\lambda}}\sum_{k=0}^{\infty}\frac{\Gamma(\alpha+k)\Gamma(\beta+vk)\Gamma(\lambda+1+k)\Gamma(\delta+k)\Gamma(\delta+\frac{1}{2}+k)}{\Gamma(\gamma+vk)\Gamma(\lambda+k)\Gamma(\frac{1+\delta+\lambda}{2}+k)\Gamma(\frac{2+\delta+\lambda}{2}+k)}\left(\frac{x}{2}\right)^k\frac{1}{k!}.$$

Upon using the result (1.2), we obtain the required result. □

Corollary 3.3. *Let $\alpha,\beta,\delta,i \in \boldsymbol{C}$ and $\gamma \in \mathbb{C}\ \mathbb{Z}_0^-$ with $v>0$, $\boldsymbol{Re}(\delta)>0$, $\boldsymbol{Re}(\delta+i)>0$, $\boldsymbol{Re}(2\delta+i)>0$, $z>0$ and $|z|<1$. Then the following formula holds true.*

$$\int_0^1 z^{\delta+i-1}(1-z)^{2\delta-1}(1-\frac{z}{3})^{2(\delta+i)-1}\left(1-\frac{z}{4}\right)^{\delta-1}{}_2\mathbf{R}_1^v\left(x\left(1-\frac{z}{4}\right)(1-z)^2\right)dz$$
$$= \left(\frac{2}{3}\right)^{2(\delta+i)}\frac{\Gamma(\gamma)\Gamma(\delta+i)}{\Gamma(\alpha)\Gamma(\beta)}{}_3\Psi_2\left[\begin{matrix}(\alpha,1),(\beta,v),(\delta,1);\\ (\gamma,v),(2\delta+i,1);\end{matrix}\ \Bigg|x\right]. \tag{3.3}$$

Proof. Let I_7 be the left hand side of (3.3). Now applying the series representation of Wright type hypergeometric function (1.5) to the integrand of (3.3), we get

$$I_7 = \int_0^1 z^{\delta+i-1}(1-z)^{2\delta-1}\left(1-\frac{z}{3}\right)^{2(\delta+i)-1}\left(1-\frac{z}{4}\right)^{\delta-1}\frac{\Gamma(\gamma)}{\Gamma(\beta)}\sum_{k=0}^{\infty}\frac{(\alpha)_k\Gamma(\beta+vk)}{\Gamma(\gamma+vk)k!}\left(x\left(1-\frac{z}{4}\right)(1-z)^2\right)^k dz.$$

By interchanging the order of integration and summation under the assumption of Corollary 3.3, we have

$$I_6 = \frac{\Gamma(\gamma)}{\Gamma(\beta)}\sum_{k=0}^{\infty}\frac{(\alpha)_k\Gamma(\beta+vk)}{\Gamma(\gamma+vk)}\frac{x^k}{k!}\int_0^1 z^{(\delta+i)-1}(1-z)^{2(\delta+k)-1}\left(1-\frac{z}{3}\right)^{2(\delta+i)-1}\left(1-\frac{z}{4}\right)^{(\delta+k)-1}dz.$$

By using the result (1.7), we have

$$I_7 = \frac{\Gamma(\gamma)}{\Gamma(\beta)}\sum_{k=0}^{\infty}\frac{(\alpha)_k\Gamma(\beta+vk)}{\Gamma(\gamma+vk)}\frac{x^k}{k!}\left(\frac{2}{3}\right)^{2(\delta+i)}\frac{\Gamma(\delta+i)\Gamma(\delta+k)}{\Gamma(2\delta+i+k)}$$
$$= \left(\frac{2}{3}\right)^{2(\delta+i)}\frac{\Gamma(\gamma)\Gamma(\delta+i)}{\Gamma(\alpha)\Gamma(\beta)}\sum_{k=0}^{\infty}\frac{\Gamma(\alpha+k)\Gamma(\beta+vk)\Gamma(\delta+k)}{\Gamma(\gamma+vk)\Gamma(2\delta+i+k)}\frac{x^k}{k!}.$$

Upon using the result (1.2), we obtain the required result. □

Corollary 3.4. *Let* $\alpha,\beta,\delta,i \in \boldsymbol{C}$ *and* $\gamma \in \mathbb{C}\ \mathbb{Z}_0^-$ *with* $v > 0$, $\boldsymbol{Re}(\delta) > 0$, $\boldsymbol{Re}(\delta+i) > 0$, $\boldsymbol{Re}(2\delta+i) > 0$, $z > 0$ *and* $|z| < 1$. *Then the following formula holds true.*

$$\int_0^1 z^{\delta-1}(1-z)^{2(\delta+i)-1}\left(1-\frac{z}{3}\right)^{2\delta-1}\left(1-\frac{z}{4}\right)^{(\delta+i)-1}{}_2\mathbf{R}_1^{v}\left(xz\left(1-\frac{z}{3}\right)^2\right)dz$$
$$= \left(\frac{2}{3}\right)^{2\delta}\frac{\Gamma(\gamma)\Gamma(\delta+i)}{\Gamma(\alpha)\Gamma(\beta)}\,{}_3\Psi_2\left[\begin{array}{l}(\alpha,1),(\beta,v),(\delta,1);\\ (\gamma,v),(2\delta+i,1);\end{array}\middle|\frac{4x}{9}\right]. \tag{3.4}$$

Proof. Let I_8 be the left hand side of (3.4). Now applying the series representation of Wright type hypergeometric function (1.5) to the integrand of (3.4), we get

$$I_8 = \int_0^1 z^{\delta-1}(1-z)^{2(\delta+i)-1}\left(1-\frac{z}{3}\right)^{2\delta-1}\left(1-\frac{z}{4}\right)^{(\delta+i)-1}\frac{\Gamma(\gamma)}{\Gamma(\beta)}\sum_{k=0}^{\infty}\frac{(\alpha)_k\Gamma(\beta+vk)}{\Gamma(\gamma+vk)k!}\left(xz\left(1-\frac{z}{3}\right)^2\right)^n dz.$$

By interchanging the order of integration and summation under the assumption of Corollary 3.4, we have

$$I_8 = \frac{\Gamma(\gamma)}{\Gamma(\beta)}\sum_{k=0}^{\infty}\frac{(\alpha)_k\Gamma(\beta+vk)}{\Gamma(\gamma+vk)}\frac{x^k}{k!}\int_0^1 z^{(\delta+k)-1}(1-z)^{2(\delta+i)-1}\left(1-\frac{z}{3}\right)^{2(\delta+k)-1}\left(1-\frac{z}{4}\right)^{(\delta+i)-1}dz.$$

By using the result (1.7), we have

$$I_8 = \frac{\Gamma(\gamma)}{\Gamma(\beta)}\sum_{k=0}^{\infty}\frac{(\alpha)_k\Gamma(\beta+vk)}{\Gamma(\gamma+vk)}\frac{x^k}{k!}\left(\frac{2}{3}\right)^{2(\delta+k)}\frac{\Gamma(\delta+k)\Gamma(\delta+i)}{\Gamma(2\delta+i+k)}$$
$$= \left(\frac{2}{3}\right)^{2\delta}\frac{\Gamma(\gamma)\Gamma(\delta+i)}{\Gamma(\alpha)\Gamma(\beta)}\sum_{k=0}^{\infty}\frac{\Gamma(\alpha+k)\Gamma(\beta+vk)\Gamma(\delta+k)}{\Gamma(\gamma+vk)\Gamma(2\delta+i+k)k!}\left(\frac{4x}{9}\right)^k.$$

Upon using the result (1.2), we obtain the required result. □

4. Conclusion

Generalized integral formulae containing the Wright type generalized hypergeometric function are explored in this section. Additionally, the special cases of main results have also been established for the Wright type hypergeometric function.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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