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On the Solution of Reduced Wave Equation with Damping

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Abstract. In this paper we find particular solutions of Reduced wave equation with damping in the form $\Delta u + k^2 n(\boldsymbol{x})u + \mu|\nabla u| = 0$, $\mathbb{R}^n$, $\mu \in \mathbb{R}$ and $n(\boldsymbol{x})$ is a continuous function on Ω, by making use of Fundamental solution $u = \frac{\exp(ikR)}{R}$ of the scalar Helmholtz equation and employing a variation of constant technique. Moreover, some examples are given to illustrate the importance of our results.

1. Introduction

Let $\boldsymbol{x}$ be an arbitrary point and $\boldsymbol{y}$ a fixed points in a domain $\Omega \subset \mathbb{R}^m$, $m \geq 2$. Let $R = d(\boldsymbol{x},\boldsymbol{y}) = |\boldsymbol{x}-\boldsymbol{y}| = \sqrt{\sum_{j=1}^m (x_j - y_j)^2}$ denote the distance between $\boldsymbol{x}$ and $\boldsymbol{y}$. Let $k > 0$ be given such that $R \neq \frac{n\pi}{k}$, $n = 1, 2, 3, \dots$

Consider the second-order partial differential equation of the form

$$\Delta u + k^2 n(\boldsymbol{x})u + \mu|\nabla u| = 0, \quad \boldsymbol{x} \in \Omega \subset \mathbb{R}^m,\ \mu \in \mathbb{R} \tag{1.1}$$

where μ is an arbitrary constant and $n(\boldsymbol{x})$ is a continuous function on Ω.

If $\mu = 0$ and $n(\boldsymbol{x}) = 0$ then (1.1) is known Laplace equation, $\mu = 0$ and $n(\boldsymbol{x}) \equiv 1$ for all $\boldsymbol{x}$ in Ω, then (1.1) is known scalar Helmholtz equation and $\mu = 0$ and $n(\boldsymbol{x}) \neq \{0, 1\}$ is a continuous function on Ω, then (1.1) is known reduced wave equation.

The reduced wave equation (or scalar Helmholtz equation) is an important partial differential equation that describes a variety of waves, such as sound, light and water waves. It arises in acoustic, electromagnetic and fluid dynamics for $m = 2, 3$. Helmholtz equation naturally appears from general conservation laws of physics and can be interpreted as a wave equation for monochromatic waves (wave equation in the frequency domain). Helmholtz equation can also be derived from the heat conduction equation, Schr dinger equation, telegraph and other wave type, or evolutionary, equations. In physically applications k and $n(\boldsymbol{x})$ are known wave number and refraction index of media respectively see details [5, 8].

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In [3, 13] some particular solution have been constructed for scalar Helmholtz equation and reduced wave equation in two dimensions. Although 3-D problems are more realistic physically, their solutions are found rarely in the literature. Hence in this paper we investigate the solution of (1.1) for $m = 3$.

The general solution of the scalar Helmholtz equation with radial and polar coordinates in three dimensions can be obtained by using separation of variable as follows

$$u(r,\theta,\varphi) = \sum_{n=0}^{\infty}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}(a_{nlm}j_n(kr) + b_{nlm}y_n(kr))Y_l^m(\theta,\varphi),$$

where $j_n(kr)$ and $y_n(kr)$ are the Bessel functions,and $Y_l^m(\theta,\varphi)$ is the spherical harmonics [1].

Very few elementary or closed-form solutions of the reduced wave equation are known for $m = 3$ for the case with variable index of refraction n. For layered media $[n = n(x_2)]$, only two solutions have been found so far. These are (i) Pekeris' solution [12] for a point source in a medium specified by $n = x_2^{-1}$,

$$u(\boldsymbol{x},\boldsymbol{y}) = \frac{2(x_2y_2)^{\frac{1}{2}}}{RR'}\exp\left[2i\left(k^2 - \frac{1}{4}\right)^{\frac{1}{2}}\tanh^{-1}\left(\frac{R}{R'}\right)\right],$$

where $i = \sqrt{-1}$ and $R'^2 = (x_1 - y_1)^2 + (x_2 + y_2)^2 + (x_3 - y_3)^2$; and (ii) Kormilitsin's solution [10] for a line source extending parallel to the x_3-axis in a media specified by $n = \sqrt{x_2}$,

$$u(\boldsymbol{x},\boldsymbol{y}) = \int_0^{\infty}\exp\left[ik\left(\frac{Q^2}{2\zeta} + (x_2 + y_2)\frac{\zeta}{4} - \frac{\zeta^3}{96}\right)\right]\frac{d\zeta}{\zeta},$$

where $Q^2 = (x_1 - y_1)^2 + (x_2 - y_2)^2$. In [9], R. L. Holford shows that an elementary solution of the reduced wave equation can be found for a line source extending parallel to the x_3-axis in a medium specified by

$$n(\boldsymbol{x}) = \sqrt{A + Bx_1 + Cx_2 + Dx_1^2 + Ex_1y_1 + Fx_2^2},$$

where A, B, C, D and F are arbitrary constants, and for a point source excitation when $n(\boldsymbol{x}) = \sqrt{A + Cx_2 + Fx_2^2}$; i.e., when the medium is layered. In both cases, the solution is obtained in the form

$$u(\boldsymbol{x},\boldsymbol{y}) = \int_C \exp[ikf(\boldsymbol{x},\boldsymbol{y},\zeta)]g(\zeta)d\zeta,$$

where f and g are elementary functions of complex variable ζ, and C is a path running from $\zeta = 0$ to ∞.

However in recent years a lot of useful numerical methods and numerical solutions have been presented for the reduced wave equation with variable coefficient (some times called nonlinear Helmholtz equation) [2, 6, 7].

Straightforward differentiation shows that for fixed $\boldsymbol{y} \in \mathbb{R}^3$

$$u(\boldsymbol{x},\boldsymbol{y}) := \frac{1}{4\pi}\frac{e^{ikR}}{R}\,, \quad \boldsymbol{x} \neq \boldsymbol{y}\,,$$

satisfies Helmholtz equation, which is known as Fundamental solution of the Helmholtz equation [5, 8].

2. **Variation of Constant Method**

Separating the fundamental solution of scalar Helmholtz equation into real and imaginary parts we have

$$c_1\frac{\cos kR}{R} \quad \text{and} \quad c_2\frac{\sin kR}{R}$$

as real solutions.

In this study our aim is to find particular solutions of (1.1) by using the above solutions of scalar Helmholtz equation. In particular, following the method of variation of constant we look for solutions of the form

$$u = \frac{1}{f(R)}\frac{\sin kR}{R}\,, \tag{2.1}$$

where $f(R)$ is to be determined. We note that if we use $u = \frac{1}{f(R)}\frac{\cos kR}{f(R)}$ we do not obtain any new solution. Therefore, we use only (2.1). Naturally, we assume that f is a continuous function having first and second derivatives on Ω.

Define

$$P(\boldsymbol{x}) := k^2(1 - n(\boldsymbol{x})) \pm \mu\left(\frac{1}{R} - k\cot kR\right),$$

where $n(\boldsymbol{x})$, μ are as defined above.

Lemma 2.1. *Let $f : \Omega \subset \mathbb{R}^3 \to \mathbb{R}\backslash\{0\}$ be a continuous function that has first and second derivatives and satisfies the equation*

$$P(R) = -\frac{f''(R)}{f(R)} + 2\frac{(f'(R))^2}{f^2(R)} + (-2k\cot kR \pm \mu)\frac{f'(R)}{f(R)} \tag{2.2}$$

then

$$u(\boldsymbol{x}) = \frac{\sin kR}{Rf(R)}, \quad R = |\boldsymbol{x} - \boldsymbol{y}| \tag{2.3}$$

satisfies (1.1).

Proof. Using the chain rule we get for $j = 1,2,3$

$$\frac{\partial u}{\partial x_j} = \left[\frac{k\cos kR}{R^2 f(R)} - \frac{\sin kR}{R^3 f(R)} - \frac{f'(R)\sin kR}{R^2 f^2(R)}\right](x_{j-}y_j).$$

Moreover,

$$\frac{\partial^2 u}{\partial x_j^2} = \frac{\cos kR}{Rf(R)}\left(\frac{k}{R}\right) - \frac{\sin kR}{Rf(R)}\left(\frac{1}{R^2} + \frac{f'(R)}{Rf(R)}\right)$$
$$+ \frac{(x_{j-}y_j)^2}{R}\left\{\frac{\sin kR}{Rf(R)}\Psi_1(R) + \frac{\cos kR}{Rf(R)}\Psi_2(R)\right\}.$$

Where

$$\Psi_1(R) = -\frac{k^2}{R} + \frac{3}{R^3} + \frac{3f'(R)}{R^2 f(R)} - \frac{f''(R)}{Rf(R)} + \frac{2f'^2(R)}{Rf^2(R)}$$

and

$$\Psi_2(R) = -\frac{3k}{R^2} - \frac{2kf'(R)}{Rf(R)}.$$

Then we get

$$\Delta u = \sum_{j=1}^{3}\frac{\partial^2 u}{\partial x_i^2} = \frac{\sin kR}{rf(R)}\left(-k^2 - \frac{f''(R)}{f(R)} + 2\frac{f'^2(R)}{f^2(R)} - 2k\cot kR\frac{f'(R)}{f(R)}\right)$$

and because of

$$\nabla u = \left(\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \frac{\partial u}{\partial x_3}\right)$$
$$= \left(\frac{k\cos kR}{R^2 f} - \frac{\sin kR}{R^3 f} - \frac{f'\sin kR}{R^2 f^2}\right)(x_{1-}y_1, x_{2-}y_2, x_{3-}y_3)$$

we get

$$\Delta u + \mu|\nabla u| = u\left[-k^2 \mp \mu\left(k\cot kR - \frac{1}{R}\right) + P\right].$$

This completes the proof of lemma because of our assumption. □

As an initial simplification, we will choose a constant k such that the (2.2) has no critical point. We can give an exponential type solution of (1.1) as follow.

Theorem 2.2. *Let P and f satisfies* (2.2) *in Lemma* 1 *and ϕ be a continuously differentiable function, such that ϕ satisfies the Riccati differential equation*

$$\phi' + \phi^2 = g, \tag{2.4}$$

where

$$g = P - k^2 \pm \mu\left(k\cot kR - \frac{1}{R}\right) \pm \frac{\mu}{R} + \frac{\mu^2}{4} = -k^2 n(\boldsymbol{x}) \pm \frac{\mu}{R} + \frac{\mu^2}{4}$$

then

$$u = c\left(\frac{1}{R}\right)\exp\left(\mp\frac{\mu R}{2}\right)\exp\left(\int^R \phi(t)dt\right), \text{ where } c \text{ is any constant} \tag{2.5}$$

is the solution of (1.1).

Proof. If we multiply the (2.2) by $f^2(R)$, then we get

$$f(R)f''(R) - 2f'^2(R) - (2k\cot kr \pm \mu)f(R)f'(R) + Pf^2(R) = 0. \qquad (2.6)$$

If we use the Riccati substitution

$$f'(R) = -f(R)\psi, \qquad (2.7)$$

where ψ is a continuously differentiable function, then we have

$$\left.\begin{aligned} &f'^2(R) = f^2(R)\psi^2,\\ &f(R)f'(R) = -f^2(R)\psi,\\ &f(R)f''(R) = f^2(R)(\psi^2 - \psi'). \end{aligned}\right\} \qquad (2.8)$$

If we substitute (2.8) in (2.6), we get

$$\psi' + \psi^2 = -(2k\cot kr \pm \mu)\psi + P. \qquad (2.9)$$

Again if we consider the change of variables

$$\psi = \phi - k\cot kr \mp \frac{\mu}{2}, \qquad (2.10)$$

where ϕ is a continuously differentiable function, then (2.9) becomes a classical Riccati equation in the form $\phi' + \phi^2 = -k^2 n(x) \pm \frac{\mu}{2} + \frac{\mu^2}{4}$, which completes the proof of the theorem. □

Corollary 2.3. *If there exists a continuous function P such that $k^2 n(\boldsymbol{x}) = k^2 - P$ and f satisfies the differential equation*

$$P(R) = -\frac{f''(R)}{f(R)} + 2\frac{(f'(R))^2}{f^2(R)} - 2k(\cot kR)\frac{f'(R)}{f(R)} \qquad (2.11)$$

then

$$u = c\left(\frac{1}{R}\right)\exp\left(\int^R \phi(t)dt\right), \quad \textit{where } c \textit{ is any constant} \qquad (2.12)$$

satisfies the partial differential equation

$$\Delta u + k^2 n(\boldsymbol{x})u = 0, \quad \boldsymbol{x} \in \Omega \subset \mathbb{R}^3. \qquad (2.13)$$

It is known that the solution of (2.4) is equivalent to an exponential solution of the linear differential equation

$$z'' = gz. \qquad (2.14)$$

Here we note that,in order to construct the solution of (1.1) by using the Theorem1,we must solve the differential equation (2.4) or equivalently (2.14). But, in general to solve the differential equation (2.4) or (2.14) is not easy. Fortunately, we present an alternative solution procedure for differential equation (2.14) which works under the some condition on $k^2 n(\boldsymbol{x})$.

3. **The Iteration Technique**

Let $w_0' = h_0 \in C^\infty(a,b)$ and consider the equation

$$z''(t) = w_0(t)z(t)\,. \tag{3.1}$$

For some w_0 function we shall give a new method to obtain the particular solutions of equation (2.14). This method depends on finding some symmetric structure as in [4] by using asymptotic behavior of equation (2.14).

Thus for this purpose if we differentiate (3.1) with respect to the t, we find that

$$z''' = w_0 z' + h_0 z\,. \tag{3.2}$$

Again if we differentiate (3.2) with respect to the t, we find that

$$z^{(4)} = w_1 z' + h_1 z\,, \tag{3.3}$$

where $w_1 = w_0' + h_0$ and $h_1 = h_0' + w_0 w_0$.

If we differentiate again (3.3) with respect to the t, we find that

$$z^{(5)} = w_2 z' + h_2 z\,, \tag{3.4}$$

where $w_2 = w_1' + h_1$ and $h_2 = h_1' + w_1 w_0$.

Thus if we continue in this way,we get for $n \geq 0$,

$$z^{(n+3)} = w_n z' + h_n z \tag{3.5}$$

and similarly

$$z^{(n+4)} = w_{n+1} z' + h_{n+1} z\,, \tag{3.6}$$

where

$$w_i = w_{i-1}' + h_{i-1} \quad \text{and} \quad h_i = h_{i-1}' + w_{i-1} w_0\,, \quad \text{for } i = 1,2,\ldots,n. \tag{3.7}$$

From the ratio of the $(n+4)$th and $(n+3)$th derivatives, we get

$$\frac{d}{dt}\left(\ln z^{(n+3)}\right) = \frac{z^{(n+4)}}{z^{(n+3)}} = \frac{w_{n+1}\left(z' + \frac{h_{n+1}}{w_{n+1}} z\right)}{w_n\left(z' + \frac{h_n}{w_n} z\right)}\,. \tag{3.8}$$

If we have, for sufficiently large $n \geq 0$,

$$\gamma(t) := \frac{h_{n+1}}{w_{n+1}} = \frac{h_n}{w_n} \tag{3.9}$$

then (3.8) reduces to

$$\frac{d}{dt}\left(\ln z^{(n+3)}\right) = \frac{w_{n+1}}{w_n}$$

which yields

$$z^{(n+3)} = c_1 \exp\left(\int^t \frac{w_{n+1(\tau)}}{w_n(\tau)} d\tau\right). \tag{3.10}$$

But in Eq. (2.10) the integrand function is

$$\frac{w_{n+1}}{w_n} = \frac{w'_n}{w_n} + \frac{h_n}{w_n} = \frac{w'_n}{w_n} + \gamma \,.$$

Then (3.10) becomes

$$z^{(n+3)} = c_1 w_n \exp\left(\int^t \gamma(\tau) d\tau \right). \tag{3.11}$$

Substituting (3.11) into (3.5) we obtain the first-order differential equation

$$z' + \gamma z = c_1 \exp\left(\int^r \gamma(\tau) d\tau \right). \tag{3.12}$$

Thus we get the general solution of (3.12) as

$$z(t) = \exp\left(- \int^t \gamma(\tau) d\tau \right) \left[c_1 \int^t \exp\left(\int^v 2\gamma(\tau) d\tau \right) dv + c_2 \right], \tag{3.13}$$

where c_1 and c_2 are arbitrary constants.

Note that in [11] a different method with same procedure has been applied and same result has been obtained for $\mu = 0$.

Remark 3.1. If we take $w_0 = g$ then (3.1) coincide with the (2.14).

Theorem 3.2. *Let $w'_0 = h_0 \in C^\infty(a, b)$. Then the differential equation*

$$z'' = w_0 z$$

has a general solution (3.13) *if for some $n \geq 0$*

$$\gamma(t) \equiv \frac{h_{n+1}}{w_{n+1}} = \frac{h_n}{w_n}$$

equality holds. Here $w_i = w'_{i-1} + h_{i-1}$ and $h_i = h'_{i-1} + w_{i-1} w_0$ for $i = 1, 2, \ldots, n$.

Example 3.3. Let $k > 0$, μ and $n(\boldsymbol{x}) = \frac{1}{k^2}\left(\frac{\mu^2}{4} \pm \frac{\mu}{R} - \frac{20R^2}{1+R^4} \right)$ be given. Then $g = \frac{20R^2}{1+R^4}$ and $\gamma = \frac{h_4}{w_4} = \frac{h_3}{w_3} = -\frac{1+4R^4}{R+R^5}$. If we substitute this value of γ in (3.13) we get the solution of the partial differential equation (1.1) of the form

$$\begin{aligned} u \;=\; \exp\left(\mp \frac{\mu R}{2} \right)(1 + R^4) \Bigg[\frac{c_1}{32} \Bigg\{ \frac{-32}{R} - \frac{8R^3}{1+R^4} + 10\sqrt{2}\, \arctan(1 - \sqrt{2}R) \\ - 10\sqrt{2}\, \arctan(1 + \sqrt{2}R) + 5\sqrt{2}\, \log \left| \frac{1 + \sqrt{2}R + R^2}{1 - \sqrt{2}R - R^2} \right| \Bigg\} + c_2 \Bigg]. \end{aligned}$$

Example 3.4. Let $k>0$, μ and $n(\boldsymbol{x}) = \frac{1}{k^2}\left(\frac{\mu^2}{4} \pm \frac{\mu}{R} + \frac{12R}{3-R^3}\right)$ be given, then $g = \frac{-12R}{3-R^3}$ and $\gamma = \frac{h_3}{w_3} = \frac{h_2}{w_2} = \frac{3-4R^3}{-3R+R^4}$. Thus if we use the same procedure of Example 1, we get the solution of the partial differential equation (1.1) of the form

$$u = \exp\left(\mp\frac{\mu R}{2}\right)(-3+R^3)\left[\frac{c_1}{243}\left\{-\frac{27}{R} - \frac{9R^3}{R^3-3} - (123)^{\frac{1}{6}} \arctan\left(\frac{1}{\sqrt{3}} + \frac{2R}{(3)^{\frac{5}{6}}}\right)\right.\right.$$

$$\left.\left. + (23)^{\frac{2}{3}} \log|3+(3)^{\frac{2}{3}}R+(3)^{\frac{1}{3}}R^2| - (43)^{\frac{2}{3}} \log|-3+(3)^{\frac{2}{3}}R| + c_2\right\}\right],$$

where c_1 and c_2 are arbitrary constants.

Theorem 3.5. *Let $g \in C^\infty(a,b)$. If there exists a γ such that the Eq.* (3.9) *holds, then the Riccati differential equation* (2.4) *is solvable and*

$$u = \frac{c}{R}\exp\left(-\int^R \gamma(\tau)d\tau\right), \quad \textit{where } c \textit{ is any constant} \tag{3.14}$$

is the solution of (1.1).

Proof. If there exists a γ, which satisfies (2.9), then a solution of (1.1) is

$$z = \exp\left(-\int^R \gamma(\tau)d\tau\right). \tag{3.15}$$

Because of the function (3.15) satisfies the Eq. (3.1), we get

$$z'' = (\beta^2-\beta')\exp\left(-\int^R \gamma(\tau)d\tau\right) = (\beta^2-\beta')z\,. \tag{3.16}$$

If we take $\beta = -\phi$ in (3.16) then (3.16) becomes (2.4). Consequently, the proof is completed because of Lemma 2. □

Example 3.6. Consider the differential equation $\phi'(t)+\phi^2(t) = \frac{20t^2}{5+t^4}$. Then $g = w_0 = \frac{20t^2}{5+t^4}$ and $\gamma = \frac{h_5}{w_5} = \frac{h_4}{w_4} = \frac{5(1+t^4)}{t(5+t^4)}$. If we substitute this value of γ in (3.13) we get the solution of the given Riccati differential equation of the form

$$\phi(t) = \frac{c_1t^4+3c_2t}{2c_1t^3-3c_2},$$

where c_1 and c_2 are arbitrary constants.

Example 3.7. Let $\phi'(t)+\phi^2(t) = \frac{42t^4}{7+t^6}$. Then $\gamma = \frac{h_1}{w_1} = \frac{h_0}{w_0} = \frac{-3t^2}{1+t^3}$. Thus the solution of the given differential equation is

$$\phi(t) = \frac{c_1\left(9+6\sqrt{3}t^2\arctan\left[\frac{2t-1}{3}\right] + 3t^2\log\left|\frac{1+2t+t^2}{1-t+t^2}\right|\right) + 27c_2t^2}{c_1\left(3t+2\sqrt{3}[1+t^3]\arctan\left[\frac{2t-1}{3}\right] + \log\left|\frac{(1+t^3)^2}{(1-t+t^2)^{t^3+1}}\right|\right) + c_2(1+t^3)}.$$

Remark 3.8. In order to calculate γ in Examples 3.1, 3.2, 3.3 and 3.4 MATHEMATICA software has been used.

Conclusion 3.9. (1.1) has many application in physics, chemistry and some branch of engineering. Thus the solution of (1.1) is important in these areas in the applications. It is shown that if the solution of (1.1) is in the form (2.1) then f must be solved from (2.2). In Section 3, a functional iteration method of a linear equation of the form $L(z) = z'' - \lambda_0 z = 0$ is given. If during the iteration process (3.9) is obtained at some step, then Theorem 1 gives the particular solutions of (1.1).

Conclusion 3.10. The boundary value problem which include Helmholtz equation or reduced wave equation (with damping term) with Dirichlet or Neumann condition in any domain can represent a physical problem. Some of them can be given as acoustic scattering, inverse acoustic scattering or inverse conductive scattering problem in any homogeneous or inhomogeneous media, ocean waves problem and ext. The arbitrary constants, which are obtained by the iteration method of general solutions have special importance for each boundary value problems which are mentioned above.

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