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On the Dynamics of Solutions of Non-linear Recursive System

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Abstract. In this paper, we have studied and examined the periodicity of solutions of rational difference equation system. Then we obtained equilibrium points of this difference equation system and investigated the behaviour of this system related to equilibrium points.

1. Introduction

In applied mathematics, we can see non-linear difference equations system. Recently, many scientists have interested in many branches of mathematics as well as other sciences with difference equation systems. There have been many investigations and interest in the field of functions of difference equations by several authors.

Nasri *et al.* introduced a deterministic model for HIV infection in the presence of combination therapy related to difference equations system [2].

Grove *et al.* in [6], studied on the behaviour and existence of the solutions of the rational equation system $x_{n+1} = \frac{a}{x_n} + \frac{b}{y_n}$, $y_{n+1} = \frac{c}{x_n} + \frac{d}{y_n}$.

In [7], Cinar and Yalcınkaya examined the periodicity of positive solutions of the difference equation system $x_{n+1} = \frac{1}{z_n}$, $y_{n+1} = \frac{1}{x_{n-1}y_{n-1}}$, $z_{n+1} = \frac{1}{x_{n-1}}$.

Clark and Kulenovic, in [1], has investigated the global stability properties and asymptotic behavior of solutions of the recursive sequence

$$x_{n+1} = \frac{x_n}{a + cy_n}, \quad y_{n+1} = \frac{y_n}{b + dx_n}, \quad n = 0, 1, 2, \dots .$$

Similar to references above, in this study, we investigated the solutions of following difference equation system

$$x_{n+1} = \frac{A}{y_n}, \quad y_{n+1} = \frac{Bx_{n-1}}{x_n y_{n-1}}, \quad n = 0, 1, 2, \tag{1.1}$$

where $x_{-1}, x_0, y_{-1}, y_0, A, B \in \mathbb{R}\backslash\{0\}$. Then we obtained equilibrium points of the difference equation systems (1.1) and investigated dynamics of solutions of system (1.1).

Now, we give basic, initial definitions and theorems firstly.

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Let I_1 and I_2 be some intervals of real numbers and let

$$F_1 : I_2 \to I_1,$$

$$F_2 : I_1 \times I_2 \to I_2$$

be two continuously differentiable functions. Then for every initial condition $(x_0, y_0) \in I_1 \times I_2$, the system of difference equations

$$x_{n+1} = F_1(y_n), \tag{1.2}$$

$$y_{n+1} = F_2(x_n, y_n) \tag{1.3}$$

has a unique solution $\{x_n, y_n\}_{n=0}^{\infty}$.

Definition 1.1. We say that a solution $\{x_n, y_n\}_{n=0}^{\infty}$ of the system of difference equations (1.2) and (1.3) is periodic if there exist a positive integer p such that

$$x_{n+p} = x_n,$$

$$y_{n+p} = y_n.$$

The smallest such positive integer p is called the prime period of the solution of difference equations (1.2) and (1.3) [2], [4].

Definition 1.2 ([2, 3])**.** A point $(\bar{x}, \bar{y}) \in I_1 \times I_2$ is called an equilibrium point of system equation (1.2) and (1.3) if

$$\bar{x} = F_1(\bar{y}), \tag{1.4}$$

$$\bar{y} = F_2(\bar{x}, \bar{y}).$$

Definition 1.3. The equilibrium point $(\bar{x}, \bar{y})$ of system equation (1.2) and (1.3) is called stable (or locally stable) if for every $\varepsilon > 0$, there exists $\delta > 0$ such that for all $(x_0, y_0) \in I_1 \times I_2$ with

$$\|(x_0, y_0) - (\bar{x}, \bar{y})\| < \delta,$$

implies

$$\|(x_n, y_n) - (\bar{x}, \bar{y})\| < \varepsilon$$

for all $n \geq 0$. Otherwise equilibrium point is called unstable [2].

Definition 1.4 ([2])**.** The equilibrium point $(\bar{x}, \bar{y})$ of system equation (1.2) and (1.3) is called asymptotically stable (or locally asymptotically stable) if it is stable and there exists $\gamma > 0$ such that for all $(x_0, y_0) \in I_1 \times I_2$ with

$$\|(x_0, y_0) - (\bar{x}, \bar{y})\| < \gamma,$$

implies

$$\lim_{n\to\infty} \|(x_n, y_n) - (\bar{x}, \bar{y})\| = 0.$$

Definition 1.5 ([2])**.** The equilibrium point $(\bar{x}, \bar{y})$ of system equation (1.2) and (1.3) is called global asymptotically stable if it is stable and for every $(x_0, y_0) \in I_1 \times I_2$, we have

$$\lim_{n\to\infty} \|(x_n, y_n) - (\bar{x}, \bar{y})\| = 0.$$

Definition 1.6 ([2])**.** The equilibrium point $(\bar{x}, \bar{y})$ of system equation (1.2) and (1.3) is called a repeller, if there exists $r > 0$ such that for all $(x_0, y_0) \in I_1 \times I_2$ with

$$0 < \|(x_0, y_0) - (\bar{x}, \bar{y})\| < r,$$

there exists $N \geq 1$ such that

$$\|(x_N, y_N) - (\bar{x}, \bar{y})\| \geq \gamma.$$

Lemma 1.7. *Consider the quadratic equation*

$$\lambda^2 + p\lambda + q = 0. \tag{1.5}$$

Then the following statements hold:

(i) *A necessary and sufficient condition for both roots of equation* (1.5) *to lie inside the open unit disk* $|\lambda| < 1$, *is* $|p| < 1 + q < 2$.

(ii) *A necessary and sufficient condition for one root of equation* (1.5) *to lie inside the open unit disk* $|\lambda| < 1$ *and for the other one to have absolute value greater than one is that* $|1 + q| < |p|$ *and* $4q < p^2$.

(iii) *A necessary and sufficient condition for both roots of equation* (1.5) *to have absolute value greater than one is that* $|p| < |1 + q|$ *and* $1 < |q|$ [2].

Theorem 1.8. *Let* $J(\bar{x}, \bar{y})$ *be the Jacobian matrix of system equation* (1.2) *and* (1.3) *at the equilibrium point* $(\bar{x}, \bar{y})$ *and* $P(\lambda)$ *denote the characteristics polynomial of matrix* $J(\bar{x}, \bar{y})$. *Then the followings are true.*

(i) *If all roots of* $P(\lambda)$ *lie inside the open unit disk* $|\lambda| < 1$, *then the equilibrium point* $(\bar{x}, \bar{y})$ *is asymptotically stable.*

(ii) *If at least one root of* $P(\lambda)$ *has absolute value greater than one, then the equilibrium point* $(\bar{x}, \bar{y})$ *is unstable.*

(iii) *If all roots of* $P(\lambda)$ *have absolute value greater than one, then the equilibrium point* $(\bar{x}, \bar{y})$ *is repeller* [2].

2. Main results

The obtained results in this section are results in [5]. Firstly we give some results related to difference equation system (1.1). In the following theorem, we show the periodicity of this solutions and obtain solutions of system (1.1).

Theorem 2.1. *Let* $\{x_n, y_n\}$ *be the solutions of the equation system* (1.1) *with initial conditions* $x_{-1}, x_0, y_{-1}, y_0 \in \mathbb{R}\backslash\{0\}$. *Then all solutions of the system* (1.1) *are periodic with period six.*

Proof. From the equation system (1.1),we obtain the following equalities

$$\begin{aligned}
x_{n+2} &= \frac{Ax_n y_{n-1}}{x_{n-1}}, & y_{n+2} &= \frac{Bx_n}{A},\\
x_{n+3} &= \frac{A^2}{Bx_n}, & y_{n+3} &= \frac{B}{y_n},\\
x_{n+4} &= \frac{Ay_n}{B}, & y_{n+4} &= \frac{x_n y_{n-1}}{x_{n-1}},\\
x_{n+5} &= \frac{Ax_{n-1}}{x_n y_{n-1}}, & y_{n+5} &= \frac{A}{x_n},\\
x_{n+6} &= x_n, & y_{n+6} &= y_n.
\end{aligned}$$

By using Definition 1.1, it is obvious that the solutions are periodic with six period. □

Theorem 2.2. *Let* $\{x_n, y_n\}$ *be the solutions of the equation system* (1.1) *with* $x_{-1} = r$, $x_0 = s$, $y_{-1} = p$, $y_0 = q \in \mathbb{R}\backslash\{0\}$. *Then, for* $n = 0, 1, 2, \ldots$, *it is obtained the solutions*

$$
\begin{array}{lll}
x_{6n+1} = \frac{A}{q}, & y_{6n+1} = \frac{Br}{sp}, & x_{6n+2} = \frac{Asp}{Br}, \\
y_{6n+2} = \frac{Bs}{A}, & x_{6n+3} = \frac{A^2}{Bs}, & y_{6n+3} = \frac{B}{q}, \\
x_{6n+4} = \frac{Aq}{B}, & y_{6n+4} = \frac{sp}{r}, & x_{6n+5} = \frac{Ar}{sp}, \\
y_{6n+5} = \frac{A}{s}, & x_{6n+6} = s, & y_{6n+6} = q.
\end{array}
$$

Proof. Let us use the principle of mathematical induction on n.

It is obvious that theorem holds for $n = 0$.

Assume that it is true the following equalities for $n - 1$. That is,

$$
\left.
\begin{array}{ll}
x_{6n-5} = \frac{A}{q}, & y_{6n-5} = \frac{Br}{sp}, \\
x_{6n-4} = \frac{Asp}{Br}, & y_{6n-4} = \frac{Bs}{A}, \\
x_{6n-3} = \frac{A^2}{Bs}, & y_{6n-3} = \frac{B}{q}, \\
x_{6n-2} = \frac{Aq}{B}, & y_{6n-2} = \frac{sp}{r}, \\
x_{6n-1} = \frac{Ar}{sp}, & y_{6n-1} = \frac{A}{s}, \\
x_{6n} = s, & y_{6n} = q.
\end{array}
\right\} \tag{2.1}
$$

Therefore, we have to show that it is true for n. Now, we can see that the following results are true by using the equations (2.1) and (1.1)

$$
\begin{array}{ll}
x_{6n+1} = \frac{A}{y_{6n}} = \frac{A}{q}, & y_{6n+1} = \frac{Bx_{6n-1}}{x_{6n}y_{6n-1}} = B\frac{\frac{Ar}{sp}}{s\frac{A}{s}} = \frac{Br}{sp}, \\
x_{6n+2} = \frac{A}{y_{6n+1}} = \frac{A}{\frac{Br}{sp}} = \frac{Asp}{Br}, & y_{6n+2} = \frac{Bx_{6n}}{A} = \frac{Bs}{A}, \\
x_{6n+3} = \frac{A}{y_{6n+2}} = \frac{A^2}{Bs}, & y_{6n+3} = \frac{Bx_{6n+1}}{A} = \frac{B}{q}, \\
x_{6n+4} = \frac{A}{y_{6n+3}} = \frac{Aq}{B}, & y_{6n+4} = \frac{Bx_{6n+2}}{A} = \frac{sp}{r}, \\
x_{6n+5} = \frac{A}{y_{6n+4}} = \frac{Ar}{sp}, & y_{6n+5} = \frac{Bx_{6n+3}}{A} = \frac{A}{s}, \\
x_{6n+6} = \frac{A}{y_{6n+5}} = s, & y_{6n+6} = \frac{Bx_{6n+4}}{A} = q,
\end{array} \tag{2.2}
$$

which ends up the induction. □

Now, in the following theorems, we give the equilibrium points of the equation system (1.1).

Theorem 2.3. *If $B > 0$, then the systems equation (1.1) have two equilibrium points which are $\left(\frac{A}{\sqrt{B}}, \sqrt{B}\right), \left(-\frac{A}{\sqrt{B}}, -\sqrt{B}\right) \in I_1 \times I_2$, where I_1 and I_2 are some intervals of real numbers.*

Proof. Let $B > 0$. For the equilibrium points of the system equation (1.1), we can write the following equalities from the system (1.1),

$$\bar{x} = F_1(\bar{y}) = \frac{A}{\bar{y}},$$

$$\bar{y} = F_2(\bar{x}, \bar{y}) = \frac{B\bar{x}}{\bar{x}\bar{y}}.$$

From above equations, we obtain the following results

$$(\bar{x}, \bar{y}) = \left(\frac{A}{\sqrt{B}}, \sqrt{B}\right), \quad (\bar{x}, \bar{y}) = \left(-\frac{A}{\sqrt{B}}, -\sqrt{B}\right). \qquad \square$$

Theorem 2.4. *For the equation system (1.1), the Jacobian matrix is*

$$J\left(\frac{A}{\sqrt{B}}, \sqrt{B}\right) = \begin{pmatrix} 0 & -\frac{A}{B} \\ 0 & -1 \end{pmatrix},$$

and the characteristic polynomial of the Jacobian matrix $J\left(\frac{A}{\sqrt{B}}, \sqrt{B}\right)$ is also $\lambda^2 + \lambda = 0$ at the equilibrium point $(\bar{x}, \bar{y}) = \left(\pm\frac{A}{\sqrt{B}}, \pm\sqrt{B}\right)$.

Proof. The Jacobian matrix at the equilibrium point $(\bar{x}, \bar{y}) = \left(\frac{A}{\sqrt{B}}, \sqrt{B}\right)$ is

$$J\left(\frac{A}{\sqrt{B}}, \sqrt{B}\right) = \begin{pmatrix} \left(\frac{\partial F_1}{\partial x}\right)_{\left(\frac{A}{\sqrt{B}}, \sqrt{B}\right)} & \left(\frac{\partial F_1}{\partial y}\right)_{\left(\frac{A}{\sqrt{B}}, \sqrt{B}\right)} \\ \left(\frac{\partial F_2}{\partial x}\right)_{\left(\frac{A}{\sqrt{B}}, \sqrt{B}\right)} & \left(\frac{\partial F_2}{\partial y}\right)_{\left(\frac{A}{\sqrt{B}}, \sqrt{B}\right)} \end{pmatrix}$$

$$= \begin{pmatrix} 0 & \left(-\frac{A}{y^2}\right)_{\left(\frac{A}{\sqrt{B}}, \sqrt{B}\right)} \\ 0 & \left(-\frac{B}{y^2}\right)_{\left(\frac{A}{\sqrt{B}}, \sqrt{B}\right)} \end{pmatrix} = \begin{pmatrix} 0 & -\frac{A}{B} \\ 0 & -1 \end{pmatrix},$$

and the characteristics polynomial of $J\left(\frac{A}{\sqrt{B}}, \sqrt{B}\right)$ is

$$|J - \lambda I| = \begin{vmatrix} -\lambda & -\frac{A}{B} \\ 0 & -1-\lambda \end{vmatrix} = \lambda^2 + \lambda = 0.$$

Similarly, the Jacobian matrix at the equilibrium point $(\bar{x},\bar{y}) = \left(-\frac{A}{\sqrt{B}},-\sqrt{B}\right)$ and the characteristics polynomial of $J\left(-\frac{A}{\sqrt{B}},-\sqrt{B}\right)$ are the same with the Jacobian matrix at the equilibrium point $(\bar{x},\bar{y}) = \left(\frac{A}{\sqrt{B}},\sqrt{B}\right)$ and the characteristics polynomial of $J\left(\frac{A}{\sqrt{B}},\sqrt{B}\right)$, respectively. □

Because all roots of characteristics polynomials of the Jacobian matrices $J\left(\pm\frac{A}{\sqrt{B}},\pm\sqrt{B}\right)$ do not lie open unit disk $|\lambda| < 1$, we can not say a result about asymptotic stability of the system (1.1) from Theorem 1.8.

After all above material, we give the following theorem and results:

Corollary 2.5. *The solution $\{x_n, y_n\}_{n=0}^{\infty}$ of the system* (1.1) *is stable because of $\lambda_1 = 0$ and $\lambda_2 = 1$ from Theorem* 1.8.

Theorem 2.6. *Let the initial values x_{-1}, x_0, y_{-1}, y_0 of the system* (1.1) *be the equilibrium points, then the following statements hold:*

(a) *The system* (1.1) *is global attractively.*
(b) *The system* (1.1) *is asymptotically stable.*

Proof. To prove (a), all of the solutions of the system (1.1) must converge to the equilibrium points $\left(\frac{A}{\sqrt{B}},\sqrt{B}\right)$ and $\left(-\frac{A}{\sqrt{B}},-\sqrt{B}\right)$. Really, for the initial values $x_{-1} = r$, $x_0 = s$, $y_{-1} = p$, $y_0 = q \in \mathbb{R}\backslash\{0\}$, the solutions of system (1.1) are $\left\{(s,q),\left(\frac{A}{q},\frac{Br}{sp}\right),\left(\frac{Asp}{Br},\frac{Bs}{A}\right),\left(\frac{A^2}{Bs},\frac{B}{q}\right),\left(\frac{Aq}{B},\frac{sp}{r}\right),\left(\frac{Ar}{sp},\frac{A}{s}\right),(s,q),\cdots\right\}$. In this case, it is obvious that $\lim_{n\to\infty}(x_n, y_n) = \left(\frac{A}{\sqrt{B}},\sqrt{B}\right)$ or $\lim_{n\to\infty}(x_n, y_n) = \left(-\frac{A}{\sqrt{B}},-\sqrt{B}\right)$.

(b) The solutions $\{x_n, y_n\}_{n=0}^{\infty}$ of system (1.1) are asymptotically stable as converge to equilibrium points $(\bar{x},\bar{y})$ and are stable from Corollary 2.5. □

Corollary 2.7. *All of the solutions $\{x_n, y_n\}_{n=0}^{\infty}$ of system* (1.1) *are not asymptotically stable* (*i.e.* $\lim_{n\to\infty}\{x_n, y_n\} \neq (\overline{x},\overline{y})$) *if it is not chosen the initial conditions as equilibrium points.*

Corollary 2.8. *The system* (1.1) *is not repeller from Theorem* 1.8 *and Definition* 1.6.

3. Numerical Results

In this section, we consider several different numerical examples to illustrate the results of the previous section and to support our theoretical discussions. Also, we see that these examples represent different types for behaviour and periodicity of solutions of equation system (1.1).

Example 3.1. It can be observed following numerical result for difference equation system (1.1) if we choose $A = 2$, $B = 1$ and initial values $x(1) = -2$, $x(2) = 1.2$, $y(1) = 5$, $y(2) = -4$. In the following, we present Table 1 and Figure 1 that were obtained using MAPLE 13.

Consequently, it is seen that the equation system (1.1) is periodic with 6 period. It is obvious that these solutions are oscillatory at equilibrium points $(2,1)$ and $(-2,-1)$.

Table 1. The stable solutions with 6 period

n	$x(n)$	$y(n)$	n	$x(n)$	$y(n)$
1	-2.0000000000	5.0000000000	11	3.3333333333	-0.2500000000
2	1.2000000000	-4.0000000000	12	-8.0000000000	-3.0000000000
3	-0.5000000000	-0.3333333334	13	-0.6666666667	1.6666666666
4	-5.9999999999	0.6000000000	14	1.2000000000	-4.0000000000
5	3.3333333333	-0.2500000000	15	-0.5000000000	-0.3333333334
6	-8.0000000000	-3.000000000	⋮	⋮	⋮
7	-0.6666666667	1.666666666	65	3.3333333333	-0.2500000000
8	1.2000000000	-4.0000000000	66	-8.0000000000	-3.0000000000
9	-0.5000000000	-0.3333333334	67	-0.6666666667	1.6666666666
10	-5.9999999999	0.6000000000	68	1.2000000000	-4.0000000000

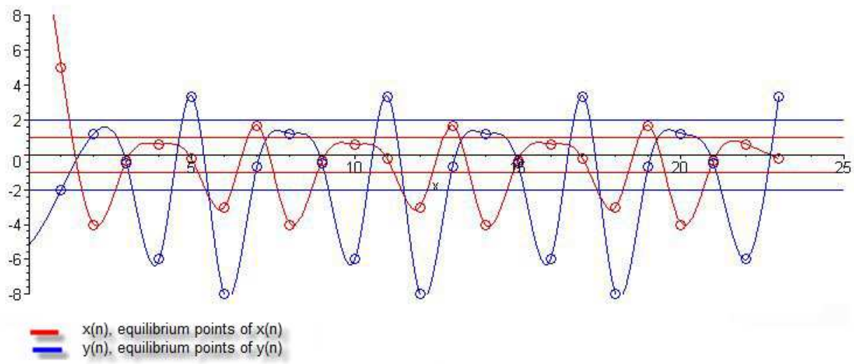

Figure 1

Example 3.2. It can be observed following numerical result for difference equation system (1.1) if we choose $A = 2$, $B = 1$ and initial values $x(1) = -2$, $x(2) = -2$, $y(1) = 1$, $y(2) = 1$. In the following, we present Table 2 and Figure 2 that were obtained using MAPLE 13.

Table 2. The asymptotically stable solutions with 6 period

n	$x(n)$	$y(n)$	n	$x(n)$	$y(n)$
1	-2.0000000000	1.0000000000	12	2.0000000000	1.0000000000
2	-2.0000000000	1.0000000000	13	2.0000000000	-1.0000000000
3	2.0000000000	1.0000000000	14	-2.0000000000	1.0000000000
4	2.0000000000	-1.0000000000	15	2.0000000000	1.0000000000
5	-2.0000000000	1.0000000000	16	2.0000000000	-1.0000000000
6	2.0000000000	1.0000000000	17	-2.0000000000	1.0000000000
7	2.0000000000	-1.0000000000	⋮	⋮	⋮
8	-2.0000000000	1.0000000000	65	-2.0000000000	1.0000000000
9	2.0000000000	1.0000000000	66	2.0000000000	1.0000000000
10	2.0000000000	-1.0000000000	67	2.0000000000	-1.0000000000
11	-2.0000000000	1.0000000000	68	-2.0000000000	1.0000000000

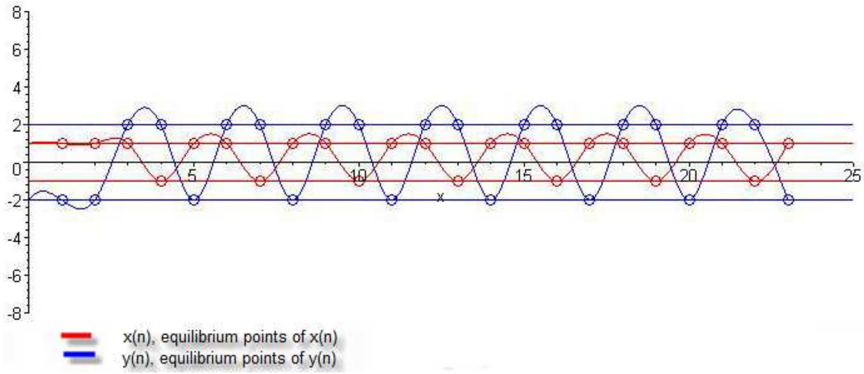

Figure 2

As a result, it is seen that solutions of difference equation system (1.1) are asymptotically stable and they are also global attractor.

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