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On The Hadwiger's Conjecture

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Abstract. The Hadwiger's conjecture (see [1] or [2] or [3] or [4] or [5] is well known. In this paper, we show an original Theorem which is equivalent to the Hadwiger's conjecture.

1. Introduction and prologue

This paper is an original investigation around the *Hadwiger conjecture*. We recall that in a graph $G = [V(G), E(G), \chi(G), \omega(G)]$, $V(G)$ is the set of vertices, $E(G)$ is the set of edges, $\chi(G)$ is the chromatic number, and $\omega(G)$ is the clique number of G. The Hadwiger conjecture states that every graph G is $\eta(G)$ colorable (i.e. we can color all vertices of G with $\eta(G)$ colors such that two adjacent vertices do not receive the same color). $\eta(G)$ is the *hadwiger number* of G and is the *maximum* of p such that G is *contractible* to the complete graph K_p). That being so, this paper is divided into four simple Sections. In Section 2 (*Standard definitions*), we present briefly some standard definitions known in Graph Theory. In Section 3, we introduce definitions that are not standard, and some elementary properties. In Section 4, we introduce a new graph parameter denoted by τ (τ is called the *hadwiger index*) and we present elementary properties of this parameter. In Section 5, using the graph parameter τ, we show a simple Theorem which is equivalent to the Hadwiger conjecture. This simple Theorem immediately implies that *the Hadwiger conjecture is true if and only if* $\tau(G) = \omega(G)$*, for every graph G which is complete* $\omega(G)$*-partite* (τ *is the graph parameter defined in Section* 4). Here, all results are completely different from all the investigations that have been done around the Hadwiger conjecture in the past. In this paper, every graph is finite, is simple and undirected.

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2. **Standard definitions**

We start by standard definitions (see [2] or [3] for instance). Recall that in a graph $G = [V(G), E(G)]$, $V(G)$ is the set of vertices and $E(G)$ is the set of edges. A graph F is a *subgraph* of G, if $V(F) \subseteq V(G)$ and $E(F) \subseteq E(G)$. We say that a graph F is an *induced subgraph* of G by Z, if F is a subgraph of G such that $V(F) = Z$, $Z \subseteq V(G)$, and two vertices of F are adjacent in F, if and only if they are adjacent in G. For $X \subseteq V(G)$, $G \setminus X$ denotes the *subgraph* of G *induced* by $V(G) \setminus X$. A *clique* of G is a subgraph of G that is *complete*; such a subgraph is necessarily an induced subgraph (recall that a graph K is complete if every pair of vertices of K is an edge of K); $\omega(G)$ is the size of a largest clique of G, and $\omega(G)$ is called the *clique number* of G. A *stable set* of a graph G is a set of vertices of G that induces a subgraph with no edges; $\alpha(G)$ is the size of a largest stable set, and $\alpha(G)$ is called the *stability number* of G. The *chromatic number* of G (denoted by $\chi(G)$) is the smallest number of colors needed to color all vertices of G such that two adjacent vertices do not receive the same color. It is easy to see:

Assertion 2.1. *Let G be a graph. Then $\omega(G) \leq \chi(G)$.*

The *hadwiger number* of a graph G (denoted by $\eta(G)$), is the maximum of p such that G is *contractible* to the complete graph K_p. (Recall that, if e is an edge of G incident to x and y, we can obtain a new graph from G by removing the edge e and identifying x and y so that the resulting vertex is incident to all those edges (other than e) originally incident to x or to y. This is called *contracting* the edge e. If a graph F can be obtained from G by a succession of such edge-contractions, then, G is *contractible* to F. The *maximum* of p such that G is contractible to the complete graph K_p is the *hadwiger number* of G, and is denoted by $\eta(G)$). *The Hadwiger conjecture states that $\chi(G) \leq \eta(G)$, for every graph G*. Clearly we have:

Assertion 2.2. *Let G and let F be a subgraph of G. Then $\eta(F) \leq \eta(G)$.*

3. **Non-standard definitions and some elementary properties**

In this section, we introduce definitions that are not standard. These definitions are determining for our final Theorem. We say that a graph G is a *true pal* of a graph F, if F is a subgraph of G and $\chi(F) = \chi(G)$; *trpl*(F) denotes the set of all true pals of F (so, $G \in$ *trpl*(F) means G is a *true pal* of F).

Recall that a set X is a *stable* subset of a graph G, if $X \subseteq V(G)$ and if the subgraph of G induced by X has no edges. A graph G is a *complete $\omega(G)$-partite* graph (or a *complete multipartite* graph), if there exists a partition $\Xi(G) = \{Y_1, \dots, Y_{\omega(G)}]$ of $V(G)$ into $\omega(G)$ stable sets such that $x \in Y_j \in \Xi(G)$, $y \in Y_k \in \Xi(G)$ and $j \neq k$, $\Rightarrow x$ and y are adjacent in G. *It is immediate that $\chi(G) = \omega(G)$, for every complete $\omega(G)$-partite graph*. Ω denotes the set of graphs G which are complete $\omega(G)$-partite. So, $G \in \Omega$ means G is a complete $\omega(G)$-partite graph (For example, if H is a *complete $\omega(H)$-partite* graph with $\omega(H) = 1$, then $H \in \Omega$; if H is a *complete $\omega(H)$-partite* graph with $\omega(H) = 2$, then $H \in \Omega$; if H is a *complete* $\omega(H)$*-partite* graph with $\omega(H) = 3$, then $H \in \Omega; \ldots$ etc). Using the definition of Ω, then the following Assertion becomes immediate.

Assertion 3.1. *Let $H \in \Omega$ and let F be a graph. Then we have the following two properties.*

(3.1.1) $\chi(H) = \omega(H)$.

(3.1.2) *There exists a graph $P \in \Omega$ such that P is a true pal of F.*

Proof. Property (3.1.1) is immediate (use the definition of Ω and note $H \in \Omega$). Property (3.1.2) is also immediate (indeed, let F be graph and let $\Xi(F) = \{Y_1, \ldots, Y_{\chi(F)}]$ be a partition of $V(F)$ into $\chi(F)$ stable sets (it is immediate that such a partition $\Xi(F)$ exists). Now let Q be a graph defined as follows: (i) $V(Q) = V(F)$, (ii) $\Xi(Q) = \{Y_1, \ldots, Y_{\chi(F)}]$ is a partition of $V(Q)$ into $\chi(F)$ stable sets such that $x \in Y_j \in \Xi(Q)$, $y \in Y_k \in \Xi(Q)$ and $j \neq k$, $\Rightarrow$ x and y are adjacent in Q. Clearly $Q \in \Omega$, $\chi(Q) = \omega(Q) = \chi(F)$, and F is visibly a subgraph of Q; in particular Q is a true pal of F such that $Q \in \Omega$ (because F is a subgraph of Q and $\chi(Q) = \chi(F)$ and $Q \in \Omega$). Now put $Q = P$; property (3.1.2) follows.) □

So, we say that a graph P is a *parent* of a graph F, if $P \in \Omega \bigcap trpl(F)$. In other words, P is a *parent* of F, if P is a complete $\omega(P)$-partite graph and P is also a true pal of F (observe that such a P exists, via property (3.1.2) of Assertion 3.1). *parent*(F) denotes the set of all parents of a graph F (so, $P \in$ *parent*(F) means P is a *parent* of F). Using the definition of a parent, then the following Assertion is immediate.

Assertion 3.2. *Let F be a graph and let $P \in$* parent(F)*. We have the following two properties.*

(3.2.1) *Suppose that $F \in \Omega$. Then $\chi(F) = \omega(F) = \omega(P) = \chi(P)$.*

(3.2.2) *Suppose that $F \notin \Omega$. Then $\chi(F) = \omega(P) = \chi(P)$.*

4. **The hadwiger index of a graph**

Here, we define the *hadwiger index* of a graph and a *son* of a graph, and we also give some elementary properties related to the *hadwiger index*. We recall (see Section 2) that $\eta(G)$ is the *hadwiger number* of G. Using the definition of a *true pal* (see Section 3), then the following assertion is immediate.

Assertion 4.1. *Let G be a graph. Then, there exists a graph S such that G is a true pal of S and $\eta(S)$ is minimum for this property.*

Now we define the *hadwiger index* and a *son*. Let G be a graph and put $\mathscr{A}(G) = [H; G \in trpl(H)]$; clearly $\mathscr{A}(G)$ is the set of all graphs H, such that G is a true pal of H. The *hadwiger index* of G is denoted by $\tau(G)$, where $\tau(G) = \min_{F \in \mathscr{A}(G)} \eta(F)$. In other words, $\tau(G) = \eta(F'')$, where $F'' \in \mathscr{A}(G)$, and $\eta(F'')$ is *minimum* for this property. We say that a graph S is a *son* of G, if $G \in trpl(S)$ and $\eta(S) = \tau(G)$. In other words, a graph S is a *son* of G, if $S \in \mathscr{A}(G)$ and $\eta(S) = \tau(G)$. In other terms again, a graph S is a *son* of G, if G is a true pal of S and $\eta(S)$ is *minimum* for this property. Observe that such a *son* exists, via Assertion 4.1. *It is immediate that, if S is a son of a graph G, then $\chi(S) = \chi(G)$ and $\eta(S) \leq \eta(G)$.*

Proposition 4.2. *Let $G \in \Omega$. We have the following three properties.*

(4.2.1) *If $\omega(G) \leq 1$, then $\eta(G) = \omega(G) = \tau(G) = \chi(G)$.*
(4.2.2) *If G is a complete graph, then $\eta(G) = \omega(G) = \tau(G) = \chi(G)$.*
(4.2.3) *$\omega(G) \geq \tau(G)$.*

Proof. Properties (4.2.1) and (4.2.2) are immediate. Now we show property (4.2.3). Indeed, recall $G \in \Omega$, and clearly $\chi(G) = \omega(G)$. Now, put $\mathscr{A}(G) = [H; G \in trpl(H)]$ and let K be a complete graph such that $\omega(K) = \omega(G)$ and $V(K) \subseteq V(G)$; clearly K is a subgraph of G and

$$\chi(G) = \omega(G) = \chi(K) = \omega(K) = \eta(K) = \tau(K). \tag{4.1}$$

In particular K is a subgraph of G with $\chi(G) = \chi(K)$, and therefore, G is a true pal of K. So $K \in \mathscr{A}(G)$ and clearly

$$\tau(G) \leq \eta(K). \tag{4.2}$$

Note $\omega(G) = \eta(K)$ (use (4.1)), and inequality (4.2) immediately becomes $\tau(G) \leq \omega(G)$. □

Proposition 4.3. *Let F be a graph and let $G \in \mathrm{trpl}(F)$. Then $\tau(G) \leq \tau(F)$.*

Proof. Put $\mathscr{A}(G) = [H; G \in trpl(H)]$, and let S be a son of F, recalling that $G \in trpl(F)$, clearly $G \in trpl(S)$; so $S \in \mathscr{A}(G)$ and clearly $\tau(G) \leq \eta(S)$. Now, observe $\eta(S) = \tau(F)$ (because S is a son of F), and the previous inequality immediately becomes $\tau(G) \leq \tau(F)$. □

Proposition 4.3 clearly says that the hadwiger index τ decreases (In the following sense: *G is a true pal of $F \Rightarrow \tau(G) \leq \tau(F)$*).

Corollary 4.4. *Let F be a graph and let $P \in \mathrm{parent}(F)$. Then $\tau(P) \leq \tau(F)$.*

Proof. Observe that $P \in trpl(F)$ and apply Proposition 4.3. □

We will see in Section 5 that the hadwiger index helps to obtain a simple Theorem which is equivalent to the Hadwiger conjecture.

5. A simple Theorem which is equivalent to the Hadwiger conjecture

In this section, we prove a simple Theorem which is equivalent to the Hadwiger conjecture. This simple Theorem immediately implies that the Hadwiger conjecture is true if and only if $\tau(G) = \omega(G)$, for every graph G which is complete $\omega(G)$-partite. We recall (see *Introduction* or see Section 2) that the *Hadwiger* *conjecture* states that $\chi(G) \leq \eta(G)$, for every graph G. Using the *hadwiger index* τ (see Section 4), then the following simple Theorem is equivalent to the Hadwiger conjecture.

Theorem 5.1. *The following are equivalent.*

(1) *The Hadwiger conjecture holds (i.e. $\chi(H) \leq \eta(H)$, for every graph H).*
(2) $\chi(F) \leq \tau(F)$*, for every graph F.*
(3) $\omega(G) = \tau(G)$*, for every $G \in \Omega$.*

Proof. (2)⇒(3). Let $G \in \Omega$, clearly G is a graph and so $\chi(G) \leq \tau(G)$. Note $\chi(G) = \omega(G)$ (since $G \in \Omega$), and the previous inequality becomes $\omega(G) \leq \tau(G)$; now, using property (4.2.3) of Proposition 4.2, we have $\omega(G) \geq \tau(G)$, and the last two inequalities imply that $\omega(G) = \tau(G)$.

(3)⇒(1). Let H be a graph and let $P \in parent(H)$, then $\tau(P) \leq \tau(H)$ (use Corollary 4.4); observe $P \in \Omega$ (since $P \in parent(H)$), clearly $\omega(P) = \tau(P)$ (since $P \in \Omega$), and $\chi(H) = \chi(P) = \omega(P)$ (since $P \in parent(H)$). Clearly $\tau(P) = \chi(H)$ and the previous inequality becomes $\chi(H) \leq \tau(H)$. Recall $\tau(H) \leq \eta(H)$, and the last two inequalities become $\chi(H) \leq \tau(H) \leq \eta(H)$. So $\chi(H) \leq \eta(H)$, and clearly (3) ⇒ (1).

(1)⇒(2). Indeed, let F be a graph and let S be a son of F, clearly $\chi(S) \leq \eta(S)$; now observing that $\chi(S) = \chi(F)$ (since $F \in trpl(S)$) and $\eta(S) = \tau(F)$ (because S is a son of F), then the previous inequality immediately becomes $\chi(F) \leq \tau(F)$. So (1) ⇒ (2) and Theorem 5.1 follows. □

From Theorem 5.1, the following Theorem immediately comes:

Theorem 5.2. *The following are equivalent.*

(i) *The Hadwiger conjecture holds.*
(ii) $\omega(G) = \tau(G)$*, for every $G \in \Omega$.*

Proof. Indeed, it is an immediate consequence of Theorem 5.1. □

Visibly, Theorem 5.2 clearly says the Hadwiger conjecture is true if and only if $\tau(G) = \omega(G)$, for every graph G which is complete $\omega(G)$-partite.

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