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Research Article

Regularity of Linear Hypersubstitutions for Algebraic Systems of Type $((n),(m))$

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Abstract. An algebraic system consisting a nonempty set together with a sequence of operations and a sequence of relations on this set. The properties of this structure are expressed by terms and formulas. In this paper we study on linear terms of type (n) for a natural number $n \geq 1$ and linear formulas of type $((n),(m))$ for natural numbers $n,m \geq 1$. Using the partial clone of linear terms and the partial clone of linear formulas, we define the new concept of linear hypersubstitutions for algebraic systems of type $((n),(m))$ and proved that the set of all linear hypersubstitutions for algebraic systems of type $((n),(m))$ with a binary operation on this set and the identity element forms a monoid. Finally, we also interest in studying the semigroup or monoid properties of its. In particular, we investigate the idempotency and regularity of linear hypersubstitutions for algebraic systems of this monoid.

Keywords. Algebraic systems; Linear terms; Linear formulas; Linear hypersubstitutions; Regular elements

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1. Introduction

The algebraic system was first introduced by A.I. Malcev in 1951 [9]. We now recall informal definition of algebraic systems. An algebraic system is a structure consisting a nonempty set together with a sequence of operations and a sequence of relations on this set.

The concept of terms is one of the fundamental concepts of universal algebra. Terms may be considered as words formed by letters. To define terms one needs variables and operation symbols, let $(f_i)_{i\in I}$ be a sequence of operation symbols, when f_i is n_i-ary and $n_i \in \mathbb{N}^+ := \mathbb{N}\setminus\{0\}$. We denote by $X := \{x_1,\dots,x_n,\dots\}$ is a countably infinite set of symbols called variables and for each $n \geq 1$ let $X_n := \{x_1,\dots,x_n\}$. The sequence $\tau := (n_i)_{i\in I}$ is called a type. Then *an n-ary term of type* τ is defined inductively as follows:

(i) Every variable $x_j \in X_n$ is an n-ary term of type τ.

(ii) If $t_1,\dots,t_{n_i}$ are n-ary terms of type τ and f_i is an n_i-ary operation symbol, then $f_i(t_1,\dots,t_{n_i})$ is an n-ary term of type τ.

Let $W_\tau(X_n)$ be the set of all n-ary terms of type τ which contains $x_1,\dots,x_n$ and is closed under finite application of (ii) and let $W_\tau(X) := \bigcup_{n\in\mathbb{N}^+} W_\tau(X_n)$ be the set of all terms of type τ.

The investigation of terms is a relatively new, actively developing field of universal algebra, computer science and several subjects. For the application of terms in algebras is to defined identities. We use identities to classify algebras into collections called varieties. Moreover, the knowledge of the identities valid in algebras could be useful for solving functional equations (see [7]). Not only the concept of terms which is used to express properties of algebraic systems but there is the other one which is called formulas, first introduced by A.I. Mal'cev in 1973 (see [9]).

To define quantifier free formulas we need terms, logical connectives and relation symbols. We now recall the definition of a formula which is defined by K. Denecke and D. Phusanga in 2013 [6]. Let J be an indexed set and A be a nonempty set. An *n_j-ary relation* on A is a relation $\gamma \subseteq A^{n_j}$ and call n_j the arity of γ. Let $(\gamma_j)_{j\in J}$ be a sequence of relation symbols and $\tau' := (n_j)_{j\in J}$ where γ_j has the arity n_j for every $j \in J$.

Definition 1 ([10])**.** Let $n \in \mathbb{N}^+$ and τ,τ' be the types of operation symbols and relation symbols, respectively. *An n-ary quantifier free formula of type* (τ,τ') (*for simply, formula*) is defined in the following way:

(i) If t_1,t_2 are n-ary terms of type τ, then the equation $t_1 \approx t_2$ is an n-ary quantifier free formula of type (τ,τ').

(ii) If $j \in J$ and $t_1,\dots,t_{n_j}$ are n-ary terms of type τ and γ_j is an n_j-ary relation symbol, then $\gamma_j(t_1,\dots,t_{n_j})$ is an n-ary quantifier free formula of type (τ,τ').

(iii) If F is an n-ary quantifier free formula of type (τ,τ'), then $\neg F$ is an n-ary quantifier free formula of type (τ,τ').

(iv) If F_1 and F_2 are n-ary quantifier free formulas of type (τ,τ'), then $F_1 \vee F_2$ is an n-ary quantifier free formula of type (τ,τ').

Let $\mathscr{F}_{(\tau,\tau')}(X_n)$ be the set of all n-ary quantifier free formulas of type (τ,τ') and let $\mathscr{F}_{(\tau,\tau')}(X) := \bigcup_{n\in\mathbb{N}^+} \mathscr{F}_{(\tau,\tau')}(X_n)$ be the set of all quantifier free formulas of type (τ,τ').

Many mathematicians are interested in a term in which each variable occur at most once which is called a linear term (see also [2]). The concept of linear terms was introduced by M. Couceiro and E. Lehtonen [3] in 2012. It is important to do research on linear terms because it is connected with several other areas of algebras. For example, a linear term may be considered on generalization of a linear expression over a vector space (see e.g. [4]).

As we already mentioned above why we are interesting in linear terms, we now recall a formal definition of linear terms of type τ as follows: let $var(t)$ be the set of all variables occurring in the term t.

Definition 2 ([3])**.** An n-ary linear term of type τ is defined in the following inductive way:

(i) Every $x_i \in X_n$ is an n-ary linear term of type τ.

(ii) If $t_1,\dots,t_{n_i}$ are n-ary linear terms of type τ with $var(t_l)\cap var(t_k)=\emptyset$ for all $1\le l<k\le n_i$ and f_i is an n_i-ary operation symbol, then $f_i(t_1,\dots,t_{n_i})$ is an n-ary linear term of type τ.

Let $W_\tau^{lin}(X_n)$ be the set of all n-ary linear terms of type τ and let $W_\tau^{lin}(X):=\bigcup_{n\in\mathbb{N}^+} W_\tau^{lin}(X_n)$ be the set of all linear terms of type τ.

To study clones, mathematicians have used many several different techniques such as combinatorics, set theory or topology. One of research directions in clone theory is the clone of terms which plays an important role in universal algebra and computer science. Clone can be given an algebraic structure, for example as many-sorted algebra of particular type. In 2016, K. Denecke [4] published the paper “The Partial Clone of Linear Terms”, which investigate the concept about the clone of linear terms and their properties. In the recently year, the authors extended the concept of clone of terms in algebra to clone of linear terms of type (n) and study clone of linear formulas for algebraic systems (see [8]).

Not only the concept of clone is important in universal algebra, the classification of algebras by identities into collections called varieties are interesting. We can also use hyperidentities to classify varieties into collections called hypervarieties. In 1991, K. Denecke, D. Lau, R. Pöschel and D. Schweigert [5] introduced the concept of a hypersubstitution for algebras which used to define hyperidentities and hypervarieties mentioned above. A hypersubstitution is a map which takes every n-ary operation symbol to an n-ary term. Any such map can be uniquely extended to a map defined on the set of all terms, and then any two such hypersubstitutions can be composed in a natural way. They proved that the set of all hypersubstitutions forms a monoid.

A hypersubstitution for algebraic systems was first introduced by K. Denecke and D. Phusanga [10] in 2008. It is a mapping which maps operation symbols to terms and relation symbols to quantifier free formulas preserving arities. They defined a binary operation on the set of all hypersubstitutions for algebraic systems and then proved that this set with the binary operation and an identity element forms a monoid. Five years later, the definition of a hypersubstitution for algebraic systems was improved by D. Phusanga (see [10]).

In 2016 Th. Changphas, K. Denecke and B. Pibaljommee [2] restricted to study a hypersubstitution for algebras which maps any operation symbols to a linear term of the same arity, called a linear hypersubstitution for algebras. As a consequence, they proved that the set of all linear hypersubstitutions forms a monoid.

Next, we want to recall some basic concepts for the discussion of our main results.

2. Preliminaries

We present the concepts about the partial clone of linear terms and the partial clone of linear formulas and recall some properties of these structures (for more detail see [8]). For the basic knowledge of hypersubstitutions, the reader is refered to [7].

Definition 3 ([8]). Let $n, p \in \mathbb{N}^+$ with $p \geq n$. *A p-ary linear term of type* (n) is defined in the following inductive way:

(i) Every $x_i \in X_p$ is a p-ary linear term of type (n).

(ii) If $t_1, \dots, t_n$ are p-ary linear terms of type (n) with $var(t_l) \cap var(t_k) = \emptyset$ for all $1 \leq l < k \leq n$ and f is an n-ary operation symbol, then $f(t_1, \dots, t_n)$ is a p-ary linear term of type (n).

Let $W^{lin}_{(n)}(X_p)$ be the set of all p-ary linear terms of type (n) and let $W^{lin}_{(n)}(X) := \bigcup_{p \in \mathbb{N}^+} W^{lin}_{(n)}(X_p)$ be the set of all linear terms of type (n).

Now, we recall the concept of superposition of linear terms of type (n), this leads us to forms the many-sorted algebra which is called the clone of linear terms.

Definition 4 ([8]). Let $p, q \in \mathbb{N}^+$ with $p \leq q$, $t \in W^{lin}_{(n)}(X_p)$ and $s_1, \dots, s_p \in W^{lin}_{(n)}(X_q)$ with $var(s_l) \cap var(s_k) = \emptyset$ for all $1 \leq l < k \leq p$. Then we define a superposition operation of linear terms

$$S^{lin}{}^{p}_{q} : W^{lin}_{(n)}(X_p) \times (W^{lin}_{(n)}(X_q))^p \multimap W^{lin}_{(n)}(X_q)$$

inductively by the following steps:

(i) If $t = x_i$ for $1 \leq i \leq p$, then $S^{lin}{}^{p}_{q}(x_i, s_1, \dots, s_p) := s_i$.

(ii) If $t = f(t_1, \dots, t_p)$, then

$$S^{lin}{}^{p}_{q}(f(t_1, \dots, t_p), s_1, \dots, s_p) := f(S^{lin}{}^{p}_{q}(t_1, s_1, \dots, s_p), \dots, S^{lin}{}^{p}_{q}(t_p, s_1, \dots, s_p)).$$

On the set $W^{lin}_{(n)}(X_p)$ of all p-ary linear terms of type (n), we establish the many-sorted algebra of type $(p+1, \dots, 0, \dots, 0)$, by using the $(p+1)$-ary superposition operation $S^{lin}{}^{p}_{q}$ as we already defined in Definition 4 and adding the variables $x_1, \dots, x_p$ as nullary operations, call projections. Then we obtain the many-sorted algebra

$$PLinClone(n) := ((W^{lin}_{(n)}(X_p))_{p \in \mathbb{N}^+}, (S^{lin}{}^{p}_{q})_{p \leq q, p, q \in \mathbb{N}^+}, (x_i)_{i \leq p, i \in \mathbb{N}^+}),$$

which is called *the partial clone of linear terms of type* (n).

Next, some properties of $PLinClone(n)$ will be presented.

Theorem 1 ([8]). *The many sorted algebra* $PLinClone(n)$ *satisfies the following equations:*

(LC1) $S^{lin}{}^{p}_{q}(S^{lin}{}^{r}_{p}(t, t_1, \dots, t_r), s_1, \dots, s_p) = S^{lin}{}^{r}_{q}(t, S^{lin}{}^{p}_{q}(t_1, s_1, \dots, s_p), \dots, S^{lin}{}^{p}_{q}(t_r, s_1, \dots, s_p))$,

(LC2) $S^{lin}{}^{p}_{q}(x_i, t_1, \dots, t_p) = t_i$ *for* $1 \leq i \leq p$,

(LC3) $S^{lin}{}^{p}_{p}(t, x_1, \dots, x_p) = t$,

where $p, q, r \in \mathbb{N}^+$ *with* $r \leq p \leq q$, $t \in W^{lin}_{(n)}(X_r)$, $t_1, \dots, t_r \in W^{lin}_{(n)}(X_p)$, $var(t_l) \cap var(t_k) = \emptyset$ *for all* $1 \leq l < k \leq r$ *and* $s_1, \dots, s_p \in W^{lin}_{(n)}(X_q)$, $var(s_l) \cap var(s_k) = \emptyset$ *for all* $1 \leq l < k \leq p$.

Using the definition of the partial clone of linear terms, we defined the new concepts of the partial clone of linear formulas.

Let $var(F)$ be the set of all variables occurring in the formula F.

Definition 5 ([8])**.** Let $m,n,p \in \mathbb{N}^+$ with $p \geq m$. *A p-ary quantifier free linear formula of type* $((n),(m))$ (for simply, linear formula) is defined as follows:

(i) If s,t are p-ary linear terms of type (n) and $var(s) \cap var(t) = \emptyset$, then the equation $s \approx t$ is a p-ary quantifier freelinear formula of type $((n),(m))$.

(ii) If $t_1,\dots,t_m$ are p-ary linear terms of type (n) with $var(t_l) \cap var(t_k) = \emptyset$ for all $1 \leq l < k \leq m$ and γ is an m-ary relation symbol, then $\gamma(t_1,\dots,t_m)$ is a p-ary quantifier free linear formula of type $((n),(m))$.

(iii) If F is a p-ary quantifier free linear formula of type $((n),(m))$, then $\neg F$ is a p-ary quantifier free linear formula of type $((n),(m))$.

(iv) If F_1 and F_2 are p-ary quantifier free linear formulas of type $((n),(m))$ and $var(F_1) \cap var(F_2) = \emptyset$, then $F_1 \vee F_2$ is a p-ary quantifier free linear formula of type $((n),(m))$.

Let $\mathscr{F}^{lin}_{((n),(m))}(W^{lin}_{(n)}(X_p))$ be the set of all p-ary quantifier free linear formulas of type $((n),(m))$ and let $\mathscr{F}^{lin}_{((n),(m))}(W^{lin}_{(n)}(X)) := \bigcup_{p\in\mathbb{N}^+} \mathscr{F}^{lin}_{((n),(m))}(W^{lin}_{(n)}(X_p))$ be the set of all quantifier free linear formulas of type $((n),(m))$.

Remark 1. The linear formulas defined by (i) and (ii) are called atomic linear formulas.

Example 1. Let $((n),(m)) = ((2),(2))$ be a type, i.e., we have one binary operation symbol f and one binary relation symbol γ and let $X_2 = \{x_1,x_2\}$. Then the binary atomic linear formulas of type $((2),(2))$ are $x_1 \approx x_2, x_2 \approx x_1, \gamma(x_1,x_2), \gamma(x_2,x_1)$. Moreover, we obtained all other linear formulas of type $((2),(2))$ from binary atomic linear formulas of type $((2),(2))$ by using the logical connections $\neg$ and $\vee$.

Lemma 1 ([1])**.** *Suppose F is a formula in $\mathscr{F}_{(\tau,\tau')}(X)$. Then the following pair of formula is equivalent: $\neg(\neg F) \equiv F$.*

Moreover, we also extended the definition of superposition of linear terms to superposition of linear formulas as follows:

Definition 6 ([8])**.** Let $p,q \in \mathbb{N}^+$ with $p \leq q$, $F \in \mathscr{F}^{lin}_{((n),(m))}(W^{lin}_{(n)}(X_p))$ and $s_1,\dots,s_p \in W^{lin}_{(n)}(X_q)$ with $var(s_l) \cap var(s_k) = \emptyset$ for all $1 \leq l < k \leq p$. Then we define the superposition operation

$$R^{lin}{}^{p}_{q} : \mathscr{F}^{lin}_{((n),(m))}(W^{lin}_{(n)}(X_p)) \times (W^{lin}_{(n)}(X_q))^p \multimap\!\!\to \mathscr{F}^{lin}_{((n),(m))}(W^{lin}_{(n)}(X_q))$$

by the following steps:

(i) If F has the form $s \approx t$, then

$$R^{lin}{}^{p}_{q}(s \approx t, s_1,\dots,s_p) := S^{lin}{}^{p}_{q}(s,s_1,\dots,s_p) \approx S^{lin}{}^{p}_{q}(t,s_1,\dots,s_p).$$

(ii) If F has the form $\gamma(t_1,\dots,t_p)$, then

$$R^{lin}{}^{p}_{q}(\gamma(t_1,\dots,t_p),s_1,\dots,s_p) := \gamma(S^{lin}{}^{p}_{q}(t_1,s_1,\dots,s_p),\dots,S^{lin}{}^{p}_{q}(t_p,s_1,\dots,s_p)).$$

(iii) If $F \in \mathscr{F}^{lin}_{((n),(m))}(W^{lin}_{(n)}(X_p))$ and assume that $R^{lin}\,{}^p_q(F,s_1,\dots,s_p)$ is already defined, then
$$R^{lin}\,{}^p_q(\neg F,s_1,\dots,s_p) := \neg R^{lin}\,{}^p_q(F,s_1,\dots,s_p).$$

(iv) If $F_1,F_2 \in \mathscr{F}^{lin}_{((n),(m))}(W^{lin}_{(n)}(X_p))$ and supposed that
$R^{lin}\,{}^p_q(F_l,s_1,\dots,s_p)$ is already defined for all $l \in \{1,2\}$, then
$$R^{lin}\,{}^p_q(F_1 \vee F_2,s_1,\dots,s_p) := R^{lin}\,{}^p_q(F_1,s_1,\dots,s_p) \vee R^{lin}\,{}^p_q(F_2,s_1,\dots,s_p).$$

Now, we may consider the many-sorted algebra:
$$PLinFormClone((n),(m)) := ((W^{lin}_{(n)}(X_p))_{p\in\mathbb{N}^+}, (\mathscr{F}^{lin}_{((n),(m))}(W^{lin}_{(n)}(X_p)))_{p\in\mathbb{N}^+}, (S^{lin}\,{}^p_q)_{p\le q,p,q\in\mathbb{N}^+}, (R^{lin}\,{}^p_q)_{p\le q,p,q\in\mathbb{N}^+}, (x_i)_{i\le p,i\in\mathbb{N}^+}),$$
which is called *the partial clone of linear formula of type* $((n),(m))$.

Theorem 2. ([8]) *The many sorted algebra PLin FormClone*$((n),(m))$ *satisfies the following properties:*
(LFC1) $R^{lin}\,{}^p_q(R^{lin}\,{}^r_p(F,t_1,\dots,t_r),s_1,\dots,s_p) = R^{lin}\,{}^r_q(F,S^{lin}\,{}^p_q(t_1,s_1,\dots,s_p),\dots,S^{lin}\,{}^p_q(t_r,s_1,\dots,s_p))$,
(LFC2) $R^{lin}\,{}^p_p(F,x_1,\dots,x_p) = F$,
where $p,q,r \in \mathbb{N}^+$ *with* $r \le p \le q$, $F \in \mathscr{F}^{lin}_{((n),(m))}(W^{lin}_{(n)}(X_r))$, $t_1,\dots,t_r \in W^{lin}_{(n)}(X_p)$, $var(t_l) \cap var(t_k) = \emptyset$ *for all* $1 \le l < k \le r$ *and* $s_1,\dots,s_p \in W^{lin}_{(n)}(X_q)$, $var(s_l) \cap var(s_k) = \emptyset$ *for all* $1 \le l < k \le p$.

After this preliminaries, we will start our main results in the next section.

3. Monoid of Linear Hypersubstitutions for Algebraic Systems of Type $((n),(m))$

The main propose of this section is to introduce the new algebraic structure and consider some semigroup properties. We will start with giving the concept of linear hypersubstitutions for algebraic systems of type $((n),(m))$ for fixed natural numbers $n,m \ge 1$ and $n \ge m$ by using the elementary concepts as we recalled in the previous section.

Let us start with the definition of the based set of our new structure.

Definition 7. Let $n \in \mathbb{N}^+$. *A linear hypersubstitution for algebraic systems of type* $((n),(m))$ is a mapping $\sigma : \{f\} \cup \{\gamma\} \to W^{lin}_{(n)}(X_n) \cup \mathscr{F}^{lin}_{((n),(m))}(W^{lin}_{(n)}(X_m))$ which maps an n-ary operation symbol f to an n-ary linear term of type (n) and maps an m-ary relation symbol γ to an m-ary quantifier free linear formula of type $((n),(m))$. We denote the set of all linear hypersubstitutions for algebraic systems of type $((n),(m))$ by $Hyp^{lin}((n),(m))$.

From now on, every element in $Hyp^{lin}((n),(m))$ is denoted by $\sigma_{t,F}$ which maps an n-ary operation symbol f and an m-ary relation symbol γ to a linear term t and a linear formula F, respectively. That is $\sigma_{t,F}(f) = t$ and $\sigma_{t,F}(\gamma) = F$.
Let S_n be the set of all permutations on $\{1,\dots,n\}$.
To define a binary operation on $Hyp^{lin}((n),(m))$, we extend a linear hypersubstitution for algebraic systems σ to a mapping $\widehat{\sigma}$.

Definition 8. Let $\sigma_{t,F} \in Hyp^{lin}((n),(m))$. Then we define a mapping

$$\widehat{\sigma}_{t,F} : W^{lin}_{(n)}(X_n) \cup \mathscr{F}^{lin}_{((n),(m))}(W^{lin}_{(n)}(X_m)) \to W^{lin}_{(n)}(X_n) \cup \mathscr{F}^{lin}_{((n),(m))}(W^{lin}_{(n)}(X_m))$$

inductively defined as follows:

(i) $\widehat{\sigma}_{t,F}[x_i] := x_i$ for every $i = 1,\dots,n$.

(ii) $\widehat{\sigma}_{t,F}[f(x_{\pi(1)},\dots,x_{\pi(n)})] := S^{lin}{}^n_n(\sigma_{t,F}(f),\widehat{\sigma}_{t,F}[x_{\pi(1)}],\dots,\widehat{\sigma}_{t,F}[x_{\pi(n)}])$ where $\pi \in S_n$.

(iii) $\widehat{\sigma}_{t,F}[x_l \approx x_k] := \widehat{\sigma}_{t,F}[x_l] \approx \widehat{\sigma}_{t,F}[x_l]$ where $l,k \in \{1,\dots,m\}$ and $l \neq k$.

(iv) $\widehat{\sigma}_{t,F}[\gamma(x_{\phi(1)},\dots,x_{\phi(m)}] := R^{lin}{}^m_m(\sigma_{t,F}(\gamma),\widehat{\sigma}_{t,F}[x_{\phi(1)}],\dots,\widehat{\sigma}_{t,F}[x_{\phi(m)}])$ where $\phi \in S_m$.

(v) $\widehat{\sigma}_{t,F}[\neg F] := \neg\widehat{\sigma}_{t,F}[F]$ for $F \in \mathscr{F}^{lin}_{((n),(m))}(W^{lin}_{(n)}(X_m))$.

(vi) $\widehat{\sigma}_{t,F}[F_1 \vee F_2] := \widehat{\sigma}_{t,F}[F_1] \vee \widehat{\sigma}_{t,F}[F_2]$.

Example 2. Let $((n),(m)) = ((3),(3))$ be a type, i.e., we have one ternary operation symbol and one ternary relation symbol, say f and γ, respectively. Let $\sigma : \{f\}\cup\{\gamma\} \to W^{lin}_{(3)}(X_3)\cup\mathscr{F}^{lin}_{((3),(3))}(W^{lin}_{(3)}(X_3))$ where $\sigma_{t,F}(f) = f(x_2,x_1,x_3)$ and $\sigma_{t,F}(\gamma) = x_3 \approx x_1$. Then, we have

$$\begin{aligned}\widehat{\sigma}_{t,F}[f(x_3,x_1,x_2)] &= S^{lin}{}^3_3(\sigma_{t,F}(f),\widehat{\sigma}_{t,F}[x_3],\widehat{\sigma}_{t,F}[x_1],\widehat{\sigma}_{t,F}[x_2]) \\ &= S^{lin}{}^3_3(f(x_2,x_1,x_3),x_3,x_1,x_2) \\ &= f(x_1,x_3,x_2),\end{aligned}$$

and

$$\begin{aligned}\widehat{\sigma}_{t,F}[\gamma(x_1,x_3,x_2)] &= R^{lin}{}^3_3(\sigma_{t,F}(\gamma),\widehat{\sigma}_{t,F}[x_1],\widehat{\sigma}_{t,F}[x_3],\widehat{\sigma}_{t,F}[x_2]) \\ &= S^{lin}{}^3_3(x_3 \approx x_1,x_1,x_3,x_2) \\ &= S^{lin}{}^3_3(x_3,x_1,x_3,x_2) \approx S^{lin}{}^3_3(x_1,x_1,x_3,x_2) \\ &= x_2 \approx x_1.\end{aligned}$$

Now, we define a binary operation $\circ_r$ on $Hyp^{lin}((n),(m))$ as follows:

Definition 9. Let $t_1,t_2 \in W^{lin}_{(n)}(X_n), F_1,F_2 \in \mathscr{F}^{lin}_{((n),(m))}(W^{lin}_{(n)}(X_m))$, $\sigma_{t_1,F_1},\sigma_{t_2,F_2} \in Hyp^{lin}((n),(m))$ and $\circ$ is the usual composition of mappings. Then we define a binary operation $\circ_r$ on $Hyp^{lin}((n),(m))$ by

$$\sigma_{t_1,F_1} \circ_r \sigma_{t_2,F_2} := \widehat{\sigma}_{t_1,F_1} \circ \sigma_{t_2,F_2}.$$

Next, we prove that a binary operation $\circ_r$ satisfies the associative law. To get our result, we need some preparation as follows:

Lemma 2. *Let $t \in W^{lin}_{(n)}(X_n)$, $\beta \in \mathscr{F}^{lin}_{((n),(m))}(W^{lin}_{(n)}(X_m))$, $\pi \in S_n$ and $\phi \in S_m$. Then for each $\sigma_{t,F} \in Hyp^{lin}((n),(m))$, we have*

(i) $\widehat{\sigma}_{t,F}[S^{lin}{}^n_n(t,x_{\pi(1)},\dots,x_{\pi(n)})] = S^{lin}{}^n_n(\widehat{\sigma}_{t,F}[t],\widehat{\sigma}_{t,F}[x_{\pi(1)}],...,\widehat{\sigma}_{t,F}[x_{\pi(n)}])$.

(ii) $\widehat{\sigma}_{t,F}[R^{lin}{}^m_m(\beta,x_{\phi(1)},\dots,x_{\phi(m)})] = R^{lin}{}^m_m(\widehat{\sigma}_{t,F}[\beta],\widehat{\sigma}_{t,F}[x_{\phi(1)}],\dots,\widehat{\sigma}_{t,F}[x_{\phi(m)}])$.

Proof. (i) Let $t \in W^{lin}_{(n)}(X_n)$. We give a proof by induction on the complexity of a linear term t. If $t = x_i$ for all $1 \le i \le n$, the proof is obvious. If $t = f(x_{\pi(1)},\dots,x_{\pi(n)})$ where $\pi \in S_n$ and for every $l \in \{1,\dots n\}$ we assume that $\widehat{\sigma}_{t,F}[S^{lin}{}^n_n(x_{\pi(l)},x_{\pi(1)},\dots,x_{\pi(n)})] =$

$S^{lin}{}^{n}_{n}(\widehat{\sigma}_{t,F}[x_{\pi(l)}],\widehat{\sigma}_{t,F}[x_{\pi(1)}],\ldots,\widehat{\sigma}_{t,F}[x_{\pi(n)}])$, then by Theorem 1, we get

$$\begin{aligned}
&\widehat{\sigma}_{t,F}[S^{lin}{}^{n}_{n}(f(x_{\pi(1)},\ldots,x_{\pi(n)}),x_{\pi(1)},\ldots,x_{\pi(n)})]\\
&\quad= S^{lin}{}^{n}_{n}(\sigma_{t,F}(f),\widehat{\sigma}_{t,F}[S^{lin}{}^{n}_{n}(x_{\pi(1)},x_{\pi(1)},\ldots,x_{\pi(n)})],\ldots,\widehat{\sigma}_{t,F}[S^{lin}{}^{n}_{n}(x_{\pi(n)},x_{\pi(1)},\ldots,x_{\pi(n)})])\\
&\quad= S^{lin}{}^{n}_{n}(\sigma_{t,F}(f),S^{lin}{}^{n}_{n}(\widehat{\sigma}_{t,F}[x_{\pi(1)}],\widehat{\sigma}_{t,F}[x_{\pi(1)}],\ldots,\widehat{\sigma}_{t,F}[x_{\pi(n)}]),\ldots,\\
&\qquad S^{lin}{}^{n}_{n}(\widehat{\sigma}_{t,F}[x_{\pi(n)}],\widehat{\sigma}_{t,F}[x_{\pi(1)}],\ldots,\widehat{\sigma}_{t,F}[x_{\pi(n)}])\\
&\quad= S^{lin}{}^{n}_{n}(S^{lin}{}^{n}_{n}(\sigma_{t,F}(f),\widehat{\sigma}_{t,F}[x_{\pi(1)}],\ldots,\widehat{\sigma}_{t,F}[x_{\pi(n)}]),\widehat{\sigma}_{t,F}[x_{\pi(1)}],\ldots,\widehat{\sigma}_{t,F}[x_{\pi(n)}])\\
&\quad= S^{lin}{}^{n}_{n}(\widehat{\sigma}_{t,F}[f(x_{\pi(1)},\ldots,x_{\pi(n)})],\widehat{\sigma}_{t,F}[x_{\pi(1)}],\ldots,\widehat{\sigma}_{t,F}[x_{\pi(n)}]).
\end{aligned}$$

(ii) Let $\beta\in\mathscr{F}^{lin}_{((n),(m))}(W^{lin}_{(n)}(X_m))$. We give a proof by the following steps.
If β has the form $x_l\approx x_k$ where $l,k\in\{1,\ldots,m\}$ and $l\neq k$, then we have

$$\begin{aligned}
\widehat{\sigma}_{t,F}[R^{lin}{}^{m}_{m}(x_l\approx x_k,x_{\phi(1)},\ldots,x_{\phi(m)})] &= S^{lin}{}^{m}_{m}(\widehat{\sigma}_{t,F}[x_l],\widehat{\sigma}_{t,F}[x_{\phi(1)}],\ldots,,\widehat{\sigma}[x_{\phi(m)}])\\
&\approx S^{lin}{}^{m}_{m}(\widehat{\sigma}_{t,F}[x_k],\widehat{\sigma}_{t,F}[x_{\phi(1)}],\ldots,\widehat{\sigma}_{t,F}[x_{\phi(m)}])\\
&= R^{lin}{}^{m}_{m}(\widehat{\sigma}_{t,F}[x_l]\approx\widehat{\sigma}_{t,F}[x_k],\widehat{\sigma}_{t,F}[x_{\phi(1)}],\ldots,\widehat{\sigma}_{t,F}[x_{\phi(m)}])\\
&= R^{lin}{}^{m}_{m}(\widehat{\sigma}_{t,F}[x_l\approx x_k],\widehat{\sigma}_{t,F}[x_{\phi(1)}],\ldots,\widehat{\sigma}_{t,F}[x_{\phi(m)}]).
\end{aligned}$$

If β has the form $\gamma(x_{\phi(1)},\ldots,x_{\phi(m)})$ where $\phi\in S_m$, then by Theorem 2 we have

$$\begin{aligned}
&\widehat{\sigma}[R^{lin}{}^{m}_{m}(\gamma(x_{\phi(1)},\ldots,x_{\phi(m)}),x_{\phi(1)},\ldots,x_{\phi(m)})]\\
&\quad= R^{lin}{}^{m}_{m}(\sigma_{t,F}(\gamma),\widehat{\sigma}_{t,F}[S^{lin}{}^{m}_{m}(x_{\phi(1)},x_{\phi(1)},\ldots,x_{\phi(m)})],\ldots,\\
&\quad= \widehat{\sigma}_{t,F}[S^{lin}{}^{m}_{m}(x_{\phi(m)},x_{\phi(1)},\ldots,x_{\phi(m)})])\\
&\quad= R^{lin}{}^{m}_{m}(\sigma_{t,F}(\gamma),S^{lin}{}^{m}_{m}(\widehat{\sigma}_{t,F}[x_{\phi(1)}],\widehat{\sigma}_{t,F}[x_{\phi(1)}],\ldots,\widehat{\sigma}_{t,F}[x_{\phi(m)}]),\ldots,\\
&\qquad S^{lin}{}^{m}_{m}(\widehat{\sigma}_{t,F}[x_{\phi(m)}],\widehat{\sigma}_{t,F}[x_{\phi(1)}],\ldots,\widehat{\sigma}_{t,F}[x_{\phi(m)}]))\\
&\quad= R^{lin}{}^{m}_{m}(R^{lin}{}^{m}_{m}(\sigma_{t,F}(\gamma),\widehat{\sigma}_{t,F}[x_{\phi(1)}],\ldots,\widehat{\sigma}_{t,F}[x_{\phi(m)}]),\widehat{\sigma}_{t,F}[x_{\phi(1)}],\ldots,\widehat{\sigma}_{t,F}[x_{\phi(m)}])\\
&\quad= R^{lin}{}^{m}_{m}(\widehat{\sigma}_{t,F}[\gamma(x_{\phi(1)},\ldots,x_{\phi(m)})],\widehat{\sigma}_{t,F}[x_{\phi(1)}],\ldots,\widehat{\sigma}_{t,F}[x_{\phi(m)}]).
\end{aligned}$$

If β has the form $\neg F$ and assume that

$$\widehat{\sigma}_{t,F}[R^{lin}{}^{m}_{m}(F,x_{\phi(1)},\ldots,x_{\phi(m)})] = R^{lin}{}^{m}_{m}(\widehat{\sigma}_{t,F}[F],\widehat{\sigma}_{t,F}[x_{\phi(1)}],\ldots,\widehat{\sigma}_{t,F}[x_{\phi(m)}]),$$

then

$$\begin{aligned}
\widehat{\sigma}_{t,F}[R^{lin}{}^{m}_{m}(\neg F,x_{\phi(1)},\ldots,x_{\phi(m)})] &= \neg(\widehat{\sigma}_{t,F}[R^{lin}{}^{m}_{m}(F,x_{\phi(1)},\ldots,x_{\phi(m)})])\\
&= \neg(R^{lin}{}^{m}_{m}(\widehat{\sigma}_{t,F}[F],\widehat{\sigma}_{t,F}[x_{\phi(1)}],\ldots,\widehat{\sigma}_{t,F}[x_{\phi(m)}]))\\
&= R^{lin}{}^{m}_{m}(\widehat{\sigma}_{t,F}[\neg F],\widehat{\sigma}_{t,F}[x_{\phi(1)}],\ldots,\widehat{\sigma}_{t,F}[x_{\phi(m)}]).
\end{aligned}$$

If β has the form $F_1\vee F_2$ and assume that

$$\widehat{\sigma}_{t,F}[R^{lin}{}^{m}_{m}(F_l,x_{\phi(1)},\ldots,x_{\phi(m)})] = R^{lin}{}^{m}_{m}(\widehat{\sigma}_{t,F}[F_l],\widehat{\sigma}_{t,F}[x_{\phi(1)}],\ldots,\widehat{\sigma}_{t,F}[x_{\phi(m)}])\quad \text{for all } l=1,2,$$

then, we consider

$$\begin{aligned}
&\widehat{\sigma}_{t,F}[R^{lin}{}^{m}_{m}(F_1\vee F_2,x_{\phi(1)},\ldots,x_{\phi(m)})]\\
&\quad= \widehat{\sigma}_{t,F}[R^{lin}{}^{m}_{m}(F_1,x_{\phi(1)},\ldots,x_{\phi(m)})]\vee\widehat{\sigma}_{t,F}[R^{lin}{}^{m}_{m}(F_2,x_{\phi(1)},\ldots,x_{\phi(m)})]\\
&\quad= R^{lin}{}^{m}_{m}(\widehat{\sigma}_{t,F}[F_1\vee F_2],\widehat{\sigma}_{t,F}[x_{\phi(1)}],\ldots,\widehat{\sigma}_{t,F}[x_{\phi(m)}]).
\end{aligned}$$

□

Now, we can say that the extension $\widehat{\sigma}_{t,F}$ of a linear hypersubstitution $\sigma_{t,F}$ of type $((n),(m))$ is an endomorphism. As a result of Lemma 2, we have the following lemma.

Lemma 3. *Let $t_1,t_2 \in W^{lin}_{(n)}(X_n)$ and $F_1,F_2 \in \mathscr{F}^{lin}_{((n),(m))}(W^{lin}_{(n)}(X_m))$. Then for any $\sigma_{t_1,F_1},\sigma_{t_2,F_2} \in Hyp^{lin}((n),(m))$, we have*

$$(\sigma_{t_1,F_1} \circ_r \sigma_{t_2,F_2})\widehat{} = \widehat{\sigma}_{t_1,F_1} \circ \widehat{\sigma}_{t_2,F_2}.$$

Proof. Let $t \in W^{lin}_{(n)}(X_n)$. We give a proof by induction on the complexity of a linear term t. If $t = x_i$; $1 \le i \le n$, then $(\sigma_{t_1,F_1} \circ_r \sigma_{t_2,F_2})\widehat{}\,[x_i] = x_i = \widehat{\sigma}_{t_1,F_1}[x_i] = \widehat{\sigma}_{t_1,F_1}[\widehat{\sigma}_{t_2,F_2}[x_i]] = (\widehat{\sigma}_{t_1,F_1} \circ \widehat{\sigma}_{t_2,F_2})[x_i]$.
If $t = f(x_{\pi(1)},\dots,x_{\pi(n)})$ where $\pi \in S_n$, then we have

$$\begin{aligned}
&(\sigma_{t_1,F_1} \circ_r \sigma_{t_2,F_2})\widehat{}\,[f(x_{\pi(1)},\dots,x_{\pi(n)})]\\
&\quad = S^{lin}{}^{n}_{n}(\widehat{\sigma}_{t_1,F_1}[\sigma_{t_2,F_2}(f)],\widehat{\sigma}_{t_1,F_1}[\widehat{\sigma}_{t_2,F_2}[x_{\pi(1)}]],\dots,\widehat{\sigma}_{t_1,F_1}[\widehat{\sigma}_{t_2,F_2}[x_{\pi(n)}]])\\
&\quad = \widehat{\sigma}_{t_1,F_1}[S^{lin}{}^{n}_{n}(\sigma_{t_2,F_2}(f),\widehat{\sigma}_{t_2,F_2}[x_{\pi(1)}],\dots,\widehat{\sigma}_{t_2,F_2}[x_{\pi(n)}])]\\
&\quad = (\widehat{\sigma}_{t_1,F_1} \circ \widehat{\sigma}_{t_2,F_2})[f(x_{\pi(1)},\dots,x_{\pi(n)})].
\end{aligned}$$

Let $\beta \in \mathscr{F}^{lin}_{((n),(m))}(W^{lin}_{(n)}(X_m))$. We give a proof by the following steps.
If β has the form $x_l \approx x_k$ where $l,k \in \{1,\dots,m\}$ and $l \neq k$, then

$$\begin{aligned}
(\sigma_{t_1,F_1} \circ_r \sigma_{t_2,F_2})\widehat{}[x_l \approx x_k] &= \widehat{\sigma}_{t_1,F_1}[\widehat{\sigma}_{t_2,F_2}[x_l]] \approx \widehat{\sigma}_{t_1,F_1}[\widehat{\sigma}_{t_2,F_2}[x_k]]\\
&= \widehat{\sigma}_{t_1,F_1}[\widehat{\sigma}_{t_2,F_2}[x_l] \approx \widehat{\sigma}_{t_2,F_2}[x_k]]\\
&= (\widehat{\sigma}_{t_1,F_1} \circ \widehat{\sigma}_{t_2,F_2})[x_l \approx x_k].
\end{aligned}$$

If β has the form $\gamma(x_{\phi(1)},\dots,x_{\phi(m)})$ where $\phi \in S_m$, then by Theorem 2 we have

$$\begin{aligned}
&(\sigma_{t_1,F_1} \circ_r \sigma_{t_2,F_2})\widehat{}\,[\gamma(x_{\phi(1)},\dots,x_{\phi(m)})]\\
&\quad = R^{lin}{}^{m}_{m}((\widehat{\sigma}_{t_1,F_1}[\sigma_{t_2,F_2}(\gamma)],\widehat{\sigma}_{t_1,F_1}[\widehat{\sigma}_{t_2,F_2}[x_{\phi(1)}]],\dots,\widehat{\sigma}_{t_1,F_1}[\widehat{\sigma}_{t_2,F_2}[x_{\phi(m)}]])\\
&\quad = \widehat{\sigma}_{t_1,F_1}[R^{lin}{}^{m}_{m}(\sigma_{t_2,F_2}[\gamma],\widehat{\sigma}_{t_1,F_1}[x_{\phi(1)}],\dots,\widehat{\sigma}_{t_1,F_1}[x_{\phi(m)}])]\\
&\quad = (\widehat{\sigma}_{t_1,F_1} \circ \widehat{\sigma}_{t_2,F_2})[\gamma(x_{\phi(1)},\dots,x_{\phi(m)})].
\end{aligned}$$

If β has the form $\neg F$ and we assume that $(\sigma_{t_1,F_1} \circ_r \sigma_{t_2,F_2})\widehat{}\,[F] = (\widehat{\sigma}_{t_1,F_1} \circ \widehat{\sigma}_{t_2,F_2})[F]$, then

$$\begin{aligned}
(\sigma_{t_1,F_1} \circ_r \sigma_{t_2,F_2})\widehat{}\,[\neg F] &= \neg(\widehat{\sigma}_{t_1,F_1}[\widehat{\sigma}_{t_2,F_2}[F]])\\
&= \widehat{\sigma}_{t_1,F_1}[\neg(\widehat{\sigma}_{t_2,F_2}[F])]\\
&= \widehat{\sigma}_{t_1,F_1}[\widehat{\sigma}_{t_2,F_2}[\neg(F)]]\\
&= (\widehat{\sigma}_{t_1,F_1} \circ \widehat{\sigma}_{t_2,F_2})[\neg(F)].
\end{aligned}$$

If β has the form $F_1 \vee F_2$ and we assume that $(\sigma_{t_1,F_1} \circ_r \sigma_{t_2,F_2})\widehat{}\,[F_l] = (\widehat{\sigma}_{t_1,F_1} \circ \widehat{\sigma}_{t_2,F_2})[F_l]$ for all $l = 1,2$, then

$$\begin{aligned}
(\sigma_{t_1,F_1} \circ_r \sigma_{t_2,F_2})\widehat{}\,[F_1 \vee F_2] &= \widehat{\sigma}_{t_1,F_1}[\widehat{\sigma}_{t_2,F_2}[F_1]] \vee \widehat{\sigma}_{t_1,F_1}[\widehat{\sigma}_{t_2,F_2}[F_2]]\\
&= \widehat{\sigma}_{t_1,F_1}[\widehat{\sigma}_{t_2,F_2}[F_1] \vee \widehat{\sigma}_{t_2,F_2}[F_2]]\\
&= \widehat{\sigma}_{t_1,F_1}[\widehat{\sigma}_{t_2,F_2}[F_1 \vee F_2]\\
&= (\widehat{\sigma}_{t_1,F_1} \circ \widehat{\sigma}_{t_2,F_2})[F_1 \vee F_2].
\end{aligned}$$

□

It follows from Lemma 3 that the binary operation $\circ_r$ satisfies the associative law. We prove this fact in the next lamma.

Lemma 4. *Let $t_1,t_2,t_3 \in W^{lin}_{(n)}(X_n), F_1,F_2,F_3 \in \mathscr{F}^{lin}_{((n),(m))}(W^{lin}_{(n)}(X_m))$. Then for any $\sigma_{t_1,F_1},\sigma_{t_2,F_2},\sigma_{t_3,F_3} \in Hyp^{lin}((n),(m))$, we have*

$$(\sigma_{t_1,F_1} \circ_r \sigma_{t_2,F_2}) \circ_r \sigma_{t_3,F_3} = \sigma_{t_1,F_1} \circ_r (\sigma_{t_2,F_2} \circ_r \sigma_{t_3,F_3}).$$

Proof. Using Lemma 3 and using the fact that $\circ$ satisfies the associative law, it can be shown that $\circ_r$ satisfies the associative law. In fact, we have

$$\begin{aligned}(\sigma_{t_1,F_1} \circ_r \sigma_{t_2,F_2}) \circ_r \sigma_{t_3,F_3} &= \widehat{(\sigma_{t_1,F_1} \circ_r \sigma_{t_2,F_2})} \circ \sigma_{t_3,F_3} \\ &= (\widehat{\sigma}_{t_1,F_1} \circ \widehat{\sigma}_{t_2,F_2}) \circ \sigma_{t_3,F_3} \\ &= \widehat{\sigma}_{t_1,F_1} \circ (\widehat{\sigma}_{t_2,F_2} \circ \sigma_{t_3,F_3}) \\ &= \widehat{\sigma}_{t_1,F_1} \circ (\sigma_{t_2,F_2} \circ_r \sigma_{t_3,F_3}) \\ &= \sigma_{t_1,F_1} \circ_r (\sigma_{t_2,F_2} \circ_r \sigma_{t_3,F_3}).\end{aligned}$$ □

Let σ_{id} be a linear hypersubstitution for algebraic systems which maps the operation symbol f to the linear term $f(x_1,\dots,x_n)$ and maps the relational symbol γ to the linear formula $\gamma(x_1,\dots,x_m)$, i.e. $\sigma_{id}(f) = f(x_1,\dots,x_n)$ and $\sigma_{id}(\gamma) = \gamma(x_1,\dots,x_m)$.

Lemma 5. *For any linear term $t \in W^{lin}_{(n)}(X_n)$ and linear formula $\beta \in \mathscr{F}^{lin}_{((n),(m))}(W^{lin}_{(n)}(X_m))$, we have $\widehat{\sigma}_{id}[t] = t$ and $\widehat{\sigma}_{id}[\beta] = \beta$.*

Proof. The proof is straightforward and hence omitted. □

A linear hypersubstitution σ_{id} is claimed to be an identity, which we will prove this fact in the next lemma.

Lemma 6. *Let $\sigma_{id} \in Hyp^{lin}((n),(m))$. Then σ_{id} is an identity element with respect to $\circ_r$.*

Proof. First, we prove that σ_{id} is a left identity element by using Lemma 5. Let $\sigma_{t,F} \in Hyp^{lin}((n),(m))$. Then we have $(\sigma_{id} \circ_r \sigma_{t,F})(f) = (\widehat{\sigma}_{id} \circ \sigma_{t,F})(f) = \widehat{\sigma}_{id}[\sigma_{t,F}(f)] = \sigma_{t,F}(f)$. Now, we show that σ_{id} is a right identity element. Let $\sigma_{t,F} \in Hyp^{lin}((n),(m))$. By Theorem 1, we obtain that $(\sigma_{t,F} \circ_r \sigma_{id})(f) = \widehat{\sigma}_{t,F}[f(x_1,\dots,x_n)] = S^{lin}{}^{n}_{n}(\sigma_{t,F}(f),x_1,\dots,x_n) = \sigma_{t,F}(f)$ and by Theorem 2 we have $(\sigma_{t,F} \circ_r \sigma_{id})(\gamma) = \widehat{\sigma}_{t,F}[\gamma(x_1,\dots,x_m)] = R^{lin}{}^{m}_{m}(\sigma_{t,F}(\gamma),x_1,\dots,x_m) = \sigma_{t,F}(\gamma)$. Therefore, $\sigma_{t,F} \circ_r \sigma_{id} = \sigma_{t,F} = \sigma_{id} \circ_r \sigma_{t,F}$. □

Theorem 3. *$\mathscr{H}yp^{lin}((n),(m)) := (Hyp^{lin}((n),(m)), \circ_r, \sigma_{id})$ is a monoid.*

Proof. From Lemma 4 and 6, the conclusion holds. □

Next, we study some semigroup properties of $\mathscr{H}yp^{lin}((n),(m))$, especially we characterize idempotency and regularity of $\sigma_{t,F} \in Hyp^{lin}((n),(m))$.

4. Regularity of $\mathscr{H}yp^{lin}((n),(m))$

Firstly, we separate the classes of all linear hypersubstitutions of type $((n),(m))$ by considering the image of a mapping $\sigma_{t,F}$ in several forms. Since the set $W^{lin}_{(n)}(X_n)$ contains elements in following forms: $x_i \in X_n$ and $f(x_{\pi(1)},\dots,x_{\pi(n)})$ where $\pi \in S_n$ and the set $\mathscr{F}^{lin}_{((n),(m))}(W^{lin}_{(n)}(X_m))$

contains all m-ary linear formulas in four forms as follows: $x_l \approx x_k$, $\gamma(x_{\phi(1)},\dots,x_{\phi(m)})$ with $\phi \in S_m$, these lead us to use the connective "negation" and "or" for the first and second forms. So we can separate the class of linear hypersubstitutions into sixteen classes and we denote by the following notations:

For any $\sigma_{t,F} \in Hyp^{lin}((n),(m)), \pi \in S_n, \phi \in S_m, l,k,i_1,i_2,i_3,i_4 \in \{1,\dots,m\}$ with $l \neq k$ and i_1,i_2,i_3,i_4 are all distinct we denote:

$$C_1 := \{\sigma_{t,F} \mid t = x_i \in X_n, F = x_l \approx x_k\},$$
$$C_2 := \{\sigma_{t,F} \mid t = x_i \in X_n, F = \gamma(x_{\phi(1)},\dots,x_{\phi(m)})\},$$
$$C_3 := \{\sigma_{t,F} \mid t = x_i \in X_n, F = \neg(x_l \approx x_k)\},$$
$$C_4 := \{\sigma_{t,F} \mid t = x_i \in X_n, F = \neg\gamma(x_{\phi(1)},\dots,x_{\phi(m)})\},$$
$$C_5 := \{\sigma_{t,F} \mid t = x_i \in X_n, F = (x_{i_1} \approx x_{i_2}) \vee (x_{i_3} \approx x_{i_4})\},$$
$$C_6 := \{\sigma_{t,F} \mid t = x_i \in X_n, F = \neg(x_{i_1} \approx x_{i_2}) \vee (x_{i_3} \approx x_{i_4})\},$$
$$C_7 := \{\sigma_{t,F} \mid t = x_i \in X_n, F = (x_{i_1} \approx x_{i_2}) \vee \neg(x_{i_3} \approx x_{i_4})\},$$
$$C_8 := \{\sigma_{t,F} \mid t = x_i \in X_n, F = \neg(x_{i_1} \approx x_{i_2}) \vee \neg(x_{i_3} \approx x_{i_4})\},$$
$$C_9 := \{\sigma_{t,F} \mid t = f(x_{\pi(1)},\dots,x_{\pi(n)}), F = x_l \approx x_k\},$$
$$C_{10} := \{\sigma_{t,F} \mid t = f(x_{\pi(1)},\dots,x_{\pi(n)}), F = \gamma(x_{\phi(1)},\dots,x_{\phi(m)})\},$$
$$C_{11} := \{\sigma_{t,F} \mid t = f(x_{\pi(1)},\dots,x_{\pi(n)}), F = \neg(x_l \approx x_k)\},$$
$$C_{12} := \{\sigma_{t,F} \mid t = f(x_{\pi(1)},\dots,x_{\pi(n)}), F = \neg\gamma(x_{\phi(1)},\dots,x_{\phi(m)})\},$$
$$C_{13} := \{\sigma_{t,F} \mid t = f(x_{\pi(1)},\dots,x_{\pi(n)}), F = (x_{i_1} \approx x_{i_2}) \vee (x_{i_3} \approx x_{i_4})\},$$
$$C_{14} := \{\sigma_{t,F} \mid t = f(x_{\pi(1)},\dots,x_{\pi(n)}), F = \neg(x_{i_1} \approx x_{i_2}) \vee (x_{i_3} \approx x_{i_4})\},$$
$$C_{15} := \{\sigma_{t,F} \mid t = f(x_{\pi(1)},\dots,x_{\pi(n)}), F = (x_{i_1} \approx x_{i_2}) \vee \neg(x_{i_3} \approx x_{i_4})\},$$
$$C_{16} := \{\sigma_{t,F} \mid t = f(x_{\pi(1)},\dots,x_{\pi(n)}), F = \neg(x_{i_1} \approx x_{i_2}) \vee \neg(x_{i_3} \approx x_{i_4})\}.$$

We note that $P = \{C_1,\dots,C_{16}\}$ is a partition of $Hyp^{lin}((n),(m))$.

We now introduce definitions of idempotent and regular elements for $Hyp^{lin}((n),(m))$ with respect to $\circ_r$. An element $\sigma_{t,F} \in Hyp^{lin}((n),(m))$ is said to be *idempotent* if $\sigma_{t,F} \circ_r \sigma_{t,F} = \sigma_{t,F}$, that is, $(\sigma_{t,F} \circ_r \sigma_{t,F})(f) = \sigma_{t,F}(f)$ and $(\sigma_{t,F} \circ_r \sigma_{t,F})(\gamma) = \sigma_{t,F}(\gamma)$ and $\sigma_{t,F} \in Hyp^{lin}((n),(m))$ is called *regular* if there is an element $\sigma_{t',F'} \in Hyp^{lin}((n),(m))$ such that $\sigma_{t,F} = \sigma_{t,F} \circ_r \sigma_{t',F'} \circ_r \sigma_{t,F}$. The semigroup $Hyp^{lin}((n),(m))$ is called regular if every element in $Hyp^{lin}((n),(m))$ is regular. Furthermore, we denote the set of all idempotent and regular elements in $Hyp^{lin}((n),(m))$ by $E(Hyp^{lin}((n),(m)))$ and $Reg(Hyp^{lin}((n),(m)))$, respectively.

First, we introduce the following lemma which is an important tool to study the idempotent elements in $\mathscr{H}yp^{lin}((n),(m))$.

Lemma 7. *For each $\sigma_{t,F} \in Hyp^{lin}((n),(m))$. Then $\sigma_{t,F}$ is idempotent in $\mathscr{H}yp^{lin}((n),(m))$ if and only if $\widehat{\sigma}_{t,F}[t] = t$ and $\widehat{\sigma}_{t,F}[F] = F$.*

Proof. Assume that $\sigma_{t,F}$ is idempotent. We now consider

$$\begin{aligned}\widehat{\sigma}_{t,F}[t] &= \widehat{\sigma}_{t,F}[\sigma_{t,F}(f)] = (\widehat{\sigma}_{t,F} \circ \sigma_{t,F})(f) = (\sigma_{t,F} \circ_r \sigma_{t,F})(f)\\ &= \sigma_{t,F}(f) = t\end{aligned}$$

and

$$\begin{aligned}\widehat{\sigma}_{t,F}[F] &= \widehat{\sigma}_{t,F}[\sigma_{t,F}(\gamma)] = (\widehat{\sigma}_{t,F} \circ \sigma_{t,F})(\gamma)\\ &= (\sigma_{t,F} \circ_r \sigma_{t,F})(\gamma) = \sigma_{t,F}(\gamma)\\ &= F.\end{aligned}$$

Conversely, let $\widehat{\sigma}_{t,F}[t] = t$ and $\widehat{\sigma}_{t,F}[F] = F$. Then, we have

$$\begin{aligned}(\sigma_{t,F} \circ_r \sigma_{t,F})(f) &= (\widehat{\sigma}_{t,F} \circ \sigma_{t,F})(f) = \widehat{\sigma}_{t,F}[\sigma_{t,F}(f)]\\ &= \widehat{\sigma}_{t,F}[t] = t\\ &= \sigma_{t,F}(f)\end{aligned}$$

and

$$\begin{aligned}(\sigma_{t,F} \circ_r \sigma_{t,F})(\gamma) &= (\widehat{\sigma}_{t,F} \circ \sigma_{t,F})(\gamma) = \widehat{\sigma}_{t,F}[\sigma_{t,F}(\gamma)]\\ &= \widehat{\sigma}_{t,F}[F] = F\\ &= \sigma_{t,F}(\gamma).\end{aligned}$$

This shows that $\sigma_{t,F}$ is idempotent. □

Theorem 4. *Let $\sigma_{t,F} \in Hyp^{lin}((n),(m))$. Then the following statements hold.*

(i) *Every $\sigma_{t,F} \in C_1$ is idempotent.*
(ii) *Every $\sigma_{t,F} \in C_3$ is idempotent.*
(iii) *Every $\sigma_{t,F} \in C_4$ is not idempotent.*
(iv) *Every $\sigma_{t,F} \in C_5$ is idempotent.*
(v) *Every $\sigma_{t,F} \in C_6$ is idempotent.*
(vi) *Every $\sigma_{t,F} \in C_7$ is idempotent.*
(vii) *Every $\sigma_{t,F} \in C_8$ is idempotent.*

Proof. (i) We first prove that $\sigma_{t,F} \in C_1$ is idempotent. To do this, let $\sigma_{t,F} \in B_1$. Then $t = x_i$, $F = x_l \approx x_k$. We consider $\widehat{\sigma}_{t,F}[x_i] = x_i$ and $\widehat{\sigma}_{t,F}[x_l \approx x_k] = \widehat{\sigma}_{t,F}[x_l] \approx \widehat{\sigma}_{t,F}[x_k] = x_l \approx x_k$. By Lemma 7, $\sigma_{t,F}$ is idempotent.

(ii) Let $\sigma_{t,F} \in C_3$. Then $t = x_i, F = \neg(x_l \approx x_k)$ so that $\widehat{\sigma}_{t,F}[x_i] = x_i$ and $\widehat{\sigma}_{t,F}[\neg(x_l \approx x_k)] = \neg(\widehat{\sigma}_{t,F}[x_l \approx x_k]) = \neg(\widehat{\sigma}_{t,F}[x_l] \approx \widehat{\sigma}_{t,F}[x_k]) = \neg(x_l \approx x_k)$. By Lemma 7, $\sigma_{t,F}$ is idempotent.

(iii) Let $\sigma_{t,F} \in C_4$. Then $t = x_i, F = \neg\gamma(x_{\phi(1)},\dots,x_{\phi(m)})$. To show that it is not idempotent, we consider

$$\begin{aligned}\widehat{\sigma}_{t,F}[\neg\gamma(x_{\phi(1)},\dots,x_{\phi(m)})] &= \neg(R^{lin}{}^{m}_{m}(\neg\gamma(x_{\phi(1)},\dots,x_{\phi(m)}), x_{\phi(1)},\dots,x_{\phi(m)}))\\ &= \neg(\neg(\gamma(x_{\phi(\phi(1))},\dots,x_{\phi(\phi(m))})))\\ &= \gamma(x_{\phi(\phi(1))},\dots,x_{\phi(\phi(m))})\\ &\neq \neg\gamma(x_{\phi(1)},\dots,x_{\phi(m)}).\end{aligned}$$

Therefore, every $\sigma_{t,F} \in C_4$ is not idempotent.

(iv) Let $\sigma_{t,F} \in C_5$. Then $t = x_i$ and $F = (x_{i_1} \approx x_{i_2}) \vee (x_{i_3} \approx x_{i_4})$. Clearly, $\widehat{\sigma}_{t,F}[x_i] = x_i$. Next, we consider

$$\begin{aligned}\widehat{\sigma}_{t,F}[(x_{i_1} \approx x_{i_2}) \vee (x_{i_3} \approx x_{i_4})] &= \widehat{\sigma}_{t,F}[x_{i_1} \approx x_{i_2}] \vee \widehat{\sigma}_{t,F}[x_{i_3} \approx x_{i_4}]\\ &= \widehat{\sigma}_{t,F}[x_{i_1}] \approx \widehat{\sigma}_{t,F}[x_{i_2}] \vee \widehat{\sigma}_{t,F}[x_{i_3}] \approx \widehat{\sigma}_{t,F}[x_{i_4}]\\ &= (x_{i_1} \approx x_{i_2}) \vee (x_{i_3} \approx x_{i_4}).\end{aligned}$$

Thus $\sigma_{t,F} \in C_5$ is idempotent.

(v)-(vii) Similarly to the proof of (iv) . □

The following example shows that there is an element in C_2 which is not idempotent.

Example 3. Let ((4),(3)) be a type, i.e., we have one quaternary operation symbol and one ternary relation symbol, say f and γ, respectively. If $\sigma_{t,F} \in B_2$ with $t = x_4$ and $F = \gamma(x_2,x_3,x_1)$, then $\widehat{\sigma}_{t,F}[F] = \widehat{\sigma}_{t,F}[\gamma(x_2,x_3,x_1)] = R^{lin}\ {}^3_3(\sigma_{t,F}(\gamma),x_2,x_3,x_1) = R^{lin}\ {}^3_3(\gamma(x_2,x_3,x_1),x_2,x_3,x_1) = \gamma(x_3,x_1,x_2,) \neq F$. So, $\sigma_{t,F}$ in this form is not idempotent.

We have to find some necessary conditions for the elements in C_2 which are idempotent. The next theorem shows such condition.

Theorem 5. *Let $\sigma_{t,F} \in C_2$. Then $\sigma_{t,F}$ is idempotent if and only if $\phi(j) = j$ for all $j = 1,\dots,m$.*

Proof. Let $\sigma_{t,F} \in C_2$. Then we have $t = x_i, F = \gamma(x_{\phi(1)},\dots,x_{\phi(m)})$. Assume that $\phi(i) \neq i$ for some $i = 1,\dots,m$. We now consider $\widehat{\sigma}_{t,F}[\gamma(x_{\phi(1)},\dots,x_{\phi(m)})] = R^{lin}\ {}^m_m(\gamma(x_{\phi(1)},\dots,x_{\phi(m)}),x_{\phi(1)},\dots,x_{\phi(m)}) = \gamma(x_{\phi(\phi(1))},\dots,x_{\phi(\phi(m))})$ and by assumption we have that $\widehat{\sigma}_{t,F}[\gamma(x_{\phi(1)},\dots,x_{\phi(m)})] = \gamma(x_{\phi(\phi(1))},\dots,x_{\phi(\phi(m))}) \neq \gamma(x_{\phi(1)},\dots,x_{\phi(m)})$ and thus $\sigma_{t,F}$ is not idempotent. Conversely, assume that the condition holds. Clearly, $\widehat{\sigma}_{t,F}[x_i] = x_i$ and it is not hard to verify that $\widehat{\sigma}_{t,F}[\gamma(x_{\phi(1)},\dots,x_{\phi(m)})] = \gamma(x_{\phi(1)},\dots,x_{\phi(m)})$. Thus by using Lemma 7, we get that $\sigma_{t,F}$ is idempotent. □

Now, it comes to characterize the idempotent elements in $C_9,\dots,C_{16}$. We first show that all elements in C_{12} are not idempotent and then show that the idempotency of $C_9,\dots,C_{16}$ need the some conditions. In fact, we have the following results.

Theorem 6. *Every $\sigma_{t,F} \in C_{12}$ is not idempotent.*

Proof. Let $\sigma_{t,F} \in C_{12}$ with $t = f(x_{\pi(1)},\dots,x_{\pi(n)}), F = \neg\gamma(x_{\phi(1)},\dots,x_{\phi(m)})$. Suppose the contrary that $\sigma_{t,F}$ is idempotent, by Lemma 7, we obtain that $\widehat{\sigma}_{t,F}[t] = t$ and $\widehat{\sigma}_{t,F}[F] = F$. Obviously, $\widehat{\sigma}_{t,F}[\neg\gamma(x_{\phi(1)},\dots,x_{\phi(m)})]\ \neg\gamma(x_{\phi(1)},\dots,x_{\phi(m)})$ since we have already shown this inequality holds in Theorem 4(iii). It contradicts to the result of our assumption. Therefore, $\sigma_{t,F}$ is not idempotent. □

We remark here that if $\sigma_{t,F} \in C_9,\dots,C_{16}$, then $\widehat{\sigma}_{t,F}[F]$ has the same situation in the previous theorems. So, we are interesting in the way to find some conditions for the idempotency of $\widehat{\sigma}_{t,F}[t]$. The next theorem shows that if we set some conditions, then we get the characterization of idempotent elements in $C_9,\dots,C_{16}$.

Theorem 7. *Let $\sigma_{t,F} \in Hyp^{lin}((n),(m))$. Then the following statements hold.*

(i) *$\sigma_{t,F} \in C_9$ is idempotent if and only if $\pi(i) = i$ for all $i = 1,\dots,n$.*

(ii) *$\sigma_{t,F} \in C_{10}$ is idempotent if and only if $\pi(i) = i$ for all $i = 1,\dots,n$ and $\phi(j) = j$ for all $j = 1,\dots,m$.*

(iii) *$\sigma_{t,F} \in C_{11}$ is idempotent if and only if $\pi(i) = i$ for all $i = 1,\dots,n$.*

(iv) *$\sigma_{t,F} \in C_{13}$ is idempotent if and only if $\pi(i) = i$ for all $i = 1,\dots,n$.*

(v) $\sigma_{t,F} \in C_{14}$ *is idempotent if and only if* $\pi(i) = i$ *for all* $i = 1,\dots,n$.

(vi) $\sigma_{t,F} \in C_{15}$ *is idempotent if and only if* $\pi(i) = i$ *for all* $i = 1,\dots,n$.

(vii) $\sigma_{t,F} \in C_{16}$ *is idempotent if and only if* $\pi(i) = i$ *for all* $i = 1,\dots,n$.

Proof. (i) Let $\sigma_{t,F} \in B_9$. Then $t = f(x_{\pi(1)},\dots,x_{\pi(n)}), F = x_l \approx x_k$. Now we may assume that $\pi(i) \neq i$ for some $i = 1,\dots,n$. Then $\widehat{\sigma}_{t,F}[f(x_{\pi(1)},\dots,x_{\pi(n)})] = S^{lin}\,{}^{n}_{n}(f(x_{\pi(1)},\dots,x_{\pi(n)}),x_{\pi(1)},\dots,x_{\pi(n)}) = f(x_{\pi(\pi(1))},\dots,x_{\pi(\pi(n))})$. By our assumption,

$$f(x_{\pi(\pi(1))},\dots,x_{\pi(\pi(n))}) \neq f(x_{\pi(1)},\dots,x_{\pi(n)})$$

and thus $\sigma_{t,F}$ is not idempotent. Conversely, assume that the condition holds. We now consider $\widehat{\sigma}_{t,F}[f(x_{\pi(1)},\dots,x_{\pi(n)})] = \widehat{\sigma}_{t,F}[f(x_1,\dots,x_n)] = f(x_1,\dots,x_n) = f(x_{\pi(1)},\dots,x_{\pi(n)})$ so that $\widehat{\sigma}_{t,F}[t] = t$. We can prove similarly to the proof of Theorem 4 (i) that $\widehat{\sigma}_{t,F}[F] = F$. Therefore, $\sigma_{t,F}$ is idempotent.

(ii) Let $\sigma_{t,F} \in C_{10}$. Then $t = f(x_{\pi(1)},\dots,x_{\pi(n)}), F = \gamma(x_{\phi(1)},\dots,x_{\phi(m)})$. We first assume that $\pi(i) \neq i$ for some $i = 1,\dots,n$ or $\phi(j) \neq j$ for some $j = 1,\dots,m$. Then by the same manner as in the proof of (i) we can show that $\sigma_{t,F}$ is not idempotent. Conversely, assume that the condition holds. Clearly, $\widehat{\sigma}_{t,F}[f(x_{\pi(1)},\dots,x_{\pi(n)})] = f(x_1,\dots,x_n) = f(x_{\pi(1)},\dots,x_{\pi(n)})$ and thus $\widehat{\sigma}_{t,F}[t] = t$. Moreover, we have that $\widehat{\sigma}_{t,F}[\gamma(x_{\phi(1)},x_{\phi(2)})] = \widehat{\sigma}_{t,F}[\gamma(x_1,\dots,x_m)] = \gamma(x_1,\dots,x_m)$, that is $\widehat{\sigma}_{t,F}[F] = F$. By Lemma 7, $\sigma_{t,F}$ is idempotent.

(iii) By using Lemma 1, we can prove similarly to the proof of (i) that this statement holds.

(iv)-(vii) It is easy to verify that these statements hold. □

Now, the characterization of idempotent linear hypersubstitutions are completed. Next, we study the regularity of linear hypersubstitutions. In general semigroups, we known that every idempotent element is regular. To characterize which linear hypersubstitutions in $Hyp^{lin}((n),(m))$ are regular, we consider only for the case $\sigma_{t,F}$ which is not idempotent. The characterization of regularity in $Hyp^{lin}((n),(m))$ can be shown in the next theorem.

Theorem 8. *Let* $\sigma_{t,F} \in Hyp^{lin}((n),(m))$. *Then the following statements hold.*

(i) *Every* $\sigma_{t,F} \in C_2$ *is regular.*

(ii) *Every* $\sigma_{t,F} \in C_4$ *is regular.*

(iii) *Every* $\sigma_{t,F} \in C_9$ *is regular.*

(iv) *Every* $\sigma_{t,F} \in C_{10}$ *is regular.*

(v) *Every* $\sigma_{t,F} \in C_{11}$ *is regular.*

(vi) *Every* $\sigma_{t,F} \in C_{12}$ *is regular.*

(vii) *Every* $\sigma_{t,F} \in C_{13}$ *is regular.*

(viii) *Every* $\sigma_{t,F} \in C_{14}$ *is regular.*

(ix) *Every* $\sigma_{t,F} \in C_{15}$ *is regular.*

(x) *Every* $\sigma_{t,F} \in C_{16}$ *is regular.*

Proof. (i) Let $\sigma_{t,F} \in C_2$ with $t = x_i, F = \gamma(x_{\phi(1)},\dots,x_{\phi(m)})$. We consider regularity of $\sigma_{t,F} \in C_2$ only the case of $\phi(j) \neq j$ for some $j = 1,\dots,m$. To do this, we choose $\sigma_{t',F'} \in C_2$ with $t' = x_i$ and $F' = \gamma(x_{\phi^{-1}(1)},\dots,x_{\phi^{-1}(m)})$ such that $(\sigma_{t,F} \circ_r \sigma_{t',F'} \circ_r \sigma_{t,F})(f) = x_i = (\sigma_{t,F})(f)$ and

$$\begin{aligned}(\sigma_{t,F} \circ_r \sigma_{t',F'} \circ_r \sigma_{t,F})(\gamma) &= \widehat{\sigma}_{t,F}[\widehat{\sigma}_{t',F'}[\gamma(x_{\phi(1)},\dots,x_{\phi(m)})]] \\ &= \widehat{\sigma}_{t,F}[\gamma(x_{\phi(\phi^{-1}(1))},\dots,x_{\phi(\phi^{-1}(m))})] \\ &= \widehat{\sigma}_{t,F}[\gamma(x_{(\phi\circ\phi^{-1})(1)},\dots,x_{(\phi\circ\phi^{-1})(m)})] \\ &= \widehat{\sigma}_{t,F}[\gamma(x_1,\dots,x_m)]\end{aligned}$$

$$= R^{lin}\,{}^{m}_{m}(\gamma(x_{\phi(1)},\dots,x_{\phi(m)}),x_1,\dots,x_m)$$
$$= \sigma_{t,F}(\gamma).$$

This implies that $\sigma_{t,F}$ is regular.

(ii) Similarly to the proof of (i) and by using Lemma 1, we can show that every $\sigma_{t,F} \in C_4$ is regular.

(iii) Let $\sigma_{t,F} \in C_9$ with $t = f(x_{\pi(1)},\dots,x_{\pi(n)}), F = x_l \approx x_k$. We consider in the case of $\pi(i) \neq i$ for some $i = 1,\dots,n$, then there exists $\sigma_{t',F'} \in C_5$ with $t' = f(x_{\pi^{-1}(1)},\dots,x_{\pi^{-1}(n)})$ and $F' = x_l \approx x_k$ such that

$$\begin{aligned}(\sigma_{t,F} \circ_r \sigma_{t',F'} \circ_r \sigma_{t,F})(f) &= \widehat{\sigma}_{t,F}[S^{lin}\,{}^{n}_{n}(f(x_{\pi^{-1}(1)},\dots,x_{\pi^{-1}(n)}),x_{\pi(1)},\dots,x_{\pi(n)})]\\
&= \widehat{\sigma}_{t,F}[f(x_{(\pi\circ\pi^{-1})(1)},\dots,x_{(\pi\circ\pi^{-1})(n)})]\\
&= \widehat{\sigma}_{t,F}[f(x_1,\dots,x_n)]\\
&= S^{lin}\,{}^{n}_{n}(f(x_{\pi(1)},\dots,x_{\pi(n)}),x_1,\dots,x_n)\\
&= f(x_{\pi(1)},\dots,x_{\pi(n)})\\
&= \sigma_{t,F}(f)\end{aligned}$$

and

$$\begin{aligned}(\sigma_{t,F} \circ_r \sigma_{t',F'} \circ_r \sigma_{t,F})(\gamma) &= \widehat{\sigma}_{t,F}[\widehat{\sigma}_{t',F'}[x_l \approx x_k]]\\
&= \widehat{\sigma}_{t,F}[\widehat{\sigma}_{t',F'}[x_l] \approx \widehat{\sigma}_{t',F'}[x_k]]\\
&= \widehat{\sigma}_{t,F}[x_l \approx x_k]\\
&= (x_l \approx x_k).\end{aligned}$$

Thus $\sigma_{t,F}$ is regular.

(iv) Let $\sigma_{t,F} \in C_{10}$. Then $t = f(x_{\pi(1)},\dots,x_{\pi(n)}), F = \gamma(x_{\phi(1)},\dots,x_{\phi(m)})$. To prove that $\sigma_{t,F}$ is regular, we consider into three cases.

Case 1: If $\pi(i) = i$ for all $i = 1,\dots,n$ and $\phi(j) \neq j$ for some $j = 1,\dots,m$. Then there exists $\sigma_{t',F'} \in C_{10}$ with $t' = f(x_{\pi(1)},\dots,x_{\pi(n)})$ and $F' = \gamma(x_{\phi^{-1}(1)},\dots,x_{\phi^{-1}(m)})$ such that

$$\begin{aligned}(\sigma_{t,F} \circ_r \sigma_{t',F'} \circ_r \sigma_{t,F})(f) &= \widehat{\sigma}_{t,F}[S^{lin}\,{}^{n}_{n}(f(x_{\pi(1)},\dots,x_{\pi(n)}),x_{\pi(1)},\dots,x_{\pi(n)})]\\
&= \widehat{\sigma}_{t,F}[f(x_{\pi(1)},\dots,x_{\pi(n)})]\\
&= S^{lin}\,{}^{n}_{n}(f(x_{\pi(1)},\dots,x_{\pi(n)}),x_{\pi(1)},\dots,x_{\pi(n)})\\
&= f(x_{\pi(1)},\dots,x_{\pi(n)})\\
&= \sigma_{t,F}(f),\end{aligned}$$

and similar to (i), it is easy to verify that $(\sigma_{t,F} \circ_r \sigma_{t',F'} \circ_r \sigma_{t,F})(\gamma) = \sigma_{t,F}(\gamma)$.

Case 2: $\pi(i) \neq i$ for some $i = 1,\dots,n$ and $\phi(j) = j$ for all $j = 1,\dots,m$. Then there exists $\sigma_{t',F'} \in C_{10}$ with $t' = f(x_{\pi^{-1}(1)},\dots,x_{\pi^{-1}(n)})$ and $F' = \gamma(x_{\phi(1)},\dots,x_{\phi(m)})$ such that $(\sigma_{t,F} \circ_r \sigma_{t',F'} \circ_r \sigma_{t,F})(f) = \sigma_{t,F}(f)$ which follows from (iii) and we have

$$\begin{aligned}(\sigma_{t,F} \circ_r \sigma_{t',F'} \circ_r \sigma_{t,F})(\gamma) &= \widehat{\sigma}_{t,F}[R^{lin}\,{}^{m}_{m}(\gamma(x_{\phi(1)},\dots,x_{\phi(m)}),x_{\phi(1)},\dots,x_{\phi(m)})]\\
&= \widehat{\sigma}_{t,F}[\gamma(x_{\phi(1)},\dots,x_{\phi(m)})]\\
&= R^{lin}\,{}^{m}_{m}(\gamma(x_{\phi(1)},\dots,x_{\phi(m)}),x_{\phi(1)},\dots,x_{\phi(m)})\end{aligned}$$

$$= \gamma(x_{\phi(1)}, \ldots, x_{\phi(m)})$$
$$= \sigma_{t,F}(\gamma).$$

Case 3: $\pi(i) \neq i$ for some $i = 1, \ldots, n$ and $\phi(j) \neq j$ for some $j = 1, \ldots, m$. Then there exists $\sigma_{t',F'} \in C_{10}$ with $t' = f(x_{\pi^{-1}(1)}, \ldots, x_{\pi^{-1}(n)})$ and $F' = \gamma(x_{\phi^{-1}(1)}, \ldots, x_{\phi^{-1}(m)})$ such that $(\sigma_{t,F} \circ_r \sigma_{t',F'} \circ_r \sigma_{t,F})(f) = \sigma_{t,F}(f)$ and $(\sigma_{t,F} \circ_r \sigma_{t',F'} \circ_r \sigma_{t,F})(\gamma) = \sigma_{t,F}(\gamma)$. Therefore, we conclude that $\sigma_{t,F}$ is regular.

(v) This statement can be proved by using Lemma 1 and the same process as we proved in (iii).

(vi) This statement can be proved by using Lemma 1 and the same process as we proved in (iv).

(vii) Let $\sigma_{t,F} \in C_{13}$. Then $t = f(x_{\pi(1)}, \ldots, x_{\pi(n)}), F = (x_{i_1} \approx x_{i_2}) \vee (x_{i_3} \approx x_{i_4})$. If $\pi(i) \neq i$ for some $i = 1, \ldots, n$, then there exists $\sigma_{t',F'} \in C_{13}$ with $t' = f(x_{\pi^{-1}(1)}, \ldots, x_{\pi^{-1}(n)})$ and $F' = (x_{i_1} \approx x_{i_2}) \vee (x_{i_3} \approx x_{i_4})$ such that $(\sigma_{t,F} \circ_r \sigma_{t',F'} \circ_r \sigma_{t,F})(f) = \sigma_{t,F}(f)$ which follows from (iii) and we consider

$$\begin{aligned}
(\sigma_{t,F} \circ_r \sigma_{t',F'} \circ_r \sigma_{t,F})(\gamma) &= \widehat{\sigma}_{t,F}[\widehat{\sigma}_{t',F'}[x_{i_1} \approx x_{i_2}] \vee \widehat{\sigma}_{t',F'}[x_{i_3} \approx x_{i_4}]] \\
&= \widehat{\sigma}_{t,F}[x_{i_1} \approx x_{i_2} \vee x_{i_3} \approx x_{i_4}] \\
&= (x_{i_1} \approx x_{i_2} \vee x_{i_3} \approx x_{i_4}) \\
&= \sigma_{t,F}(\gamma).
\end{aligned}$$

Hence $\sigma_{t,F}$ is regular.

(viii) Let $\sigma_{t,F} \in C_{14}$. Then $t = f(x_{\pi(1)}, \ldots, x_{\pi(n)}), F = \neg(x_{i_1} \approx x_{i_2}) \vee (x_{i_3} \approx x_{i_4})$. If $\pi(i) \neq i$ for some $i = 1, \ldots, n$, then there exists $\sigma_{t',F'} \in C_{14}$ with $t' = f(x_{\pi^{-1}(1)}, \ldots, x_{\pi^{-1}(n)})$ and $F' = \neg(x_{i_1} \approx x_{i_2}) \vee (x_{i_3} \approx x_{i_4})$ such that $(\sigma_{t,F} \circ_r \sigma_{t',F'} \circ_r \sigma_{t,F})(f) = \sigma_{t,F}(f)$ which follows from (iii) and we consider

$$\begin{aligned}
(\sigma_{t,F} \circ_r \sigma_{t',F'} \circ_r \sigma_{t,F})(\gamma) &= \widehat{\sigma}_{t,F}[\widehat{\sigma}_{t',F'}[\neg(x_{i_1} \approx x_{i_2})] \vee \widehat{\sigma}_{t',F'}[x_{i_3} \approx x_{i_4}]] \\
&= \widehat{\sigma}_{t,F}[\neg(x_{i_1} \approx x_{i_2}) \vee x_{i_3} \approx x_{i_4}] \\
&= \widehat{\sigma}_{t,F}[\neg(x_{i_1} \approx x_{i_2})] \vee \widehat{\sigma}_{t,F}[x_{i_3} \approx x_{i_4}] \\
&= \neg(x_{i_1} \approx x_{i_2}) \vee (x_{i_3} \approx x_{i_4}) \\
&= \sigma_{t,F}(\gamma).
\end{aligned}$$

Hence $\sigma_{t,F}$ is regular.

(ix) Let $\sigma_{t,F} \in C_{15}$. Then $t = f(x_{\pi(1)}, \ldots, x_{\pi(n)}), F = (x_{i_1} \approx x_{i_2}) \vee \neg(x_{i_3} \approx x_{i_4})$. If $\pi(i) \neq i$ for some $i = 1, \ldots, n$, then there exists $\sigma_{t',F'} \in B_{15}$ with $t' = f(x_{\pi^{-1}(1)}, \ldots, x_{\pi^{-1}(n)})$ and $F' = (x_{i_1} \approx x_{i_2}) \vee \neg(x_{j_1} \approx x_{j_2})$ such that $(\sigma_{t,F} \circ_r \sigma_{t',F'} \circ_r \sigma_{t,F})(f) = \sigma_{t,F}(f)$ which follows from (iii) and we consider

$$\begin{aligned}
(\sigma_{t,F} \circ_r \sigma_{t',F'} \circ_r \sigma_{t,F})(\gamma) &= \widehat{\sigma}_{t,F}[\widehat{\sigma}_{t',F'}[x_{i_1} \approx x_{i_2}] \vee \widehat{\sigma}_{t',F'}[\neg(x_{i_3} \approx x_{i_4})]] \\
&= \widehat{\sigma}_{t,F}[\widehat{\sigma}_{t',F'}[x_{i_1} \approx x_{i_2}] \vee \neg(\widehat{\sigma}_{t',F'}[x_{i_3} \approx x_{i_4}])] \\
&= \widehat{\sigma}_{t,F}[x_{i_1} \approx x_{i_2} \vee \neg(x_{i_3} \approx x_{i_4})] \\
&= \widehat{\sigma}_{t,F}[x_{i_1} \approx x_{i_2}] \vee \neg(\widehat{\sigma}_{t,F}[x_{i_3} \approx x_{i_4}]) \\
&= (x_{i_1} \approx x_{i_2}) \vee \neg(x_{i_3} \approx x_{i_4}) \\
&= \sigma_{t,F}(\gamma).
\end{aligned}$$

Hence $\sigma_{t,F}$ is regular.

(x) Let $\sigma_{t,F} \in C_{16}$. Then $t = f(x_{\pi(1)},\dots,x_{\pi(n)}), F = \neg(x_{i_1} \approx x_{i_2}) \vee \neg(x_{i_3} \approx x_{i_4})$. If $\pi(i) \neq i$ for some $i = 1,\dots,n$, then there exists $\sigma_{t',F'} \in C_{16}$ with $t' = f(x_{\pi^{-1}(1)},\dots,x_{\pi^{-1}(n)})$ and $F' = \neg(x_{i_1} \approx x_{i_2}) \vee \neg(x_{j_1} \approx x_{j_2})$ such that $(\sigma_{t,F} \circ_r \sigma_{t',F'} \circ_r \sigma_{t,F})(f) = \sigma_{t,F}(f)$ which follows from (iii) and we consider

$$\begin{aligned}(\sigma_{t,F} \circ_r \sigma_{t',F'} \circ_r \sigma_{t,F})(\gamma) &= \widehat{\sigma}_{t,F}[\widehat{\sigma}_{t',F'}[\neg(x_{i_1} \approx x_{i_2})] \vee \widehat{\sigma}_{t',F'}[\neg(x_{i_3} \approx x_{i_4})]] \\ &= \widehat{\sigma}_{t,F}[\neg(\widehat{\sigma}_{t',F'}[x_{i_1} \approx x_{i_2}]) \vee \neg(\widehat{\sigma}_{t',F'}[x_{i_3} \approx x_{i_4}])] \\ &= \widehat{\sigma}_{t,F}[\neg(x_{i_1} \approx x_{i_2}) \vee \neg(x_{i_3} \approx x_{i_4})] \\ &= \neg(\widehat{\sigma}_{t,F}[x_{i_1} \approx x_{i_2}]) \vee \neg(\widehat{\sigma}_{t,F}[x_{i_3} \approx x_{i_4}]) \\ &= \neg(x_{i_1} \approx x_{i_2}) \vee \neg(x_{i_3} \approx x_{i_4}) \\ &= \sigma_{t,F}(\gamma).\end{aligned}$$

Hence $\sigma_{t,F}$ is regular. □

We have now characterized all idempotent and regular elements of linear hypersubstitutions for algebraic systems of type $((n),(m))$. As we remarked earlier, we separated and described the classes of linear hypersubstitutions into sixteen classes and given the charaterization of idempotent elements in these classes. The situation is more comfortable than to consider the set of all linear hypersubstitutions. We applied these results to investigate the regularity. As a consequence of this section, we can describe the regularity of $Hyp^{lin}((n),(m))$. Every linear hypersubstitution is regular and then $\mathscr{H}yp^{lin}((n),(m))$ is a regular semigroup.

5. Conclusion

We use the concepts of the partial clone of linear terms and the partial clone of linear formulas to define a mapping which is called a linear hypersubstitution for algebraic systems of type $((n),(m))$.

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Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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