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Research Article

Coupled Random Fixed Point Theorems for Mixed Monotone Nonlinear Operators

Chayut Kongban[1,2,*], Poom Kumam[1,2,*] and Juan Martínez-Moreno[3]

[1] *KMUTTFixed Point Research Laboratory, Department of Mathematics, Room SCL 802 Fixed Point Laboratory, Science Laboratory Building, Faculty of Science, King Mongkut's University of Technology Thonburi (KMUTT), 126 Pracha-Uthit Road, Bang Mod, Thrung Khru, Bangkok 10140, Thailand*

[2] *KMUTT-Fixed Point Theory and Applications Research Group (KMUTT-FPTA), Theoretical and Computational Science Center (TaCS), Science Laboratory Building, Faculty of Science, King Mongkut's University of Technology Thonburi (KMUTT), 126 Pracha-Uthit Road, Bang Mod, Thrung Khru, Bangkok 10140, Thailand*

[3] *Department of Mathematics, Faculty of Experimental Science, University of Jaén, Campus Las Lagunillas, s/n, 23071 Jaén, Spain*

*__Corresponding author:__ chayut_kb@hotmail.com

Abstract. In this paper, we prove the existence of a random coupled coincidence and coupled random fixed point theorems in complete separable metric space without the mixed g-monotone property. The results are used to prove existence of random solutions for random integral equation.

Keywords. Coupled random coincidence; Coupled random fixed point; Measurable mapping; Mixed monotone mapping; Random operator

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1. Introduction

Random fixed point theorems are stochastic generalization of a classical fixed point theorems. Random fixed point theorems for contraction mapping in a Polish space, i.e., a separable complete metric space, were proved by Špaček [36], Hanš [10, 11]. Some random fixed point theorems play a main role in developing theory of random differential and random integral equations (see, [5, 14, 23]). Thereafter, many authors have focused on varions existence and uniqueness theorems of random fixed point and applications (see, [2,3,6,13,17–20,22,26,27,33]).

In 2004, Ran and Reurings [31] proved the existence of fixed points of nonlinear contraction mappings in partially ordered metric space presented applications of their results to matrix equations. In 2006, Bhaskar and Lakshmikantham [9] proved some coupled fixed point theorems for mixed monotone mappings in ordered metric space. In 2009, Lakshmikantham and Ćirić [21] introduced the concept of mixed g-monotone mapping and proved coupled coincidence and coupled common fixed point theorems in ordered metric space. Following this initial paper, many authors produced remarkable results in this direction, (see [1, 7, 24, 28, 29, 35]). In 2011, Berinde [4] extended the coupled fixed point theorems for mixed monotone operators in partially ordered metric spaces.

Recently, Ćirić and Lakshmikantham [8] proved random couple coincidence and random coupled fixed point theorem in partially ordered complete separable metric space. Thereafter, many researchers have obtained random coupled coincidence and random coupled fixed point theorem in partially ordered complete separable metric space, for details, (see [15, 16, 34]).

The aim of this paper is to prove some random coupled coincidence and coupled random fixed point theorems for a pair of random mappings $F:\Omega\times(X\times X)\to X$ and $g:\Omega\times X\to X$. Our result is a generalization of main result of Ćirić and Lakshmikantham [8].

2. Preliminaries

In this section, we give some definitions which are useful for main results in this paper.

Definition 2.1 ([9]). Let $(X,\preceq)$ be a partially ordered set and $F:X\times X\to X$. The mapping F is said to have the *mixed monotone property* if F is monotone non-decreasing in its first argument and is monotone non-increasing in its second argument, that is, for any $x,y\in X$,

$$x_1,x_2\in X,\quad x_1\preceq x_2\Rightarrow F(x_1,y)\preceq F(x_2,y) \tag{2.1}$$

and

$$y_1,y_2\in X,\quad y_1\preceq y_2\Rightarrow F(x,y_1)\succeq F(x,y_2). \tag{2.2}$$

Definition 2.2 ([21]). Let $(X,\preceq)$ be a partially ordered set and $F:X\times X\to X$ and $g:X\to X$. The mapping F is said to have the *mixed g-monotone property* if F is monotone g-non-decreasing in its first argument and is monotone g-non-increasing in its second argument, that is, for any $x,y\in X$,

$$x_1,x_2\in X,\quad g(x_1)\preceq g(x_2)\Rightarrow F(x_1,y)\preceq F(x_2,y) \tag{2.3}$$

and

$$y_1,y_2\in X,\quad g(y_1)\preceq g(y_2)\Rightarrow F(x,y_1)\succeq F(x,y_2). \tag{2.4}$$

Clearly, if g is the identity mapping, then Definition 2.2 reduces to Definition 2.1.

Definition 2.3 ([9])**.** Let X be a nonempty set. An element $(x,y) \in X \times X$ is called a *coupled fixed point* of the mapping $F : X \times X \to X$ if $x = F(x,y)$ and $y = F(y,x)$.

Let (Ω,Σ) be a measurable space with Σ a sigma algebra of subsets of Ω and let (X,d) be a metric space. A mapping $T : \Omega \to X$ is called Σ-*measurable* if for any open subset U of X, $T^{-1}(U) = \{\omega : T(\omega \in U)\} \in \Sigma$. In what follows, when we speak of measurability we will mean Σ-measurability. A mapping $T : \Omega \times X \to X$ is said to be a *random operator* if and only if for each fixed $x \in X$, the mapping $T(\cdot,x) : \Omega \to X$ is measurable. A random operator $T : \Omega \times X \to X$ is *continuous* if for each $\omega \in \Omega$, the mapping $S(\omega,\cdot) : X \to X$ is continuous.

Definition 2.4. A mapping $T : \Omega \times X \to X$ is called a random operator if for any $x \in X$, $T(\cdot,x)$ is measurable. A measursble mapping $\xi : \Omega \to X$ is said to be

(a) a *random fixed point* of a random function $T : \Omega \times X \to X$, if for each $\omega \in \Omega$
$$\xi(\omega) = T(\omega,\xi(\omega)).$$

(b) A *random coincidence* of a random function $T : \Omega \times X \to X$ and $g : \Omega \times X \to X$, if for each $\omega \in \Omega$
$$g(\omega,\xi(\omega)) = T(\omega,\xi(\omega)).$$

Definition 2.5 ([8])**.** Let (X,d) be a separable metric space, (Ω,Σ) be a measurable space and $F : \Omega \times (X \times X) \to X$ and $g : \Omega \times X \to X$ be mappings. We say F and g are *commutative* if
$$F(\omega,(g(\omega,x),g(\omega,y))) = g(\omega,F(\omega,(x,y))),$$
for all $\omega \in \Omega$ and $x,y \in X$.

Definition 2.6. Let (Ω,Σ) be a measurable space, X and Y be two metric spaces. A mapping $f : \Omega \times X \to Y$ is called *Carathéodory* if, for all $x \in X$, the mapping $\omega \to f(\omega,x)$ is Σ-measurable and, for all $\omega \in \Omega$, the mapping $x \to f(\omega,x)$ is continuous.

Let M, N be two locally compact metric spaces and $f : \Omega \times M \to N$. By $C(M,N)$, we denote the space of continuous functions from M into N endowed with the compact-open topology.

Lemma 2.7 ([30])**.** *f is a Carathéodory function if and only if $\omega \to r(\omega)(\cdot) = f(\omega,\cdot)$ is a measurable function from Ω to $C(M,N)$.*

3. Main Results

Denote with Φ the set of all function $\varphi : [0,\infty) \to [0,\infty)$ satisfying

(i_φ) $\varphi(t) < t$ for all $t \in (0,\infty)$,

(ii_φ) $\lim\limits_{r \to t^+} \varphi(r) < t$ for all $t \in (0,\infty)$.

Theorem 3.1. *Let $(X,\preceq)$ be a partially ordered set, (X,d) be a complete separable metric space, (Σ,Ω) be a measurable space and $F : \Omega \times (X \times X) \to X$ and $g : \Omega \times X \to X$ mappings such that*

(1) *$F(\omega,\cdot)$ and $g(\omega,\cdot)$ are continuous for all $\omega \in \Omega$,*

(2) *$F(\cdot,v)$ and $g(\cdot,x)$ are measurable for all $v \in X \times X$ and $x \in X$, respectively,*

(3) *$F:\Omega\times(X\times X)\to X$ and $g:\Omega\times X\to X$ are such that F has the mixed g-monotone property and*

$$d(F(\omega,(x,y)),F(\omega,(u,v)))+d(F(\omega,(y,x)),F(\omega,(v,u))) \leq 2\varphi\Big(\frac{d(g(\omega,x),g(\omega,u))+d(g(\omega,y),g(\omega,v))}{2}\Big) \tag{3.1}$$

for all $x,y,u,v\in X$ with $g(\omega,x)\preceq g(\omega,u)$ and $g(\omega,y)\succeq g(\omega,v)$ for all $\omega\in\Omega$, where $\varphi\in\Phi$. Suppose $F(\omega\times(X\times X))\subseteq g(\omega\times X)$, for each $\omega\in\Omega$, g is continuous and commutes with F and also suppose that either

(a) *F is continuous or*

(b) *X has the following properties:*

(i) *if a non-decreasing sequence $\{x_n\}_{n=0}^{\infty}\subset X$ converges to x, then $x_n\preceq x$ for all n;*

(ii) *if a non-increasing sequence $\{x_n\}_{n=0}^{\infty}\subset X$ converges to x, then $x_n\succeq x$ for all n.*

If there exist measurable mapping $\xi_0,\eta_0:\Omega\to X$ such that

$$g(\omega,\xi_0(\omega))\preceq F(\omega,(\xi_0(\omega),\eta_0(\omega))) \ \text{ and } \ g(\omega,\eta_0(\omega))\succeq F(\omega,(\eta_0(\omega)\xi_0(\omega))) \tag{3.2}$$

or

$$g(\omega,\xi_0(\omega))\succeq F(\omega,(\xi_0(\omega),\eta_0(\omega))) \ \text{ and } \ g(\omega,\eta_0(\omega))\preceq F(\omega,(\eta_0(\omega),\xi_0(\omega))) \tag{3.3}$$

for all $\omega\in\Omega$, then there are measurable mappings $\xi,\eta:\Omega\to X$ such that

$$F(\omega,(\xi(\omega),\eta(\omega)))=g(\omega,\xi(\omega)) \ \text{ and } \ F(\omega,(\eta(\omega),\xi(\omega)))=g(\omega,\eta(\omega))$$

for all $\omega\in\Omega$, that is, F and g have a random coupled coincidence.

Proof. Let $\Theta=\{\xi:\Omega\to X\}$ be a family of measurable mapping. Define a function $h:\Omega\times X\to\mathbb{R}^+$ as follows:

$$h(\omega,x)=d(x,g(\omega,x)).$$

Since $x\to g(\omega,x)$ is continuous for all $\omega\in\Omega$, we conclude that $h(\omega,\cdot)$ is continuous for all $\omega\in\Omega$. Also, since $\omega\to g(\omega,x)$ is measurable for all $x\in X$, we conclude that $h(\cdot,\omega)$ is measurable for all $\omega\in\Omega$ (see [37]). Thus, $h(\omega,x)$ is the Caratheéodory function. Therefore, if $\xi:\Omega\to X$ is a measurable mapping, then $\omega\to h(\omega,\xi(\omega))$ is also measurable (see [32]). Also, for each $\xi\in\Theta$ the function $\eta:\Omega\to X$ defined by $\eta(\omega)=g(\omega,\xi(\omega))$ is measurable, that is, $\eta\in\Theta$.

Now we shall construct two sequences of measurable mappings $\{\xi_n\}$ and $\{\eta_n\}$ in Θ and two sequences $\{g(\omega,\xi_n(\omega))\}$ and $\{g(\omega,\eta_n(\omega))\}$ in X as follows:

Let $\xi_0,\eta_0\in\Theta$ be such that

$$g(\omega,\xi_0(\omega))\preceq F(\omega,(\xi_0(\omega),\eta_0(\omega)))$$

and

$$g(\omega,\eta_0(\omega))\succeq F(\omega,(\eta_0(\omega),\xi_0(\omega))),$$

for all $\omega\in\Omega$. Since $F(\omega,(\xi_0(\omega),\eta_0(\omega)))\in X=g(\omega\times X)$, by a sort of Filippov measurable implicit function theorem [3,12,13,25] there is $\xi_1(\omega)\in\Theta$ such that

$$g(\omega,\xi_1(\omega))=F(\omega,(\xi_0(\omega),\eta_0(\omega))).$$

Similarly, as $F(\omega,(\eta_0(\omega),\xi_0(\omega))) \in g(\omega \times X)$, there is $\eta_1(\omega) \in \Theta$ such that

$$g(\omega,\eta_1(\omega)) = F(\omega,(\eta_0(\omega),\xi_0(\omega))).$$

Now, $F(\omega,(\xi_1(\omega),\eta_1(\omega)))$ and $F(\omega,(\eta_1(\omega),\xi_1(\omega)))$ are well defined. Again from

$$F(\omega,(\xi_1(\omega),\eta_1(\omega))), F(\omega,(\eta_1(\omega),\xi_1(\omega))) \in g(\omega \times X),$$

there are $\xi_2(\omega),\eta_2(\omega) \in \Theta$ such that

$$g(\omega,\xi_2(\omega)) = F(\omega,(\xi_1(\omega),\eta_1(\omega)))$$

and

$$g(\omega,\eta_2(\omega)) = F(\omega,(\eta_1(\omega),\xi_1(\omega))).$$

Continuing this process we can construct sequences $\{\xi_n(\omega)\}$ and $\{\eta_n(\omega)\}$ in X such that

$$g(\omega,\xi_{n+1}(\omega)) = F(\omega,(\xi_n(\omega),\eta_n(\omega))) \tag{3.4}$$

and

$$g(\omega,\eta_{n+1}(\omega)) = F(\omega,(\eta_n(\omega),\xi_n(\omega))). \tag{3.5}$$

for all $n \geq 0$.

Step I. We shall prove that

$$g(\omega,\xi_n(\omega)) \preceq g(\omega,\xi_{n+1}(\omega)), \quad \text{for all } n \geq 0, \tag{3.6}$$

and

$$g(\omega,\eta_n(\omega)) \succeq g(\omega,\eta_{n+1}(\omega)), \quad \text{for all } n \geq 0. \tag{3.7}$$

Assume (3.2) holds (the case (3.3) is similar). We have

$$g(\omega,\xi_0(\omega)) \preceq F(\omega,(\xi_0(\omega),\eta_0(\omega))),$$

and

$$g(\omega,\eta_0(\omega)) \succeq F(\omega,(\eta_0(\omega),\xi_0(\omega))).$$

Since $g(\omega,\xi_1(\omega)) = F(\omega,(\xi_0(\omega),\eta_0(\omega)))$ and $g(\omega,\eta_1(\omega)) = F(\omega,(\eta_0(\omega),\xi_0(\omega)))$, we have $g(\omega,\xi_0(\omega)) \preceq g(\omega,\xi_1(\omega))$ and $g(\omega,\eta_0(\omega)) \succeq g(\omega,\eta_1(\omega))$. Therefore, (3.6) and (3.7) hold for all $n = 0$. Suppose now that (3.6) and (3.7) hold for some fixed $n \geq 0$. Then, since $g(\omega,\xi_n(\omega)) \preceq g(\omega,\xi_{n+1}(\omega))$ and $g(\omega,\eta_{n+1}(\omega)) \preceq g(\omega,\eta_n(\omega))$ and as F is g-mixed monotone, we have

$$F(\omega,(\xi_n(\omega),\eta_n(\omega))) \preceq F(\omega,(\xi_{n+1}(\omega),\eta_{n+1}(\omega))) \tag{3.8}$$

and

$$F(\omega,(\eta_{n+1}(\omega),\xi_n(\omega))) \preceq F(\omega,(\eta_n(\omega),\xi_{n+1}(\omega))). \tag{3.9}$$

Similarly, from (2.4), (3.4), and (3.5) as $g(\omega,\eta_{n+1}(\omega)) \preceq g(\omega,\eta_n(\omega)$ and $g(\omega,\xi_n(\omega)) \preceq g(\omega,\xi_{n+1}(\omega))$,

$$F(\omega,(\xi_{n+1}(\omega),\eta_{n+1}(\omega))) \succeq F(\omega,(\xi_{n+1}(\omega),\eta_n(\omega))) \tag{3.10}$$

and

$$F(\omega,(\eta_{n+1}(\omega),\xi_n(\omega))) \succeq F(\omega,(\eta_{n+1}(\omega),\xi_{n+1}(\omega))). \tag{3.11}$$

Now from (3.8), (3.9), (3.10), (3.11), (3.4), and (3.5), we get

$$g(\omega,\xi_{n+1}(\omega)) \preceq g(\omega,\xi_{n+2}(\omega)) \tag{3.12}$$

and

$$g(\omega,\eta_{n+1}(\omega)) \succeq g(\omega,\eta_{n+2}(\omega)). \tag{3.13}$$

Thus, by the mathematical induction we conclude that (3.6) and (3.7) hold for all $n \geq 0$.

Step II. Denote

$$\delta_{n+1} = d(g(\omega,\xi_n(\omega)), g(\omega,\xi_{n+1}(\omega))) + d(g(\omega,\eta_n(\omega)), g(\omega,\eta_{n+1}(\omega))).$$

Since from (3.6) and (3.7) we have $g(\omega,\xi_n(\omega)) \preceq g(\omega,\xi_{n+1}(\omega))$ and $g(\omega,\eta_n(\omega)) \succeq g(\omega,\eta_{n+1}(\omega))$, the from (3.4), (3.5), and (3.1) we get

$$\begin{aligned}
&d(g(\omega,\xi_{n+1}(\omega)), g(\omega,\xi_n(\omega))) + d(g(\omega,\eta_{n+1}(\omega)), g(\omega,\eta_n(\omega))) \\
&\quad = d(F(\omega,(\xi_n(\omega),\eta_n(\omega))), F(\omega,(\xi_{n-1}(\omega),\eta_{n-1}(\omega)))) \\
&\qquad + d(F(\omega,(\eta_n(\omega),\xi_n(\omega))), F(\omega,(\eta_{n-1}(\omega),\xi_{n-1}(\omega)))) \\
&\quad \leq 2\varphi\Big(\frac{d(g(\omega,\xi_n(\omega)), g(\omega,\xi_{n-1}(\omega))) + d(g(\omega,\eta_n(\omega)), g(\omega,\eta_{n-1}(\omega)))}{2}\Big) \\
&\quad = 2\varphi\Big(\frac{\delta_n}{2}\Big).
\end{aligned}$$

Therefore, the sequence $\{\delta_n\}_{n=1}^{\infty}$ satisfies

$$\delta_{n+1} \leq 2\varphi\Big(\frac{\delta_n}{2}\Big), \quad \text{for all } n \geq 0. \tag{3.14}$$

From (3.14) and (φ_i) it follows that the sequence $\{\delta_n\}_{n=0}^{\infty}$ is non-increasing. Therefore, there exists some $\delta > 0$ such that

$$\lim_{n\to\infty} \delta_n = \lim_{n\to\infty} [d(g(\omega,\xi_n(\omega)), g(\omega,\xi_{n-1}(\omega))) + d(g(\omega,\eta_n(\omega)), g(\omega,\eta_{n-1}(\omega)))] = \delta.$$

We shall prove that $\delta = 0$. Assume, to the contrary, that $\delta > 0$. Then bt letting $n \to \infty$ in (3.14), in view of (φ_{ii}) we have

$$\delta = \lim_{n\to\infty} \delta_{n+1} \leq 2 \lim_{n\to\infty} \varphi\Big(\frac{\eta_n}{2}\Big) = 2 \lim_{\delta_n\to\delta+} \varphi\Big(\frac{\eta_n}{2}\Big) < \delta,$$

a contradiction. Thus $\delta = 0$ and hence

$$\lim_{n\to\infty} \delta_n = 0. \tag{3.15}$$

Step III. Now, we prove that $\{g(\omega,\xi_\omega)\}$ and $\{g(\omega,\eta_\omega)\}$ are Cauchy sequences. Suppose, to the contrary, that at least one of sequences $\{g(\omega,\xi_\omega)\}_{n=0}^{\infty}$ and $\{g(\omega,\eta_\omega)\}_{n=0}^{\infty}$ is not a Cauchy sequences. Then there exists $\epsilon > 0$ for which we can find subsequences $\{g(\omega,\xi_{n(k)}(\omega))\}$, $\{g(\omega,\xi_{m(k)}(\omega))\}$ of $\{g(\omega,\xi_\omega)\}_{n=0}^{\infty}$ and $\{g(\omega,\eta_{n(k)}(\omega))\}$, $\{g(\omega,\eta_{m(k)}(\omega))\}$ of $\{g(\omega,\eta_\omega)\}_{n=0}^{\infty}$, respectively, which $n(k) > m(k) \geq k$ such that, $k = 1,2,\cdots$.

$$r_k = d(g(\omega,\xi_{n(k)}(\omega)), g(\omega,\xi_{m(k)}(\omega))) + d(g(\omega,\eta_{n(k)}(\omega)), g(\omega,\eta_{m(k)}(\omega))) \geq \epsilon, \tag{3.16}$$

Note that we can choose $n(k)$ to be the smallest integer with property $n(k) > m(k) \geq k$ and satisfying (3.16). Then

$$d(g(\omega,\xi_{n(k)-1}(\omega)), g(\omega,\xi_{m(k)}(\omega))) + d(g(\omega,\eta_{n(k)-1}(\omega)), g(\omega,\eta_{m(k)}(\omega))) < \epsilon. \tag{3.17}$$

By (3.16) and (3.17) and the triangle inequality, we have

$$\begin{aligned}
&\epsilon \leq r_k \\
&\quad \leq d(g(\omega,\xi_{n(k)}(\omega)), g(\omega,\xi_{n(k)-1}(\omega))) + d(g(\omega,\eta_{n(k)}(\omega)), g(\omega,\eta_{n(k)-1}(\omega)))
\end{aligned}$$

$$+d(g(\omega,\xi_{n(k)-1}(\omega)),g(\omega,\xi_{m(k)}(\omega)))+d(g(\omega,\eta_{n(k)-1}(\omega)),g(\omega,\eta_{m(k)}(\omega)))$$
$$\leq d(g(\omega,\xi_{n(k)}(\omega)),g(\omega,\xi_{n(k)-1}(\omega)))+d(g(\omega,\eta_{n(k)}(\omega)),g(\omega,\eta_{n(k)-1}(\omega)))+\epsilon.$$

Letting $k\to\infty$ in the above inequality and using (3.15), we get

$$\lim_{k\to\infty} r_k=\epsilon. \tag{3.18}$$

On the other hand

$$\begin{aligned}
r_k &\leq d(g(\omega,\xi_{n(k)}(\omega)),g(\omega,\xi_{n(k)+1}(\omega)))+d(g(\omega,\xi_{n(k)+1}(\omega)),g(\omega,\xi_{m(k)}(\omega)))\\
&\quad +d(g(\omega,\eta_{n(k)}(\omega)),g(\omega,\eta_{n(k)+1}(\omega)))+d(g(\omega,\eta_{n(k)+1}(\omega)),g(\omega,\eta_{m(k)}(\omega)))\\
&=\delta_{n(k)}+d(g(\omega,\xi_{n(k)+1}(\omega)),g(\omega,\xi_{m(k)}(\omega)))+d(g(\omega,\eta_{n(k)+1}(\omega)),g(\omega,\eta_{m(k)}(\omega)))\\
&=2\delta_{n(k)}+2\delta_{m(k)}+d(g(\omega,\xi_{n(k)+1}(\omega)),g(\omega,\xi_{m(k)+1}(\omega)))\\
&\quad +d(g(\omega,\eta_{n(k)+1}(\omega)),g(\omega,\eta_{m(k)+1}(\omega))).
\end{aligned} \tag{3.19}$$

Since $n(k)>m(k)$, by (3.12) and (3.13), we have

$$g(\omega,\xi_{n(k)}(\omega))\preceq g(\omega,\xi_{m(k)}(\omega))$$

and

$$g(\omega,\eta_{n(k)}(\omega))\succeq g(\omega,\eta_{m(k)}(\omega)),$$

and hence by (3.1) one obtains

$$\begin{aligned}
&d(g(\omega,\xi_{n(k)+1}(\omega)),g(\omega,\xi_{m(k)}(\omega)))+d(g(\omega,\eta_{n(k)+1}(\omega)),g(\omega,\eta_{m(k)}(\omega)))\\
&\quad =d(F(\omega,(\xi_{n(k)}(\omega),\eta_{n(k)}(\omega))),F(\omega,(\xi_{m(k)}(\omega),\eta_{m(k)}(\omega))))\\
&\qquad +d(F(\omega,(\eta_{n(k)}(\omega),\xi_{n(k)}(\omega))),F(\omega,(\eta_{m(k)}(\omega),\xi_{m(k)}(\omega))))\\
&\quad \leq 2\varphi\Big(\frac{d(g(\omega,\xi_{n(k)}(\omega)),g(\omega,\xi_{m(k)}(\omega)))+d(g(\omega,\eta_{n(k)}(\omega)),g(\omega,\eta_{m(k)}(\omega)))}{2}\Big)\\
&\quad \leq 2\varphi\Big(\frac{r_k}{2}\Big)
\end{aligned}$$

which, by (3.19), yields

$$r_k\leq 2\delta_{n(k)}+2\delta_{m(k)}+2\varphi\Big(\frac{r_k}{2}\Big).$$

Letting $k\to\infty$ in the above inequality and using (3.18), we get

$$\epsilon\leq 2\lim_{k\to\infty}\varphi\Big(\frac{r_k}{2}\Big)=2\lim_{r_k\to\epsilon}\varphi\Big(\frac{r_k}{2}\Big)<\epsilon,$$

a contradiction. Therefore, our supposition (3.16) was wrong. Thus, we proved that $\{g(\omega,\xi_n(\omega))\}$ and $\{g(\omega,\eta_n(\omega))\}$ are Cauchy sequences in X. Since X is complete any $g(\omega\times X)=X$, there exist $\beta,\gamma\in\Theta$ such that

$$\lim_{n\to\infty} g(\omega,\xi_n(\omega))=g(\omega,\beta_0(\omega))$$

and

$$\lim_{n\to\infty} g(\omega,\eta_n(\omega))=g(\omega,\gamma_0(\omega)).$$

Since $g(\omega,\beta_0(\omega))$ and $g(\omega,\gamma_0(\omega))$ are measurable, then the functions $\beta(\omega)$ and $\gamma(\omega)$, defined by $\beta(\omega)=g(\omega,\beta_0(\omega))$ and $\gamma(\omega)=g(\omega,\gamma_0(\omega))$ are measurable. Thus

$$\lim_{n\to\infty} g(\omega,\xi_n(\omega))=\beta(\omega) \tag{3.20}$$

and

$$\lim_{n\to\infty} g(\omega,\eta_n(\omega)) = \gamma(\omega). \tag{3.21}$$

From (3.20) and (3.21) and continuity of g

$$\lim_{n\to\infty} g(\omega,g(\omega,\xi_n(\omega))) = g(\omega,\beta(\omega)) \tag{3.22}$$

and

$$\lim_{n\to\infty} g(\omega,g(\omega,\eta_n(\omega))) = g(\omega,\gamma(\omega)). \tag{3.23}$$

On the other hand, by the commutativity of F and g

$$\begin{aligned} F(\omega,(g(\omega,\xi_n(\omega)),g(\omega,\eta_n(\omega)))) &= g(\omega,F(\omega,(\xi_n(\omega),\eta_n(\omega)))) \\ &= g(\omega,g(\omega,\beta_{n+1}(\omega))) \end{aligned} \tag{3.24}$$

and

$$\begin{aligned} F(\omega,(g(\omega,\eta_n(\omega)),g(\omega,\xi_n(\omega)))) &= g(\omega,F(\omega,(\eta_n(\omega),\xi_n(\omega)))) \\ &= g(\omega,g(\omega,\gamma_{n+1}(\omega))). \end{aligned} \tag{3.25}$$

Step IV. We now prove that

$$g(\omega,\beta(\omega)) = F(\omega,(\beta(\omega),\gamma(\omega)))$$

and

$$g(\omega,\gamma(\omega)) = F(\omega,(\gamma(\omega),\beta(\omega))).$$

Suppose first that assumption (a) holds. By letting $n\to\infty$ in (3.24) and (3.25), in view of (3.22) and (3.23), and continuity of F, we get

$$\begin{aligned} g(\omega,\beta(\omega)) &= \lim_{n\to\infty} g(\omega,\xi_{n+1}(\omega)) \\ &= \lim_{n\to\infty} F(\omega,(g(\omega,\xi_n(\omega)),g(\omega,\eta_n(\omega)))) \\ &= F(\omega,(\lim_{n\to\infty} g(\omega,\xi_n(\omega)),\lim_{n\to\infty} g(\omega,\eta_n(\omega)))) \\ &= F(\omega,(\beta(\omega),\gamma(\omega))), \end{aligned}$$

$$\begin{aligned} g(\omega,\gamma(\omega)) &= \lim_{n\to\infty} g(\omega,\eta_{n+1}(\omega)) \\ &= \lim_{n\to\infty} F(\omega,(g(\omega,\eta_n(\omega)),g(\omega,\xi_n(\omega)))) \\ &= F(\omega,(\lim_{n\to\infty} g(\omega,\eta_n(\omega)),\lim_{n\to\infty} g(\omega,\xi_n(\omega)))) \\ &= F(\omega,(\gamma(\omega),\beta(\omega))), \end{aligned}$$

that is, $(\beta(\omega),\gamma(\omega)) \in X\times X$ is coupled random coincidence of F and g. Suppose now assumption (b) holds. Since $\{g(\omega,\xi_n(\omega))\}$ is non-decreasing and as $g(\omega,\xi_n(\omega)) \to g(\omega,\beta(\omega))$ we have that $g(\omega,\xi_n(\omega)) \preceq g(\omega,\beta(\omega))$ for all n. Also, $\{g(\omega,\eta_n(\omega))\}$ is non-increasing and as $g(\omega,\eta_n(\omega)) \to g(\omega,\gamma(\omega))$, we have that $g(\omega,\eta_n(\omega)) \succeq g(\omega,\gamma(\omega))$ for all n. Then, by triangle inequality and contractive conditions (3.1)

$$\begin{aligned} &d(g(\omega,\beta(\omega)),F(\omega,(\beta(\omega),\gamma(\omega)))) + d(g(\omega,\gamma(\omega)),F(\omega,(\gamma(\omega),\beta(\omega)))) \\ &\quad \le d(g(\omega,\beta(\omega)),g(\omega,g(\omega,\xi_{n+1}(\omega)))) + d(g(\omega,g(\omega,\xi_{n+1}(\omega))),F(\omega,(\beta(\omega),\gamma(\omega)))) \end{aligned}$$

$$
\begin{aligned}
&+d(g(\omega,\gamma(\omega)),g(\omega,g(\omega,\eta_{n+1}(\omega))))+d(g(\omega,g(\omega,\eta_{n+1}(\omega))),F(\omega,(\gamma(\omega),\beta(\omega))))\\
=\,&d(g(\omega,\beta(\omega)),g(\omega,g(\omega,\xi_{n+1}(\omega))))+d(g(\omega,\gamma(\omega)),g(\omega,g(\omega,\eta_{n+1}(\omega))))\\
&+d(F(\omega,(g(\omega,\xi_n(\omega)),g(\omega,\eta_n(\omega)))),F(\omega,(\beta(\omega),\gamma(\omega))))\\
&+d(F(\omega,(g(\omega,\eta_n(\omega)),g(\omega,\xi_n(\omega)))),F(\omega,(\gamma(\omega),\beta(\omega))))\\
\le\,&d(g(\omega,\beta(\omega)),g(\omega,g(\omega,\xi_{n+1}(\omega))))+d(g(\omega,\gamma(\omega)),g(\omega,g(\omega,\eta_{n+1}(\omega))))\\
&+2\varphi\Big(\frac{d(g(\omega,g(\omega,\xi_n(\omega))),g(\omega,\beta(\omega)))+d(g(\omega,g(\omega,\eta_n(\omega))),g(\omega,\gamma(\omega)))}{2}\Big).
\end{aligned}
$$

Letting now $n\to\infty$, in the above inequality and taking into account that, by property (i_φ), $\lim\limits_{r\to 0+}\varphi(r)=0$, we obtain

$$d(g(\omega,\beta(\omega)),F(\omega,(\beta(\omega),\gamma(\omega))))+d(g(\omega,\gamma(\omega)),F(\omega,(\gamma(\omega),\beta(\omega))))=0,$$

which implies that

$$d(g(\omega,\beta(\omega)),F(\omega,(\beta(\omega),\gamma(\omega))))=0$$

and

$$d(g(\omega,\gamma(\omega)),F(\omega,(\gamma(\omega),\beta(\omega))))=0.$$

□

Example 3.2. Let $X=\mathbb{R}$ with the usual ordering and usual metric. Let $\Omega=[0,1]$ and let σ be the sigma algebra of Lebesgue's measurable subset of $[0,1]$. Define $g:\Omega\times X\to X$ and $F:\Omega\times(X\times X)\to X$ as follows,

$$g(\omega,x)=\frac{5}{6}(1-\omega)x$$

and

$$F(\omega,(x,y))=\frac{1}{4}(1-\omega)(x-2y),$$

$(x,y)\in X$, $\omega\in\Omega$, and $\varphi_\omega(t)=\frac{3}{10}t$ for all $t\in[0,\infty)$. We will check that the contraction (3.1) is satisfied for all $x,y,u,v\in X$ satisfying $g(\omega,x)\le g(\omega,u)$ and $g(\omega,y)\ge g(\omega,v)$, for all $\omega\in\Omega$. Then, from (3.1) we have,

$$
\begin{aligned}
&d(F(\omega,(x,y)),F(\omega,(u,v)))+d(F(\omega,(y,x)),F(\omega,(v,u)))\\
&\quad=\left|\frac{1}{4}(1-\omega)(x-2y)-\frac{1}{4}(1-\omega)(u-2v)\right|+\left|\frac{1}{4}(1-\omega)(y-2x)-\frac{1}{4}(1-\omega)(v-2u)\right|\\
&\quad\le\frac{1}{4}(1-\omega)[|x-u|+|v-y|]\\
&\quad\le\frac{3}{4}t(1-\omega)[|x-u|+|v-y|]\\
&\quad=\frac{9}{10}t\cdot\frac{5}{6}(1-\omega)[|x-u|+|v-y|]\\
&\quad=\varphi[d(g(\omega,x),g(\omega,u))+d(g(\omega,y),g(\omega,v))],
\end{aligned}
$$

that is, contraction (3.1) is satisfied.

If we take $g(\omega,x)=x$ in Theorem 3.1, then we get the following:

Corollary 3.3. *Let $(X,\preceq)$ be a partially ordered set, (X,d) be a complete separable metric space, (Σ,Ω) be a measurable space and $F:\Omega\times(X\times X)\to X$ mapping such that*

(1) *$F(\omega,\cdot)$ is continuous for all $\omega\in\Omega$,*

(2) *$F(\cdot,v)$ is measurable for all $v\in X\times X$ and $x\in X$,*

(3) *$F:\Omega\times(X\times X)\to X$ has the mixed monotone property and*
$$d(F(\omega,(x,y)),F(\omega,(u,v)))+d(F(\omega,(y,x)),F(\omega,(v,u)))\le 2\varphi\Big(\frac{d(x,u)+d(y,v)}{2}\Big)$$
for all $x,y,u,v\in X$ with $x\preceq u$ and $y\succeq v$ for all $\omega\in\Omega$, where $\varphi\in\Phi$. Also, suppose that either

 (a) *F is continuous, or*

 (b) *X has the following properties:*

  (i) *if a non-decreasing sequence $\{x_n\}_{n=0}^{\infty}\subset X$ converges to x, then $x_n\preceq x$ for all n;*

  (ii) *if a non-increasing sequence $\{x_n\}_{n=0}^{\infty}\subset X$ converges to x, then $x_n\succeq x$ for all n.*

If there exist measurable mapping $\xi_0,\eta_0:\Omega\to X$ such that
$$\xi_0(\omega)\preceq F(\omega,(\xi_0(\omega),\eta_0(\omega)))\ \text{ and }\ \eta_0(\omega)\succeq F(\omega,(\eta_0(\omega),\xi_0(\omega)))$$
or
$$\xi_0(\omega)\succeq F(\omega,(\xi_0(\omega),\eta_0(\omega)))\ \text{ and }\ \eta_0(\omega)\preceq F(\omega,(\eta_0(\omega),\xi_0(\omega)))$$
for all $\omega\in\Omega$, then there are measurable mappings $\xi,\eta:\Omega\to X$ such that
$$F(\omega,(\xi(\omega),\eta(\omega)))=\xi(\omega)\ \text{ and }\ F(\omega,(\eta(\omega),\xi(\omega)))=\eta(\omega)$$
for all $\omega\in\Omega$, that is, F has a random coupled fixed point.

If we take $g(\omega,x)=x$ and $\varphi_\omega(t)=k(t)$, $0\le k<1$ in Theorem (3.1), then we get the following

Corollary 3.4. *Let $(X,\preceq)$ be a partially ordered set, (X,d) be a complete separable metric space, (Σ,Ω) be a measurable space and $F:\Omega\times(X\times X)\to X$ has the mixed monotone property and such that*

(1) *$F(\omega,\cdot)$ is continuous for all $\omega\in\Omega$.*

(2) *$F(\cdot,v)$ is measurable for all $v\in X\times X$.*

(3) *There exists a $k\in[0,1)$ such that F satisfies the following condition:*
$$d(F(\omega,(x,y)),F(\omega,(u,v)))+d(F(\omega,(y,x)),F(\omega,(v,u)))\le kd(x,u)+d(y,v)$$
for all $x,y,u,v\in X$ with $x\preceq u$ and $y\succeq v$ for all $\omega\in\Omega$. Also suppose either

 (a) *F is continuous, or*

 (b) *X has the following properties:*

  (i) *if a non-decreasing sequence $\{x_n\}_{n=0}^{\infty}\subset X$ converges to x, then $x_n\preceq x$ for all n;*

  (ii) *if a non-increasing sequence $\{x_n\}_{n=0}^{\infty}\subset X$ converges to x, then $x_n\succeq x$ for all n.*

If there exist measurable mapping $\xi_0,\eta_0:\Omega\to X$ such that
$$\xi_0(\omega)\preceq F(\omega,(\xi_0(\omega),\eta_0(\omega)))\ \text{ and }\ \eta_0(\omega)\succeq F(\omega,(\eta_0(\omega),\xi_0(\omega)))$$
or
$$\xi_0(\omega)\succeq F(\omega,(\xi_0(\omega),\eta_0(\omega)))\ \text{ and }\ \eta_0(\omega)\preceq F(\omega,(\eta_0(\omega),\xi_0(\omega)))$$
for all $\omega\in\Omega$, then there are measurable mappings $\xi,\eta:\Omega\to X$ such that
$$F(\omega,(\xi(\omega),\eta(\omega)))=\xi(\omega)\ \text{ and }\ F(\omega,(\eta(\omega),\xi(\omega)))=\eta(\omega)$$
for all $\omega\in\Omega$, that is, F has a random coupled fixed point.

4. Application

In this section, we study the existence of the solution to a random integral equation using Corollary 3.3. Consider the random integral equations

$$x(\omega,t)=\int_0^1 f(\omega,t,x(s),y(s))ds, \quad t\in[0,1],$$

$$y(\omega,t)=\int_0^1 f(\omega,t,y(s),x(s))ds, \quad t\in[0,1], \tag{4.1}$$

where $f:\Omega\times[0,1]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ and Ω is a nonempty set. Let $C([0,1],\mathbb{R})$ denote the space of all continuous functions defined on $[0,1]$.

The space $C([0,1],\mathbb{R})$, endowed with metric

$$d_\infty(x,y)=\|x-y\|_\infty=\sup_{t\in[0,1]}|x(t)-y(t)|, \quad \text{for all } x,y\in X,$$

is a complete metric space.

Definition 4.1. An element $\alpha,\beta\in C([0,1],\mathbb{R})\times C([0,1],\mathbb{R})$ is called a random coupled lower and upper solution of the integral equation (4.1) if $\alpha(t)\le\beta(t)$ and

$$\alpha(t)=\int_0^1 f(\omega,t,\alpha(s),\beta(s))ds$$

and

$$\beta(t)=\int_0^1 f(\omega,t,\beta(s),\alpha(s))ds,$$

for all $t\in[0,1]$ and $\omega\in\Omega$.

Then hypotheses are the following:

(A_1) $f:\Omega\times[0,1]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ be a carathéodory function.

(A_2) For each $\omega\in\Omega$, for all $t\in[0,1]$, and for all $x,y,u,v\in\mathbb{R}$ for which $x\ge u$ and $y\le v$, we have

$$0\le f(\omega,t,x,y)-f(\omega,t,u,v)\le\varphi_\omega(x-u+2v-2y),$$

where $\varphi_\omega:[0,\infty)\to[0,\infty)$ is continuous, non-derreasing and satisfies $0=\varphi(0)<\varphi(t)<t$ and $\lim_{n\to t^+}\varphi(r)<t$ for each $t>0$.

Theorem 4.2. *If hypotheses* (A_1) *and* (A_2) *hold, then tha random operator equation* (4.1) *have the random solution* $(\bar{x},\bar{y})\in C([0,1],\mathbb{R})\times C([0,1],\mathbb{R})$ *if there exists a random coupled lower and upper solution for* (4.1)*.*

Proof. Define the mapping $F:\Omega\times C([0,1],\mathbb{R})\times C([0,1],\mathbb{R})\to C([0,1],\mathbb{R})$,

$$F(\omega,x,y)(t)\int_0^1 f(\omega,t,x(s),y(s))ds,\ x,y\in C([0,1],\mathbb{R}),t\in[0,1].$$

Step I. We show that F is a random operator on $C([0,1],\mathbb{R})$. Given $(x,y)\in C([0,1],\mathbb{R})\times C([0,1],\mathbb{R})$, since f is Carathéodory function, then $\omega\to f(\omega,t,x(t),y(t))$ is measurable map in view of Lemma 2.7. Further, the integral is a limit of a finite sum of measurable function, therefore, the map $\omega\to F(\omega,x,y)(t)$ is measurable and hence F is a random operator.

Step II. We show that F is a continuous. For fixed $\omega\in\Omega$, let (x_n,y_n) be a sequence in $C([0,1],\mathbb{R})\times C([0,1],\mathbb{R})$, as $n\to\infty$ there exist $[a,b]\times[a,b]\subset\mathbb{R}$ such that $(x_n(s),y_n(s))$, $(x(s),y(s)\in[a,b]\times[a,b]$ for all $s\in[0,1]$. In addition, the function $f(\omega,\cdot,\cdot,\cdot)$ is uniformly continuous in

$[0,1]\times[a,b]\times[a,b]$. Thus, for fixed $\epsilon>0$, there exists $\delta>0$ such that

$$|f(\omega,s_1,x_1,y_2)-f(\omega,s_2,x_2,y_2)|<\epsilon,$$

for all $s_1,s_2\in[0,1]$ and $x_1,x_2,y_1,y_2\in[a,b]$ such that

$$|s_1-s_2|+|x_1-x_2|+|y_1-y_2|<\delta.$$

Now, let $n(\delta)\in\mathbb{N}$ such that $\|(x)n,y_n)-(x,y)\|_\infty<\delta$, whenever $n\geq n(\delta)$. Then, for every $n\geq n(\delta)$, we have

$$|f(\omega,s,x_n,y_n)-f(\omega,s,x,y)|<\epsilon.$$

Consequently, for $t\in[0,1]$ and $n\geq n(\delta)$, we have

$$\begin{aligned}|F(\omega,x_n,y_n)(t)-F(\omega,x,y)(t)| &\leq \int_0^1 |f(\omega,s,x_n(s),y_n(s))-f(\omega,t,x,y)|ds\\ &\leq \epsilon.\end{aligned}$$

Then

$$\|F(\omega,x_n,y_n)(t)-F(\omega,x,y)(t)\|_\infty\leq\epsilon.$$

So, $\|F(\omega,x_n,y_n)(t)-F(\omega,x,y)(t)\|\to 0$ as $n\to\infty$. Thus, F is continuous operator for each fixed $\omega\in\Omega$.

Step III. We show that F is a monotone operator. For each $\omega\in\Omega$, then function $F(\omega,\cdot,\cdot)$ is monotone operator. Let $\omega\in\Omega$ be fixed. Let (x,y), $(u,v)\in C([0,1],\mathbb{R})\times C([0,1],\mathbb{R})$ such that $(x,y)\preceq(u,v)$, that is,

$$x(t)\preceq u(t),\quad y(t)\preceq v(t),\quad \text{for all } t\in[0,1].$$

If, for every $t\in[0,1]$, $f(\omega,t,\cdot,\cdot)$ is nondecreasing opertor, then

$$f(\omega,t,x(t),y(t))\leq f(\omega,t,u(t),v(t)),\quad \text{for all } t\in[0,1],$$

this implies that

$$F(\omega,x,y)(t)\leq F(\omega,u,v)(t),\quad \text{for all } t\in[0,1].$$

Hence

$$F(\omega,x,y)(t)\leq F(\omega,u,v)(t),\quad \text{for all } t\in[0,1].$$

On the other hand, if, for every $t\in[0,1]$, $f(\omega,t,\cdot,\cdot)$ is nonincreasing operators, then

$$F(\omega,x,y)(t)\geq F(\omega,u,v)(t),\quad \text{for all } t\in[0,1].$$

Step IV. We prove that F has random coupled lower and upper solution. Let $(\bar{x},\bar{y})\in C([0,1],\mathbb{R})\times C([0,1],\mathbb{R})$ such that $(x,y)\preceq(\bar{x},\bar{y})$.

Using (A_2) for all $t\in[0,1]$, we have

$$\begin{aligned}&|F(\omega,x,y)(t)|-|F(\omega,u,v)(t)|+|F(\omega,y,x)(t)|-|F(\omega,v,u)(t)|\\ &\quad=\int_0^1[f(\omega,t,x(s),y(s))-f(\omega,t,u(s),s(s))]ds+\int_0^1[f(\omega,t,y(s),x(s))-f(\omega,t,v(s),u(s))]ds\\ &\quad\leq\int_0^1\varphi_\omega(x(s)-u(s)+2v(s)-2y(s))ds+\int_0^1\varphi_\omega(y(s)-v(s)+2u(s)-2x(s))ds\\ &\quad\leq\int_0^1\Big[\varphi_\omega\Big(\sup_{z\in[0,1]}|x(z)-u(z)|\Big)+\varphi_\omega\Big(\sup_{z\in[0,1]}|y(z)-v(z)|\Big)\Big]ds\\ &\quad=\varphi_\omega\Big(sup_{z\in[0,1]}|x(z)-u(z)|+\sup_{z\in[0,1]}|y(z)-v(z)|\Big),\end{aligned}$$

which implies that

$$\sup_{t\in[0,1]} |F(\omega,x,y)(t)-F(\omega,u,v)(t)| + \sup_{t\in[0,1]} |F(\omega,y,x)(t)-F(\omega,v,u)(t)|$$
$$\le \varphi_\omega\Big(sup_{z\in[0,1]}|x(z)-u(z)| + \sup_{z\in[0,1]} |y(z)-v(z)|\Big).$$

Therefore, we get

$$d(F(\omega,x,y),F(\omega,u,v)) + d(F(\omega,y,x),F(\omega,v,u)) \le \varphi(d(x,u)+d(y,v)).$$

Then, from Lemma 2.7, there exists a random solution of (4.1). □

5. Conclusion

We introduced the new concept and the new nation of random coupled coincidence and coupled random fixed point theorems in complete separable metric spaces and also proved random coupled coincidence and coupled random fixed point theorems for a pair of random mappings $F:\Omega\times(X\times X)\to X$ and $g:\Omega\times X\to X$ and proved random solution of random integral equation. The presented theorems extend and improve the corresponding results which given in the literature. In particular, Theorem 3.1 extend, generalize and improve the results given of Ćirić and Lakshmikantham [8].

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Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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