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Research Article

A Modified Inertial Shrinking Projection Method for Solving Inclusion Problems and Split Equilibrium Problems in Hilbert Spaces

Watcharaporn Cholamjiak[1,*], Suhel Ahmad Khan[2] and Suthep Suantai[3]

[1] *School of Science, University of Phayao, Phayao 56000, Thailand*
[2] *Department of Mathematics, BITS-Pilani, Dubai Campus, P.O. Box 345055, Dubai, UAE*
[3] *Department of Mathematics, Faculty of Science, Chiang Mai University, Chiang Mai 50200, Thailand*
[*] **Corresponding author:** c-wchp007@hotmail.com

Abstract. In this paper, we propose a modified inertial forward-backward splitting method for solving the split equilibrium problem and the inclusion problem. Then we establish the weak convergence theorem of the proposed method. Using the shrinking projection method, we obtain strong convergence theorem. Moreover, we provide some numerical experiments to show the efficiency and the comparison.

Keywords. Inertial method; Inclusion problem; SP-iteration; Forward-backward algorithm; Split equilibrium problem

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1. Introduction

Throughout the paper unless otherwise stated, let H_1 and H_2 be real Hilbert spaces with the inner product $\langle\cdot,\cdot\rangle$ and the norm $\|\cdot\|$. Let C be a nonempty subset of H_1 and let $F_1 : C \times C \to \mathbb{R}$

be a bifunction. We study the equilibrium problem which introduced by Blum and Oettli [6] in 1994. The *equilibrium problem* is to find a point $\widehat{x} \in C$ such that

$$F_1(\widehat{x}, y) \geq 0 \tag{1.1}$$

for all $y \in C$. We know that the equilibrium problem (1.1) has received much attention due to its applications in a large variety of problems arising in numerous problems in physics, optimizations and economics. Some methods have been rapidly established for solving this problem (see [11, 14, 35, 45]).

Very recently, Kazmi and Rizvi [23] introduced and studied the following split equilibrium problem:

Let Q be a nonempty subset of H_2 and $F_2 : Q \times Q \to \mathbb{R}$ be a bifunction. Let $A : H_1 \to H_2$ be a bounded linear operator. The *split equilibrium problem* is to find $\widehat{x} \in C$ such that

$$F_1(\widehat{x}, x) \geq 0 \text{ for all } x \in C \tag{1.2}$$

and such that

$$\widehat{y} = A\widehat{x} \in Q \text{ solves } F_2(\widehat{y}, y) \geq 0 \text{ for all } y \in Q. \tag{1.3}$$

Note that the problem (1.2) is the classical equilibrium problem and we denote its solution set by $EP(F_1)$. The inequalities (1.2) and (1.3) constitute a pair of equilibrium problems which have to find the image $\widehat{y} = A\widehat{x}$, under a given bounded linear operator A, of the solution $\widehat{x}$ of (1.2) in H_1 is the solution of (1.3) in H_2. We denote the solution set of (1.3) by $EP(F_2)$. The solution set of the split equilibrium problem (1.2) and (1.3) is denoted by $\Omega = \{z \in EP(F_1) : Az \in EP(F_2)\}$.

In the recent years, the problem of finding a common element of the set of solution of split equilibriums and the set of fixed points for a mapping in the framework of Hilbert spaces and Banach spaces have been intensively studied by many authors, for instance, (see [18, 21, 23, 41, 43, 49, 50]) and the references cited therein.

Moreover, we study the following inclusion problem: find $\widehat{x} \in H_1$ such that

$$0 \in A\widehat{x} + B\widehat{x}, \tag{1.4}$$

where $A : H_1 \to H_1$ is an operator and $B : H_1 \to 2^{H_1}$ is a set-valued operator. We denote the solution set of (1.4) by $(A+B)^{-1}(0)$. This problem has received much attention due to its applications in large variety of problems arising in convex programming, variational inequalities, split feasibility problem and minimization problem. To be more precise, some concrete problems in machine learning, image processing and linear inverse problem can be modeled mathematically as this formulation.

For solving the problem (1.4), the forward-backward splitting method [4, 12, 13, 25, 26, 33, 48] is usually employed and is defined by the following manner: $x_1 \in H_1$ and

$$x_{n+1} = (I + rB)^{-1}(x_n - rAx_n), \ n \geq 1, \tag{1.5}$$

where $r > 0$. In this case, each step of iterates involves only with A as the forward step and B as the backward step, but not the sum of operators. This method includes, as special cases, the proximal point algorithm [39] and the gradient method. In [24], Lions and Mercier introduced the following splitting iterative methods in a real Hilbert space:

$$x_{n+1} = (2J_r^A - I)(2J_r^B - I)x_n, \ n \geq 1 \tag{1.6}$$

and

$$x_{n+1} = J_r^A(2J_r^B - I)x_n + (I - J_r^B)x_n, \ n \geq 1, \tag{1.7}$$

where $J_r^T = (I + rT)^{-1}$ with $r > 0$. The first one is often called Peaceman-Rachford algorithm [34] and the second one is called Douglas-Rachford algorithm [20]. We note that both algorithms are weakly convergent in general [5,24].

Many problems can be formulated as a problem of from (1.4). For instance, a stationary solution to the initial valued problem of the evolution equation

$$0 \in \frac{\partial u}{\partial t} - Fu, \quad u(0) = u_0 \tag{1.8}$$

can be recast as (1.4) when the governing maximal monotone F is of the form $F = A + B$ (see [24]). In optimization, it often needs (see [13]) to solve a minimization problem of the form

$$\min_{x \in H_1} f(x) + g(x) \tag{1.9}$$

where f, g are proper and lower semicontinuous convex functions from H_1 to the extended real line $\bar{\mathbb{R}} = (-\infty, \infty]$ such that f is differentiable with L-Lipschitz continuous gradient, and the proximal mapping of g is

$$x \mapsto \operatorname*{argmin}_{y \in H_1} g(y) + \frac{\|x - y\|^2}{2r}. \tag{1.10}$$

In particular, if $A := \nabla f$ and $B := \partial g$, where ∇f is the gradient of f and ∂g is the subdifferential of g which is defined by $\partial g(x) := \{s \in H_1 : g(y) \geq g(x) + \langle s, y - x \rangle, \ \forall \ y \in H_1\}$ then problem (1.4) becomes (1.9) and (1.5) also becomes

$$x_{n+1} = \operatorname{prox}_{rg}(x_n - r\nabla f(x_n)), n \geq 1, \tag{1.11}$$

where $r > 0$ is the stepsize and $\operatorname{prox}_{rg} = (I + r\partial g)^{-1}$ is the proximity operator of g.

In 2001, Alvarez and Attouch [2] employed the heavy ball method which was studied in [37,38] for maximal monotone operators by the proximal point algorithm. This algorithm is called the inertial proximal point algorithm and it is of the following form:

$$\begin{cases} y_n = x_n + \theta_n(x_n - x_{n-1}) \\ x_{n+1} = (I + r_n B)^{-1} y_n, \quad n \geq 1. \end{cases} \tag{1.12}$$

It was proved that if $\{r_n\}$ is non-decreasing and $\{\theta_n\} \subset [0,1)$ with

$$\sum_{n=1}^{\infty} \theta_n \|x_n - x_{n-1}\|^2 < \infty, \tag{1.13}$$

then algorithm (1.12) converges weakly to a zero of B. In particular, condition (1.13) is true for $\theta_n < 1/3$. Here θ_n is an extrapolation factor and the inertia is represented by the term $\theta_n(x_n - x_{n-1})$. It is remarkable that the inertial terminology greatly improves the performance of the algorithm and has a nice convergence properties [1,17,19,31].

Recently, Moudafi and Oliny [29] proposed the following inertial proximal point algorithm for solving the zero-finding problem of the sum of two monotone operators:

$$\begin{cases} y_n = x_n + \theta_n(x_n - x_{n-1}) \\ x_{n+1} = (I + r_n B)^{-1}(y_n - r_n A x_n), \quad n \geq 1, \end{cases} \tag{1.14}$$

where $A : H_1 \to H_1$ and $B : H_1 \to 2^{H_1}$. They obtained the weak convergence theorem provided $r_n < 2/L$ with L the Lipschitz constant of A and the condition (1.13) holds. It is observed that, for $\theta_n > 0$, the algorithm (1.14) does not take the form of a forward-backward splitting algorithm, since operator A is still evaluated at the point x_n.

Recently, Lorenz and Pock [26] proposed the following inertial forward-backward algorithm for monotone operators:

$$\begin{cases} y_n = x_n + \theta_n(x_n - x_{n-1}) \\ x_{n+1} = (I + r_n B)^{-1}(y_n - r_n A y_n), \quad n \geq 1, \end{cases} \tag{1.15}$$

where $\{r_n\}$ is a positive real sequence. It is observed that algorithm (1.15) differs from that of Moudafi and Oliny insofar that they evaluated the operator B as the inertial extrapolate y_n. The algorithms involving the inertial term mentioned above have weak convergence, and however, in some applied disciplines, the norm convergence is more desirable that the weak convergence [5].

For solving the fixed point problem of a nonlinear mapping T, the Noor iteration (see [32]) is defined by $x_1 \in H$ and

$$\begin{cases} y_n = \gamma_n x_n + (1-\gamma_n)Tx_n \\ z_n = \beta_n x_n + (1-\beta_n)Ty_n \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n)Tz_n, \end{cases} \tag{1.16}$$

for all $n \geq 1$, where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ are sequences in $[0,1]$. The iterative process (1.16) is generalized form of the Mann (one-step) iterative process by Mann [27] and the Ishikawa (two-step) iterative process by Ishikawa [22]. Phuengrattana and Suantai [36], in 2011, introduced the new process by using the concept of the Noor iteration and it is called the SP-iteration. These iteration is generated by $x_1 \in H$ and

$$\begin{cases} y_n = \gamma_n x_n + (1-\gamma_n)Tx_n \\ z_n = \beta_n y_n + (1-\beta_n)Ty_n \\ x_{n+1} = \alpha_n z_n + (1-\alpha_n)Tz_n \end{cases} \tag{1.17}$$

for all $n \geq 1$, where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ are sequences in $[0,1]$. They compared the convergence speed of Mann, Ishikawa, Noor and SP-iteration and obtained the SP-iteration converges faster than the others for the class of continuous and nondecreasing functions. However, the Noor iteration and the SP-iteration have only weak convergence even in a Hilbert space.

In this work, we introduce a new combining the SP-iteration with the inertial technical term for approximating common elements of the set of solutions of split equilibrium problems and the set of solutions of inclusion problems. We prove some weak convergence theorems of the sequences generated by our iterative process under appropriate additional assumptions in Hilbert spaces. We aim to introduce an algorithm that ensures the strong convergence. To this end, using the idea of Takahashi et al. [47], we employ the following projection method which is defined by: For $C_1 = C$, $x_1 = P_{C_1}x_0$ and

$$\begin{cases} y_n = \alpha_n x_n + (1-\alpha_n)Tx_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \leq \|x_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}}x_0, \quad \forall n \in \mathbb{N}, \end{cases} \tag{1.18}$$

where $0 \leq \alpha_n \leq a < 1$ for all $n \in \mathbb{N}$. It was proved that the sequence $\{x_n\}$ generated by (1.18) converges strongly to a fixed point of a nonexpansive mapping T. This method is usually called the shrinking projection method (see also Nakajo and Takahashi [30]). Further, we apply our main result to find the common elements of the set of solutions of split feasibility problems and variational inequality problems and also find the common elements of the set of solutions of split feasibility problems and the minimization problems. Finally, we show numerically that our proposed scheme converges faster than the algorithm without the inertial technical term.

2. Preliminaries and Lemmas

Let C be a nonempty, closed and convex subset of a Hilbert space H_1. We write $x_n \rightharpoonup x$ to indicate that the sequence $\{x_n\}$ converges weakly to x and $x_n \to x$ implies that $\{x_n\}$ converges strongly to x. A mapping $T : C \to C$ is said to be *nonexpansive* if $\|Tx - Ty\| \le \|x - y\|$ for all $x, y \in C$. An element $p \in C$ is called a *fixed point* of a mapping $T : C \to C$ if $p = Tp$. The fixed point set of T is denoted by $F(T)$. If $F(T) \neq \emptyset$ and $\|Tx - p\| \le \|x - p\|$ for all $x \in C$ and $p \in F(T)$, then T is said to be *quasi-nonexpansive*. The nearest point projection of H_1 onto C is denoted by P_C, that is, $\|x - P_C x\| \le \|x - y\|$ for all $x \in H_1$ and $y \in C$. Such P_C is called the *metric projection* of H_1 onto C. We know that the metric projection P_C is firmly nonexpansive, i.e.,

$$\|P_C x - P_C y\|^2 \le \langle P_C x - P_C y, x - y\rangle$$

for all $x, y \in H_1$. Furthermore, $\langle x - P_C x, y - P_C x\rangle \le 0$ holds for all $x \in H_1$ and $y \in C$ (see [46]).

Lemma 2.1 ([46]). *Let H_1 be a real Hilbert space. Then the following equations hold:*

(1) $\|x - y\|^2 = \|x\|^2 - \|y\|^2 - 2\langle x - y, y\rangle$ *for all* $x, y \in H_1$;

(2) $\|x + y\|^2 \le \|x\|^2 + 2\langle y, x + y\rangle$ *for all* $x, y \in H_1$;

(3) $\|tx + (1-t)y\|^2 = t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)\|x - y\|^2$ *for all* $t \in [0,1]$ *and* $x, y \in H_1$.

It is well know that every nonexpansive operator $T : H_1 \to H_1$ satisfies, for all $(x, y) \in H_1 \times H_1$, the inequality

$$\langle (x - T(x)) - (y - T(y)), T(y) - T(x)\rangle \ge \frac{1}{2}\|(T(x) - x) - (T(y) - y)\|^2$$

and therefore we get, for all $(x, y) \in H_1 \times F(T)$,

$$\langle (x - T(x)), (y - T(y)), \rangle \ge \frac{1}{2}\|T(x) - x\|^2.$$

(see, e.g, [15, 16]).

Assumption 2.2 ([6]). *Let $F_1 : C \times C \to \mathbb{R}$ be a bifunction satisfying the following assumptions:*

(1) $F_1(x, x) = 0$ *for all* $x \in C$;

(2) F_1 *is monotone, i.e.,* $F_1(x, y) + F_1(y, x) \le 0$ *for all* $x \in C$;

(3) *For each* $x, y, z \in C$, $\limsup_{t \to 0} F_1(tz + (1-t)x, y) \le F_1(x, y)$;

(4) *For each* $x \in C$, $y \to F_1(x, y)$ *is convex and lower semi-continuous.*

Lemma 2.3 ([14]). *Let $F_1 : C \times C \to \mathbb{R}$ be a bifunction satisfying Assumption 2.2. For any $r > 0$ and $x \in H_1$, define a mapping $T_r^{F_1} : H_1 \to C$ as follows:*

$$T_r^{F_1}(x) = \left\{ z \in C : F_1(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \ge 0, \forall y \in C \right\}.$$

Then, we have the following:

(1) $T_r^{F_1}$ *is nonempty and single-value;*

(2) $T_r^{F_1}$ *is firmly nonexpansive;*

(3) $F(T_r^{F_1}) = EP(F_1)$;

(4) $EP(F_1)$ *is closed and convex.*

Further, assume that $F_2 : Q \times Q \to \mathbb{R}$ satisfying Assumption 2.2. For each $s > 0$ and $w \in H_2$, define a mapping $T_s^{F_2} : H_2 \to Q$ as follows:

$$T_s^{F_2}(w) = \left\{ d \in Q : F_2(d,e) + \frac{1}{s}\langle e-d, d-w\rangle \geq 0, \forall e \in Q \right\}.$$

Then, we have the following:

(5) *$T_s^{F_2}$ is nonempty and single-value;*

(6) *$T_s^{F_2}$ is firmly nonexpansive;*

(7) *$F(T_s^{F_2}) = EP(F_2,Q)$;*

(8) *$EP(F_2,Q)$ is closed and convex.*

Lemma 2.4 ([28])**.** *Let C be a nonempty closed and convex subset of a real Hilbert space H_1. For each x, $y \in H_1$ and $a \in \mathbb{R}$, the set*

$$D = \{v \in C : \|y-v\|^2 \leq \|x-v\|^2 + \langle z, v\rangle + a\}$$

is closed and convex.

In what follows, we shall use the following notation:

$$T_r^{A,B} = J_r^B(I-rA) = (I+rB)^{-1}(I-rA), \quad r > 0. \tag{2.1}$$

Lemma 2.5 ([25])**.** *Let X be a Banach space. Let $A : X \to X$ be an α-inverse strongly accretive of order q and $B : X \to 2^X$ an m-accretive operator. Then we have*

(i) *For $r > 0$, $F(T_r^{A,B}) = (A+B)^{-1}(0)$.*

(ii) *For $0 < s \leq r$ and $x \in X$, $\|x - T_s^{A,B}x\| \leq 2\|x - T_r^{A,B}x\|$.*

Lemma 2.6 ([25])**.** *Let X be a uniformly convex and q-uniformly smooth Banach space for some $q \in (0,2]$. Assume that A is a single-valued α-inverse strongly accretive of order q in X. Then, given $r > 0$, there exists a continuous, strictly increasing and convex function $\phi_q : \mathbb{R}^+ \to \mathbb{R}^+$ with $\phi_q(0) = 0$ such that, for all $x, y \in B_r$,*

$$\begin{aligned}\|T_r^{A,B}x - T_r^{A,B}y\|^q &\leq \|x-y\|^q - r(\alpha q - r^{q-1}k_q)\|Ax - Ay\|^q \\ &\quad - \phi_q(\|(I-J_r^B)(I-rA)x - (I-J_r^B)(I-rA)y\|),\end{aligned}$$

where k_q is the q-uniform smoothness coefficient of X.

Lemma 2.7 ([2])**.** *Let $\{\psi_n\}$, $\{\delta_n\}$ and $\{\alpha_n\}$ be the sequences in $[0,+\infty)$ such that $\psi_{n+1} \leq \psi_n + \alpha_n(\psi_n - \psi_{n-1}) + \delta_n$ for all $n \geq 1$, $\sum\limits_{n=1}^{\infty} \delta_n < +\infty$ and there exists a real number α with $0 \leq \alpha_n \leq \alpha < 1$ for all $n \geq 1$. Then the followings hold:*

(i) *$\Sigma_{n\geq 1}[\psi_n - \psi_{n-1}]_+ < +\infty$, where $[t]_+ = \max\{t,0\}$;*

(ii) *there exists $\psi^* \in [0,+\infty)$ such that $\lim\limits_{n\to+\infty} \psi_n = \psi^*$.*

Lemma 2.8 ([7])**.** *Let C be a nonempty closed convex subset of a uniformly convex space X and T a nonexpansive mapping with $F(T) \neq \emptyset$. If $\{x_n\}$ is a sequence in C such that $x_n \rightharpoonup x$ and $(I-T)x_n \to y$, then $(I-T)x = y$. In particular, if $y = 0$, then $x \in F(T)$.*

Lemma 2.9 ([42])**.** *Let X be a Banach space satisfying Opial's condition and let $\{x_n\}$ be a sequence in X. Let u, $v \in X$ be such that*

$$\lim_{n\to\infty} \|x_n - u\| \quad \text{and} \quad \lim_{n\to\infty} \|x_n - v\| \ \text{exist.}$$

If $\{x_{n_k}\}$ and $\{x_{m_k}\}$ are subsequences of $\{x_n\}$ which converge weakly to u and v, respectively, then $u = v$.

Proposition 2.10 ([12]). *Let $q > 1$ and let X be a real smooth Banach space with the generalized duality mapping j_q. Let $m \in \mathbb{N}$ be fixed. Let $\{x_i\}_{i=1}^m \subset X$ and $t_i \geq 0$ for all $i = 1,2,\dots,m$ with $\sum\limits_{i=1}^m t_i \leq 1$. Then we have*

$$\left\|\sum_{i=1}^m t_i x_i\right\|^q \leq \frac{\sum_{i=1}^m t_i \|x_i\|^q}{q-(q-1)\left(\sum_{i=1}^m t_i\right)}.$$

3. Main Results

In this section, we aim to introduce and prove the strong convergence of an inertial method with a forward-backward method for solving inclusion problems and split equilibrium problems in Hilbert spaces. We first prove the following weak convergence theorem:

Theorem 3.1. *Let H_1, H_2 be two real Hilbert spaces and $C \subset H_1$, $Q \subset H_2$ be nonempty closed convex subsets of Hilbert spaces H_1 and H_2, respectively. Let $A : H_1 \to H_2$ be a bounded linear operator, and let $F_1 : C \times C \to \mathbb{R}$, $F_2 : Q \times Q \to \mathbb{R}$ be bifunctions satisfying Assumption 2.2 and F_2 is upper semi-continuous in the first argument. Let $B : H_1 \to H_1$ be an α-inverse strongly monotone operator and $D : H_1 \to 2^{H_1}$ a maximal monotone operator such that $S = (B+D)^{-1}(0) \cap \Omega \neq \emptyset$, where $\Omega = \{z \in C : z \in EP(F_1) \text{ and } Az \in EP(F_2)\}$. Let $\{x_n\}, \{y_n\}$ and $\{z_n\}$ be sequences generated by $x_0, x_1 \in H_1$ and*

$$\begin{cases} y_n = x_n + \theta_n(x_n - x_{n-1}) \\ z_n = \alpha_n y_n + (1-\alpha_n) T_{r_n}^{F_1}(I - \gamma A^*(I - T_{r_n}^{F_2})A) y_n, \\ x_{n+1} = \beta_n z_n + (1-\beta_n) J_{s_n}^D (I - s_n B) z_n, \quad n \geq 1, \end{cases} \tag{3.1}$$

where $J_{s_n}^D = (I + s_n D)^{-1}$, $\{s_n\} \subset (0, 2\alpha)$, $\{\theta_n\} \subset [0, \theta]$, $\theta \in [0,1)$, $\{r_n\} \subset (0, \infty)$ with $\gamma \in (0, 1/L)$ such that L is the spectral radius of A^*A and $\{\alpha_n\}$, $\{\beta_n\}$ are sequences in $[0,1]$. Assume that the following conditions hold:

(i) $\sum\limits_{n=1}^\infty \theta_n \|x_n - x_{n-1}\| < \infty$;

(ii) $0 < \liminf\limits_{n\to\infty} \alpha_n \leq \limsup\limits_{n\to\infty} \alpha_n < 1$;

(iii) $0 < \liminf\limits_{n\to\infty} \beta_n \leq \limsup\limits_{n\to\infty} \beta_n < 1$;

(iv) $\liminf\limits_{n\to\infty} r_n > 0$;

(v) $0 < \liminf\limits_{n\to\infty} s_n \leq \limsup\limits_{n\to\infty} s_n < 2\alpha$.

Then the sequence $\{x_n\}$ converges weakly to $q \in S$.

Proof. We split the proof into three steps.

Step 1: Show that $\lim\limits_{n\to\infty} \|x_n - p\|$ exists for every $p \in S = (B+D)^{-1}(0) \cap \Omega$.

Write $T_n = T_{r_n}^{F_1}(I - \gamma A^*(I - T_{r_n}^{F_2})A)$ and $J_n = (I + s_n D)^{-1}(I - s_n B)$.

Notice that we can write

$$z_n = \alpha_n y_n + (1-\alpha_n) T_n y_n \tag{3.2}$$

and

$$x_{n+1} = \beta_n z_n + (1-\beta_n) J_n z_n. \tag{3.3}$$

By the proof in Theorem 3.3 of [44], we know that T_n is quasi-nonexpansive. Let $p \in S$, we get

$$\begin{aligned}\|z_n - p\| &\le \alpha_n \|y_n - p\| + (1-\alpha_n)\|T_n y_n - p\| \\ &\le \|y_n - p\| \\ &\le \|x_n - p\| + \theta_n \|x_n - x_{n-1}\|.\end{aligned} \tag{3.4}$$

By Lemma 2.6, we have

$$\begin{aligned}\|x_{n+1} - p\| &\le \beta_n \|z_n - p\| + (1-\beta_n)\|J_n z_n - p\| \\ &\le \|z_n - p\| \\ &\le \|x_n - p\| + \theta_n \|x_n - x_{n-1}\|.\end{aligned} \tag{3.5}$$

From Lemma 2.7 and the assumption (i), we obtain $\lim_{n\to\infty} \|x_n - p\|$ exists, in particular, $\{x_n\}$ is bounded and also are $\{y_n\}$ and $\{z_n\}$.

Step 2: Show that $x_n \rightharpoonup q \in (B+D)^{-1}(0)$. By Lemma 2.1, 2.6 and T_n is quasi-nonexpansive, we have

$$\begin{aligned}\|x_{n+1} - p\|^2 &= \|\beta_n (z_n - p) + (1-\beta_n)(J_n z_n - p)\|^2 \\ &\le \beta_n \|z_n - p\|^2 + (1-\beta_n)\|J_n z_n - p\|^2 \\ &\le \|z_n - p\|^2 - (1-\beta_n)s_n(2\alpha - s_n)\|Az_n - Ap\|^2 - \|z_n - s_n Az_n - J_n z_n + s_n Ap\| \\ &\le \alpha_n \|y_n - p\|^2 + (1-\alpha_n)\|T_n y_n - p\|^2 - (1-\beta_n)s_n(2\alpha - s_n)\|Az_n - Ap\|^2 \\ &\quad - \|z_n - s_n Az_n - J_n z_n + s_n Ap\| \\ &\le \|y_n - p\|^2 - (1-\beta_n)s_n(2\alpha - s_n)\|Az_n - Ap\|^2 - \|z_n - s_n Az_n - J_n z_n + s_n Ap\| \\ &\le \|x_n - p\|^2 + 2\theta_n \langle x_n - x_{n-1}, y_n - p\rangle - (1-\beta_n)s_n(2\alpha - s_n)\|Az_n - Ap\|^2 \\ &\quad - \|z_n - s_n Az_n - J_n z_n + s_n Ap\|.\end{aligned} \tag{3.6}$$

Since $\lim_{n\to\infty} \|x_n - p\|$ exists, it follows from (3.6), the assumptions (i), (iii) and (v) that

$$\lim_{n\to\infty} \|Az_n - Ap\| = \lim_{n\to\infty} \|z_n - s_n Az_n - J_n z_n + s_n Ap\| = 0. \tag{3.7}$$

This give, by the triangle inequality, that

$$\lim_{n\to\infty} \|J_n z_n - z_n\| = 0. \tag{3.8}$$

Since $\liminf_{n\to\infty} s_n > 0$, there is $s > 0$ such that $s_n \ge s$ for all $n \ge 1$. Lemma 2.5 (ii) yields that

$$\|T_s^{B,D} z_n - z_n\| \le 2\|J_n z_n - z_n\|. \tag{3.9}$$

Then, by (3.8) and (3.9), we obtain

$$\lim_{n\to\infty} \|T_s^{B,D} z_n - z_n\| = 0. \tag{3.10}$$

From (3.8), we have

$$\lim_{n\to\infty} \|x_{n+1} - z_n\| = \lim_{n\to\infty} (1-\beta_n)\|J_n z_n - z_n\| = 0. \tag{3.11}$$

Again by Lemma 2.1, 2.6 and T_n is quasi-nonexpansive, we have

$$\begin{aligned}\|x_{n+1} - p\|^2 &\le \beta_n \|z_n - p\|^2 + (1-\beta_n)\|J_n z_n - p\|^2 \\ &\le \|z_n - p\|^2 \\ &\le \alpha_n \|y_n - p\|^2 + (1-\alpha_n)\|T_n y_n - p\|^2 - \alpha_n(1-\alpha_n)\|T_n y_n - y_n\|^2\end{aligned}$$

$$\le \|y_n - p\|^2 - \alpha_n(1-\alpha_n)\|T_n y_n - y_n\|^2$$
$$\le \|x_n - p\|^2 + 2\theta_n\langle x_n - x_{n-1}, y_n - p\rangle - \alpha_n(1-\alpha_n)\|T_n y_n - y_n\|^2. \tag{3.12}$$

Since $\lim_{n\to\infty}\|x_n - p\|$ exists and the assumption (ii), it follows from (3.12) that

$$\lim_{n\to\infty}\|T_n y_n - y_n\| = 0. \tag{3.13}$$

This implies that

$$\lim_{n\to\infty}\|z_n - y_n\| = \lim_{n\to\infty}(1-\alpha_n)\|T_n y_n - y_n\| = 0. \tag{3.14}$$

From the definition of $\{y_n\}$ and the assumption (i), we have

$$\lim_{n\to\infty}\|y_n - x_n\| = \lim_{n\to\infty}\theta_n\|x_n - x_{n-1}\| = 0. \tag{3.15}$$

It follows from (3.11), (3.14) and (3.15) that

$$\|x_{n+1} - x_n\| \le \|x_{n+1} - z_n\| + \|z_n - y_n\| + \|y_n - x_n\| \to 0 \tag{3.16}$$

as $n \to \infty$. From (3.11) and (3.16), we obtain

$$\|z_n - x_n\| \le \|z_n - x_{n+1}\| + \|x_{n+1} - x_n\| \to 0 \tag{3.17}$$

as $n \to \infty$. Since $\{x_n\}$ is bounded and H_1 is reflexive, $\omega_w(x_n) = \{x \in H_1 : x_{n_i} \rightharpoonup x, \{x_{n_i}\} \subset \{x_n\}\}$ is nonempty. Let $q \in \omega_w(x_n)$ be an arbitrary element. Then there exists a subsequence $\{x_{n_i}\} \subset \{x_n\}$ converging weakly to q. Let $p \in \omega_w(x_n)$ and $\{x_{n_m}\} \subset \{x_n\}$ be such that $x_{n_m} \rightharpoonup p$. From (3.14), we also have $z_{n_i} \rightharpoonup q$ and $z_{n_m} \rightharpoonup p$. Since $T_s^{B,D}$ is nonexpansive, by Lemma 2.8, we have $p, q \in (B+D)^{-1}(0)$. Applying Lemma 2.9, we obtain $p = q$.

Step 3: Show that $q \in \Omega$. Setting $u_n = T_{r_n}^{F_1}(I - \gamma A^*(I - T_{r_n}^{F_2})A)y_n$. For any $p \in S$, we estimate

$$\begin{aligned}
\|u_n - p\|^2 &= \|T_{r_n}^{F_1}(I - \gamma A^*(I - T_{r_n}^{F_2})A)y_n - p\|^2 \\
&= \|T_{r_n}^{F_1}(I - \gamma A^*(I - T_{r_n}^{F_2})A)y_n - T_{r_n}^{F_1}p\|^2 \\
&\le \|y_n - \gamma A^*(I - T_{r_n}^{F_2})Ay_n - p\|^2 \\
&\le \|y_n - p\|^2 + \gamma^2\|A^*(I - T_{r_n}^{F_2})Ay_n\|^2 + 2\gamma\langle p - y_n, A^*(I - T_{r_n}^{F_2})Ay_n\rangle.
\end{aligned} \tag{3.18}$$

Thus, we have

$$\begin{aligned}
\|u_n - p\|^2 &\le \|x_n - p\|^2 + 2\theta_n\langle x_n - x_{n-1}, y_n - p\rangle + \gamma^2\langle Ay_n - T_{r_n}^{F_2}Ay_n, AA^*(I - T_{r_n}^{F_2})Ay_n\rangle \\
&\quad + 2\gamma\langle p - y_n, A^*(I - T_{r_n}^{F_2})Ay_n\rangle.
\end{aligned} \tag{3.19}$$

On the other hand, we have

$$\begin{aligned}
\gamma^2\langle Ay_n - T_{r_n}^{F_2}Ay_n, AA^*(I - T_{r_n}^{F_2})Ay_n\rangle &\le L\gamma^2\langle Ay_n - T_{r_n}^{F_2}Ay_n, Ay_n - T_{r_n}^{F_2}Ay_n\rangle \\
&= L\gamma^2\|Ay_n - T_{r_n}^{F_2}Ay_n\|^2
\end{aligned} \tag{3.20}$$

and

$$\begin{aligned}
2\gamma\langle p - y_n, A^*(I - T_{r_n}^{F_2})Ay_n\rangle &= 2\gamma\langle A(p - y_n), Ay_n - T_{r_n}^{F_2}Ay_n\rangle \\
&= 2\gamma\langle A(p - y_n) + (Ay_n - T_{r_n}^{F_2}Ay_n) - (Ay_n - T_{r_n}^{F_2}Ay_n), Ay_n - T_{r_n}^{F_2}Ay_n\rangle \\
&= 2\gamma\{\langle Ap - T_{r_n}^{F_2}Ay_n, Ay_n - T_{r_n}^{F_2}Ay_n\rangle - \|Ay_n - T_{r_n}^{F_2}Ay_n\|^2\} \\
&\le 2\gamma\left\{\frac{1}{2}\|Ay_n - T_{r_n}^{F_2}Ay_n\|^2 - \|Ay_n - T_{r_n}^{F_2}Ay_n\|^2\right\} \\
&= -\gamma\|Ay_n - T_{r_n}^{F_2}Ay_n\|^2.
\end{aligned} \tag{3.21}$$

Using (3.19), (3.20) and (3.21), we have

$$\|u_n - p\|^2 \le \|x_n - p\|^2 + 2\theta_n\langle x_n - x_{n-1}, y_n - p\rangle + L\gamma^2\|Ay_n - T_{r_n}^{F_2}Ay_n\|^2 - \gamma\|Ay_n - T_{r_n}^{F_2}Ay_n\|^2$$

$$= \|x_n - p\|^2 + 2\theta_n\langle x_n - x_{n-1}, y_n - p\rangle + \gamma(L\gamma - 1)\|Ay_n - T_{r_n}^{F_2}Ay_n\|^2. \tag{3.22}$$

This implies that

$$\begin{aligned}\|z_n - p\|^2 &\le \alpha_n\|y_n - p\|^2 + (1-\alpha_n)\|u_n - p\|^2\\ &\le \|x_n - p\|^2 + 2\theta_n\langle x_n - x_{n-1}, y_n - p\rangle + \gamma(L\gamma - 1)\|Ay_n - T_{r_n}^{F_2}Ay_n\|^2.\end{aligned} \tag{3.23}$$

Therefore, we have

$$-\gamma(L\gamma - 1)\|Ay_n - T_{r_n}^{F_2}Ay_n\|^2 \le \|x_n - p\|^2 - \|z_n - p\|^2 + 2\theta_n\langle x_n - x_{n-1}, y_n - p\rangle. \tag{3.24}$$

Since $\gamma(L\gamma - 1) < 0$, it follows from (3.14), we obtain

$$\lim_{n\to\infty}\|Ay_n - T_{r_n}^{F_2}Ay_n\| = 0. \tag{3.25}$$

Since $T_{r_n}^{F_1}$ is firmly nonexpansive and $I - \gamma A^*(T_{r_n}^{F_2} - I)A$ is nonexpansive [44], it follows that

$$\begin{aligned}\|u_n - p\|^2 &= \|T_{r_n}^{F_1}(y_n - \gamma A^*(I - T_{r_n}^{F_2})Ay_n) - T_{r_n}^{F_1}p\|^2\\ &\le \langle T_{r_n}^{F_1}(y_n - \gamma A^*(I - T_{r_n}^{F_2})Ay_n) - T_{r_n}^{F_1}p, y_n - \gamma A^*(I - T_{r_n}^{F_2})Ay_n - p\rangle\\ &= \langle u_n - p, y_n - \gamma A^*(I - T_{r_n}^{F_2})Ay_n - p\rangle\\ &= \frac{1}{2}\{\|u_n - p\|^2 + \|y_n - \gamma A^*(I - T_{r_n}^{F_2})Ay_n - p\|^2 - \|u_n - y_n + \gamma A^*(I - T_{r_n}^{F_2})Ay_n\|^2\}\\ &\le \frac{1}{2}\{\|u_n - p\|^2 + \|y_n - p\|^2 - \|u_n - y_n + \gamma A^*(I - T_{r_n}^{F_2})Ay_n\|^2\}\\ &= \frac{1}{2}\{\|u_n - p\|^2 + \|y_n - p\|^2 - (\|u_n - y_n\|^2 + \gamma^2\|A^*(I - T_{r_n}^{F_2})Ay_n\|^2\\ &\quad + 2\gamma\langle u_n - y_n, A^*(I - T_{r_n}^{F_2})Ay_n\rangle)\},\end{aligned}$$

which implies that

$$\begin{aligned}\|u_n - p\|^2 &\le \|y_n - p\|^2 - \|u_n - y_n\|^2 - 2\gamma\langle u_n - y_n, A^*(I - T_{r_n}^{F_2})Ay_n\rangle\\ &\le \|y_n - p\|^2 - \|u_n - y_n\|^2 + 2\gamma\|u_n - y_n\|\|A^*(I - T_{r_n}^{F_2})Ay_n\|.\end{aligned} \tag{3.26}$$

This implies that

$$\begin{aligned}\|z_n - p\|^2 &\le \alpha_n\|y_n - p\|^2 + (1-\alpha_n)\|u_n - p\|^2\\ &\le \alpha_n\|y_n - p\|^2 + (1-\alpha_n)\big(\|y_n - p\|^2 - \|u_n - y_n\|^2 + 2\gamma\|u_n - y_n\|\|A^*(I - T_{r_n}^{F_2})Ay_n\|\big).\end{aligned}$$

Therefore, we have

$$(1-\alpha_n)\|u_n - y_n\|^2 \le 2\gamma\|u_n - y_n\|\|A^*(I - T_{r_n}^{F_2})Ay_n\| + \|y_n - p\|^2 - \|z_n - p\|^2. \tag{3.27}$$

From the condition (ii), (3.17) and (3.25), we have

$$\lim_{n\to\infty}\|u_n - y_n\| = 0. \tag{3.28}$$

It follows from (3.15) and (3.28) that

$$\lim_{n\to\infty}\|u_n - x_n\| = 0. \tag{3.29}$$

This implies that $u_n \rightharpoonup q$ as $n \to \infty$. We next show that $q \in EP(F_1)$.
From $u_n = T_{r_n}^{F_1}(I - \gamma A^*(I - T_{r_n}^{F_2})A)y_n$, we have

$$F_1(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - y_n + \gamma A^*(I - T_{r_n}^{F_2})Ay_n\rangle \ge 0 \tag{3.30}$$

for all $y \in C$, which implies that

$$F_1(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - y_n\rangle + \frac{1}{r_n}\langle y - u_n, \gamma A^*(I - T_{r_n}^{F_2})Ay_n\rangle \ge 0$$

for all $y \in C$. By Assumption 2.2 (2), we have

$$\frac{1}{r_n}\langle y-u_n, u_n-y_n\rangle + \frac{1}{r_n}\langle y-u_n, \gamma A^*(I-T_{r_n}^{F_1})Ay_n\rangle \geq F_1(y,u_n)$$

for all $y \in C$. From $\liminf_{n\to\infty} r_n > 0$, from (3.25), (3.28) and the Assumption 2.2 (4), we obtain

$$F_1(y,q) \leq 0$$

for all $y \in C$. For any $0 < t \leq 1$ and $y \in C$, let $y_t = ty + (1-t)q$. Since $y \in C$ and $q \in C$, $y_t \in C$ and hence $F_1(y_t,q) \leq 0$. So, by Assumption 2.2 (1) and (4), we have

$$0 = F_1(y_t,y_t) \leq tF_1(y_t,y) + (1-t)F_1(y_t,w) \leq tF_1(y_t,y)$$

and hence $F_1(y_t,y) \geq 0$. So $F_1(q,y) \geq 0$ for all $y \in C$. By Assumption 2.2 (3) we obtain $q \in EP(F_1)$. Since A is a bounded linear operator, $Ay_n \rightharpoonup Aq$. Then it follows from (3.25) that

$$T_{r_n}^{F_2}Ay_n \rightharpoonup Aq \tag{3.31}$$

as $n \to \infty$. By the definition of $T_{r_n}^{F_2}Ay_n$, we have

$$F_2(T_{r_n}^{F_2}Ay_n, y) + \frac{1}{r_n}\langle y - T_{r_n}^{F_2}Ay_n, T_{r_n}^{F_2}Ay_n - Ay_n\rangle \geq 0$$

for all $y \in C$. Since F_2 is upper semi-continuous in the first argument and (3.31), it follows that

$$F_2(Aq,y) \geq 0$$

for all $y \in C$. This shows that $Aq \in EP(F_2)$. Hence $q \in S$. □

Theorem 3.2. *Let H_1, H_2 be two real Hilbert spaces and $C \subset H_1$, $Q \subset H_2$ be nonempty closed convex subsets of Hilbert spaces H_1 and H_2, respectively. Let $A : H_1 \to H_2$ be a bounded linear operator, and let $F_1 : C \times C \to \mathbb{R}$, $F_2 : Q \times Q \to \mathbb{R}$ be bifunctions satisfying Assumption 2.2 and F_2 is upper semi-continuous in the first argument. Let $B : H_1 \to H_1$ be an α-inverse strongly monotone operator and $D : H_1 \to 2^{H_1}$ a maximal monotone operator such that $S = (B+D)^{-1}(0) \cap \Omega \neq \emptyset$, where $\Omega = \{z \in C : z \in EP(F_1) \text{ and } Az \in EP(F_2)\}$. Let $\{x_n\}, \{y_n\}$ and $\{z_n\}$ be sequences generated by $x_0, x_1 \in H_1$ and*

$$\begin{cases} y_n = x_n + \theta_n(x_n - x_{n-1}), \\ z_n = \alpha_n y_n + (1-\alpha_n)T_{r_n}^{F_1}(I - \gamma A^*(I - T_{r_n}^{F_2})A)y_n, \\ w_n = \beta_n z_n + (1-\beta_n)J_{s_n}^D(I - s_n B)z_n, \\ C_{n+1} = \{z \in C_n : \|w_n - z\|^2 \leq \|x_n - z\|^2 + 2\theta_n^2\|x_n - x_{n-1}\|^2 - 2\theta_n\langle x_n - z, x_{n-1} - x_n\rangle\} \\ x_{n+1} = P_{C_{n+1}}x_1, \quad n \geq 1 \end{cases} \tag{3.32}$$

where $J_{s_n}^D = (I + s_n D)^{-1}$, $\{s_n\} \subset (0, 2\alpha)$, $\{\theta_n\} \subset [0,\theta]$, $\theta \in [0,1)$, $\{r_n\} \subset (0,\infty)$ with $\gamma \in (0, 1/L)$ such that L is the spectral radius of A^*A and $\{\alpha_n\}$, $\{\beta_n\}$ are sequences in $[0,1]$. Assume that the following conditions hold:

(i) $\sum_{n=1}^{\infty} \theta_n \|x_n - x_{n-1}\| < \infty$;

(ii) $\limsup_{n\to\infty} \alpha_n < 1$;

(iii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;

(iv) $\liminf_{n\to\infty} r_n > 0$;

(v) $0 < \liminf_{n\to\infty} s_n \leq \limsup_{n\to\infty} s_n < 2\alpha$.

Then the sequence $\{x_n\}$ converges strongly to $q = P_S x_1$.

Proof. We split the proof into six steps.

Step 1: Show that $P_{C_{n+1}}x_1$ is well-defined for every $x \in H_1$. We know that $(B+D)^{-1}(0)$ and Ω are closed and convex by Lemma 2.5 and Lemma 2.3, respectively. From the definition of C_{n+1}, from Lemma 2.4, C_{n+1} is closed and convex for each $n \geq 1$. For each $n \in \mathbb{N}$, we put $T_n = T_{r_n}^{F_1}(I - \gamma A^*(I - T_{r_n}^{F_2})A)$ and $J_n = (I + s_n D)^{-1}(I - s_n B)$ and let $p \in S$. By the proof in Theorem 3.3 of [44], we know that T_n is quasi-nonexpansive, and since J_n is nonexpansive, we have

$$
\begin{aligned}
\|w_n - p\|^2 &\leq \beta_n \|z_n - p\|^2 + (1-\beta_n)\|J_n z_n - p\|^2 \\
&\leq \|z_n - p\|^2 \\
&\leq \alpha_n \|y_n - p\|^2 + (1-\alpha_n)\|T_n y_n - p\|^2 \\
&\leq \|y_n - p\|^2 \\
&\leq \|x_n - p\|^2 + 2\theta_n \langle x_n - x_{n-1}, y_n - p\rangle \\
&\leq \|x_n - p\|^2 + 2\theta_n^2 \|x_n - x_{n-1}\|^2 - 2\theta_n \langle x_n - p, x_{n-1} - x_n\rangle. \qquad (3.33)
\end{aligned}
$$

So, we have $p \in C_{n+1}$, thus $S \subset C_{n+1}$. Therefore $P_{C_{n+1}}x_1$ is well-defined.

Step 2: Show that $\lim_{n\to\infty} \|x_n - x_1\|$ exists. Since S is nonempty, closed and convex subset of H_1, there exists a unique $v \in S$ such that

$$v = P_S x_1. \qquad (3.34)$$

From $x_n = P_{C_n}x_1$, $C_{n+1} \subset C_n$ and $x_{n+1} \in C_{n+1}$, $\forall\ n \geq 1$, we get

$$\|x_n - x_1\| \leq \|x_{n+1} - x_1\|, \quad \forall\ n \geq 1. \qquad (3.35)$$

On the other hand, as $S \subset C_n$, we obtain

$$\|x_n - x_1\| \leq \|v - x_1\|, \quad \forall\ n \geq 1. \qquad (3.36)$$

It follows that the sequence $\{x_n\}$ is bounded and nondecreasing. Therefore, $\lim_{n\to\infty} \|x_n - x_1\|$ exists.

Step 3: Show that $x_n \to q \in C$ as $n \to \infty$. For $m > n$, by the definition of C_n, we have $x_m = P_{C_m}x_1 \in C_m \subseteq C_n$. By Lemma 2.6, we obtain that

$$\|x_m - x_n\|^2 \leq \|x_m - x_1\|^2 - \|x_n - x_1\|^2. \qquad (3.37)$$

Since $\lim_{n\to\infty} \|x_n - x_1\|$ exists, it follows from (3.37) that $\lim_{n\to\infty} \|x_m - x_n\| = 0$. Hence $\{x_n\}$ is Cauchy sequence in C and so $x_n \to q \in C$ as $n \to \infty$.

Step 4: Show that $q \in (B+D)^{-1}(0)$. From Step 3, we have that $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$. Since $x_{n+1} \in C_n$, we have

$$
\begin{aligned}
\|w_n - x_n\| &\leq \|w_n - x_{n+1}\| + \|x_{n+1} - x_n\| \\
&\leq \sqrt{\|x_n - x_{n+1}\|^2 + 2\theta_n^2\|x_n - x_{n-1}\|^2 - 2\theta_n\langle x_n - x_{n+1}, x_{n-1} - x_n\rangle} + \|x_{n+1} - x_n\|. \qquad (3.38)
\end{aligned}
$$

By the assumption (i) and (3.38), we obtain

$$\lim_{n\to\infty} \|w_n - x_n\| = 0. \qquad (3.39)$$

By Lemma 2.1 and J_n is nonexpansive and T_n is quasi-nonexpansive, we have

$$
\begin{aligned}
\|w_n - p\|^2 &= \beta_n \|z_n - p\|^2 + (1-\beta_n)\|J_n z_n - p\|^2 - \beta_n(1-\beta_n)\|J_n z_n - z_n\|^2 \\
&\leq \|z_n - p\|^2 - \beta_n(1-\beta_n)\|J_n z_n - z_n\|^2 \\
&\leq \alpha_n \|y_n - p\|^2 + (1-\alpha_n)\|T_n y_n - p\|^2 - \beta_n(1-\beta_n)\|J_n z_n - z_n\|^2
\end{aligned}
$$

$$\le \|y_n - p\|^2 - \beta_n(1-\beta_n)\|J_n z_n - z_n\|^2$$
$$\le \|x_n - p\|^2 + 2\theta_n\langle x_n - x_{n-1}, y_n - p\rangle - \beta_n(1-\beta_n)\|J_n z_n - z_n\|^2. \tag{3.40}$$

This implies that

$$\beta_n(1-\beta_n)\|J_n z_n - z_n\|^2 \le \|x_n - p\|^2 - \|w_n - p\|^2 + 2\theta_n\langle x_n - x_{n-1}, y_n - p\rangle. \tag{3.41}$$

Then, by the assumption (i), (iii) and (3.39), we obtain

$$\lim_{n\to\infty}\|J_n z_n - z_n\| = 0. \tag{3.42}$$

This implies that

$$\lim_{n\to\infty}\|w_n - z_n\| = \lim_{n\to\infty}(1-\beta_n)\|J_n z_n - z_n\| = 0. \tag{3.43}$$

It follows from (3.39) and (3.43) that

$$\lim_{n\to\infty}\|z_n - x_n\| = 0. \tag{3.44}$$

Since $\liminf_{n\to\infty} s_n > 0$, there is $s > 0$ such that $s_n \ge s$ for all $n \ge 1$. Lemma 2.5 (ii) yields that

$$\|T_s^{B,D} z_n - z_n\| \le 2\|J_n z_n - z_n\|. \tag{3.45}$$

Then, by (3.42) and (3.45), we obtain

$$\lim_{n\to\infty}\|T_s^{B,D} z_n - z_n\| = 0. \tag{3.46}$$

Since $\{x_n\}$ is bounded and H_1 is reflexive, $\omega_w(x_n) = \{x \in H_1 : x_{n_i} \rightharpoonup x, \{x_{n_i}\} \subset \{x_n\}\}$ is nonempty. Let $q \in \omega_w(x_n)$ be an arbitrary element. Then there exists a subsequence $\{x_{n_i}\} \subset \{x_n\}$ converging weakly to q. Let $p \in \omega_w(x_n)$ and $\{x_{n_m}\} \subset \{x_n\}$ be such that $x_{n_m} \rightharpoonup p$. From (3.44), we also have $z_{n_i} \rightharpoonup q$ and $z_{n_m} \rightharpoonup p$. Since $T_s^{B,D}$ is nonexpansive, by Lemma 2.8, we have $p, q \in (B+D)^{-1}(0)$. Applying Lemma 2.9, we obtain $p = q$.

Step 5: Show that $q \in \Omega$. By the same proof of Step 3 in Theorem 3.1, we have $q \in \Omega$.

Step 6: Show that $q = P_S x_1$. Since $x_n = P_{C_n} x_1$ and $S \subset C_n$, we obtain

$$\langle x_1 - x_n, x_n - z\rangle \ge 0, \ \forall\, z \in S. \tag{3.47}$$

By taking the limit in (3.47), we obtain

$$\langle x_1 - q, q - z\rangle \ge 0, \ \forall\, z \in S. \tag{3.48}$$

This shows that $q = P_S x_1$, which completes the proof. □

Remark 3.3. We remark here that the condition (i) is easily implemented in numerical computation since the valued of $\|x_n - x_{n-1}\|$ is known before choosing θ_n. Indeed, the parameter θ_n can be chosen such that $0 \le \theta_n \le \bar{\theta}_n$, where

$$\bar{\theta}_n = \begin{cases} \min\left\{\frac{\omega_n}{\|x_n - x_{n-1}\|}, \theta\right\} & \text{if } x_n \ne x_{n-1}, \\ \theta & \text{otherwise,} \end{cases}$$

where $\{\omega_n\}$ is a positive sequence such that $\sum_{n=1}^{\infty}\omega_n < \infty$.

We now give an example in Euclidean space $\mathbb{R}^3$ to support the main theorem.

Example 3.4. Let $H_1 = H_2 = \mathbb{R}^3$, $C = \{(x,y,z) \in \mathbb{R}^3 | \sqrt{x^2+y^2+z^2} \le 1\}$ and $Q = \{(x,y) \in \mathbb{R}^3 | \langle a, x\rangle \ge b\}$ where $a = (2,-1,3)$, $b = 1$. For $r > 0$, let $T_r^{F_1}x = P_C x$ and $T_r^{F_2}x = P_Q x$. Let $A, B, D : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by $Ax = \begin{pmatrix} 1 & -1 & 5 \\ 0 & 1 & 3 \\ 0 & 0 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$, $Bx = 3x + (1,2,1)$ and $Dx = 4x$ where $x = (x_1, x_2, x_3) \in \mathbb{R}^3$. We see that B is 1/3-inverse strongly monotone and D is maximal monotone. Moreover, by a direct calculation, we have for $s > 0$

$$J_s^D(x - sBx) = (I + sD)^{-1}(x - sBx) = \frac{1-3s}{1+4s}x - \frac{s}{1+4s}(1,2,1)$$

where $x = (x_1, x_2, x_3) \in \mathbb{R}^3$. Since $\alpha = \frac{1}{3}$, we can choose $s_n = 0.1$ for all $n \in \mathbb{N}$. Since $L = 3$, we can also choose $\gamma = 0.1$. Let $\alpha_n = \beta_n = r_n = \frac{n}{100n+1}$ and

$$\theta_n = \begin{cases} \min\left\{\frac{1}{n^2\|x_n - x_{n-1}\|}, 0.5\right\} & \text{if } x_n \neq x_{n-1}, \\ 0.5 & \text{otherwise.} \end{cases}$$

We provide a numerical test of a comparison between our inertial forward-backward method defined in Theorem 3.1 and a standard forward-backward method (i.e. $\theta_n = 0$). The stoping criterion is defined by $E_n = \|x_{n+1} - x_n\| < 10^{-9}$. The different choices of x_0 and x_1 are giving as follow:

Choice 1: $x_0 = (1, -5, 8)^T$ and $x_1 = (8, -5, 3)^T$;

Choice 2: $x_0 = (-1, 6, 7)^T$ and $x_1 = (-3, 5, -3)^T$;

Choice 3: $x_0 = (-2.3, 3.2, -4.5)^T$ and $x_1 = (6.1, -5.2, -1.1)^T$.

Table 1. Comparison of $\theta_n \neq 0$ and $\theta_n = 0$ in Example 3.4

	No. of Iter.		cpu (Time).	
	$\theta_n \neq 0$	$\theta_n = 0$	$\theta_n \neq 0$	$\theta_n = 0$
Choice 1: $x_0 = (1, -5, 8)^T$ $x_1 = (8, -5, 3)^T$	54	70	0.013965	0.022513
Choice 2: $x_0 = (-1, 6, 7)^T$, $x_1 = (-3, 5, -3)^T$	60	76	0.017255	0.022136
Choice 3: $x_0 = (-2.3, 3.2, -4.5)^T$, $x_1 = (6.1, -5.2, -1.1)^T$	62	74	0.019725	0.022350

The error plotting E_n of $\theta_n \neq 0$ and $\theta_n = 0$ for each of the choices in Table 1 is shown in Figure 1-3, respectively.

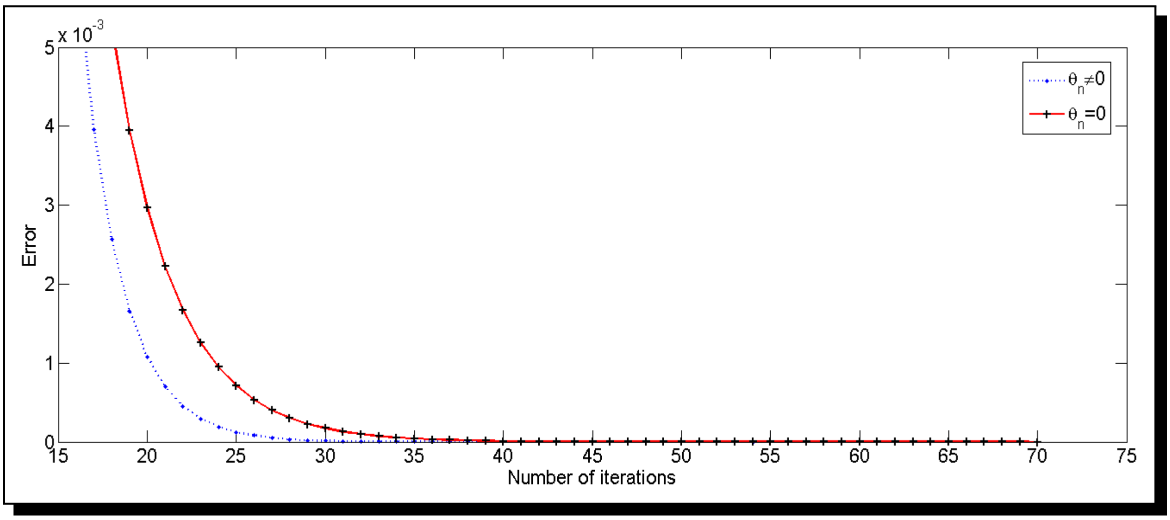

Figure 1. Comparison of iterations for Choice 1 in Example 3.4

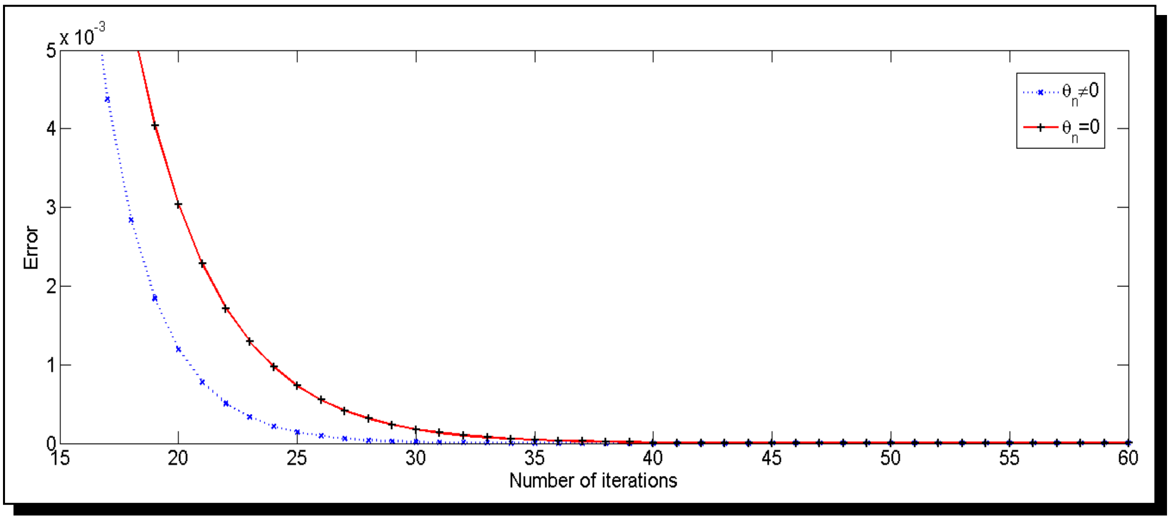

Figure 2. Comparison of iterations for Choice 2 in Example 3.4

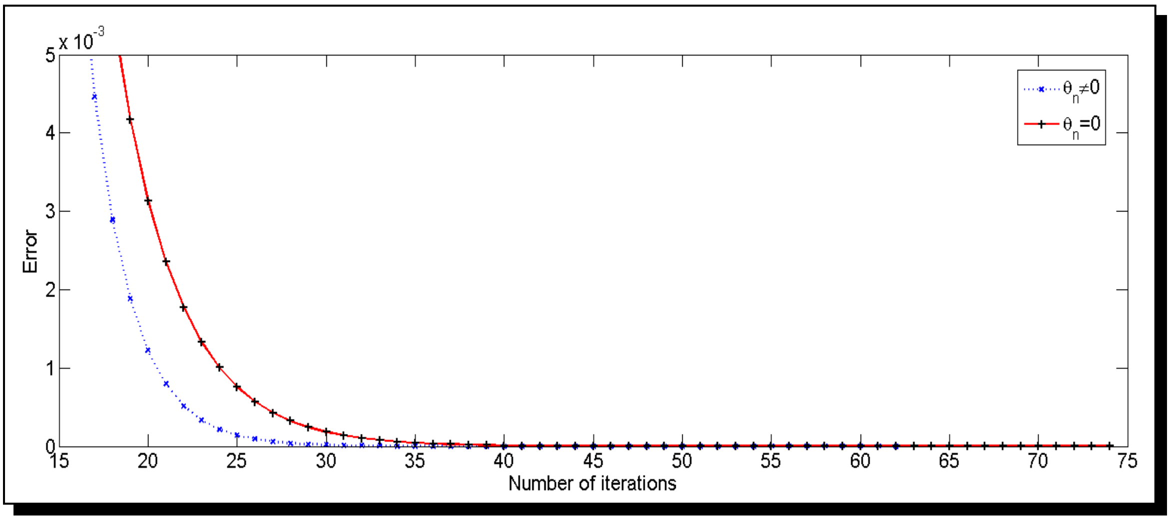

Figure 3. Comparison of iterations for Choice 3 in Example 3.4

Remark 3.5. From Figure 1-3, it is shown that our inertial forward-backward method has a good convergence speed and requires small number of iterations than the standard forward-backward method.

4. Applications and Numerical Experiments

In this section, we apply our main result to solve the common problems between the split feasibility problem and the convex minimization problem and also solve the common problems between the split feasibility problem and the variational inequality problem. The *split feasibility problem* (SFP) [10] is to find a point $\hat{x}$ such that

$$\hat{x} \in C, \quad T\hat{x} \in Q, \tag{4.1}$$

where C and Q are, respectively, closed convex subsets of Hilbert spaces H_1 and H_2 and $T : H_1 \to H_2$ is a bounded linear operator with its adjoint T^*. For solving the SFP (4.1), Byrne [8] proposed the following CQ algorithm:

$$x_{n+1} = P_C(x_n - \lambda T^*(I - P_Q)Tx_n), \tag{4.2}$$

where $0 < \lambda < 2\alpha$ with $\alpha = 1/\|T\|^2$. Here $\|T\|^2$ is the spectral radius of T^*T. We know that $T^*(I-P_Q)T$ is $1/\|T\|^2$-inverse strongly monotone [9]. So, we now obtain immediately the following strong convergence theorem for solving the SFP (4.1).

4.1 Convex Minimization Problem

Let $F : H \to \mathbb{R}$ be a convex smooth function and $G : H \to \mathbb{R}$ be a convex, lower-semicontinuous and nonsmooth function. We consider the problem of finding $\widehat{x} \in H$ such that

$$F(\widehat{x}) + G(\widehat{x}) \le F(x) + G(x) \tag{4.3}$$

for all $x \in H$. This problem (4.3) is equivalent, by Fermat's rule, to the problem of finding $\widehat{x} \in H$ such that

$$0 \in \nabla F(\widehat{x}) + \partial G(\widehat{x}) \tag{4.4}$$

where ∇F is a gradient of F and ∂G is a subdifferential of G. The minimizer of $F+G$ will be denoted by S. We know that if ∇F is $\frac{1}{L}$-Lipschitz continuous, then it is L-inverse strongly monotone [3, Corollary 10]. Moreover, ∂G is maximal monotone [40, Theorem A]. If we set $B = \nabla F$ and $C = \partial G$ in Theorem 3.2, then we obtain the following result.

Theorem 4.1. *Let H_1, H_2 be two real Hilbert spaces and $C \subset H_1$, $Q \subset H_2$ be nonempty closed convex subsets of Hilbert spaces H_1 and H_2, respectively. Let $T : H_1 \to H_2$ be a bounded linear operator. Let $F : H \to \mathbb{R}$ be a convex and differentiable function with $\frac{1}{L}$-Lipschitz continuous gradient ∇F and $G : H \to \mathbb{R}$ be a convex and lower semi-continuous function which $F+G$ attains a minimizer. Assume that $S \cap SFP \neq \emptyset$. Let $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ be sequences generated by $x_0, x_1 \in H_1$ and*

$$\begin{cases} y_n = x_n + \theta_n(x_n - x_{n-1}) \\ z_n = \alpha_n y_n + (1-\alpha_n)P_C(y_n - \gamma T^*(I-P_Q)Ty_n), \\ w_n = \beta_n z_n + (1-\beta_n)J_{s_n}^{\partial G}(z_n - s_n\nabla F(z_n)), \\ C_{n+1} = \{z \in C_n : \|w_n - z\|^2 \le \|x_n - z\|^2 + 2\theta_n^2\|x_n - x_{n-1}\|^2 - 2\theta_n\langle x_n - z, x_{n-1} - x_n\rangle\} \\ x_{n+1} = P_{C_{n+1}}x_1, \quad n \ge 1 \end{cases} \tag{4.5}$$

where $J_{r_n}^{\partial G} = (I + s_n\partial G)^{-1}$, $\{s_n\} \subset (0, 2\alpha)$, $\{\theta_n\} \subset [0, \theta]$, $\theta \in [0, 1)$ with $\gamma \in (0, 1/\|T\|^2)$ and $\{\alpha_n\}$, $\{\beta_n\}$ are sequences in $[0, 1]$. Assume that the following conditions hold:

(i) $\sum_{n=1}^{\infty} \theta_n\|x_n - x_{n-1}\| < \infty$;

(ii) $\limsup_{n\to\infty} \alpha_n < 1$;

(iii) $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$;

(iv) $0 < \liminf_{n\to\infty} s_n \le \limsup_{n\to\infty} s_n < 2\alpha$.

Then the sequence $\{x_n\}$ converges strongly to $q = P_{S\cap SFP}x_1$.

Example 4.2. Let $H_1 = H_2 = \mathbb{R}^3$, $C = \left\{(x_1, x_2, x_3) \in \mathbb{R}^3 : \sqrt{x_1^2 + x_2^2 + x_3^2} \le 1\right\}$ and $Q = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : \langle a, x\rangle \ge b\}$, where $a = (2, 4, 5)$ and $b = 5$. Let $T = \begin{pmatrix} 1 & 2 & 1 \\ -1 & 0 & 1 \\ 0 & -2 & 2 \end{pmatrix}$. Solve the following minimization problem:

$$\min_{x\in\mathbb{R}^3} \|x\|_2^2 + (3, 5, -1)x + \|x\|_1, \tag{4.6}$$

where $x = (x_1, x_2, x_3) \in \mathbb{R}^3$.

Set $F(x) = \|x\|_2^2 + (3,5,-1)x$ and $G(x) = \|x\|_1$ for all $x \in \mathbb{R}^3$. We have for $x \in \mathbb{R}^3$ and $s > 0$, $\nabla F = 2x+(3,5,-1)$ and $J_s^{\partial G}(x) = (\max\{|x_1|-s,0\}\text{sign}(x_1), \max\{|x_2|-s,0\}\text{sign}(x_2), \max\{|x_3|-s,0\}\text{sign}(x_3))$. We see that ∇F is 2-Lipschitz continuous, consequently, it is 1/2-inverse strongly monotone. In this example, we choose $\gamma = 0.1$. Let α_n, β_n, s_n and θ_n be as in Example 3.4. The stopping criterion is defined by $\|x_{n+1} - x_n\| < 10^{-9}$. The different choices of x_0 and x_1 are given as follows:

Choice 1: $x_0 = (-2,8,-5)^T$ and $x_1 = (-3,-5,8)^T$;

Choice 2: $x_0 = (-1,7,6)^T$ and $x_1 = (-3,1,-1)^T$;

Choice 3: $x_0 = (-2.3,3.2,-4.5)^T$ and $x_1 = (6.1,-5.2,-1.1)^T$.

Table 2. Comparison of $\theta_n \neq 0$ and $\theta_n = 0$ in Example 4.2

	No. of Iter.		cpu (Time).	
	$\theta_n \neq 0$	$\theta_n = 0$	$\theta_n \neq 0$	$\theta_n = 0$
Choice 1: $x_0 = (-2,8,-5)^T$ $x_1 = (-3,-5,8)^T$	48	67	0.016348	0.017879
Choice 2: $x_0 = (-1,7,6)^T$, $x_1 = (-3,1,-1)^T$	44	61	0.022234	0.031774
Choice 3: $x_0 = (-2.3,3.2,-4.5)^T$, $x_1 = (6.1,-5.2,-1.1)^T$	47	59	0.015349	0.029733

The error plotting E_n of $\theta_n \neq 0$ and $\theta_n = 0$ for each of the choices in Table 2 is shown in Figures 4-6, respectively.

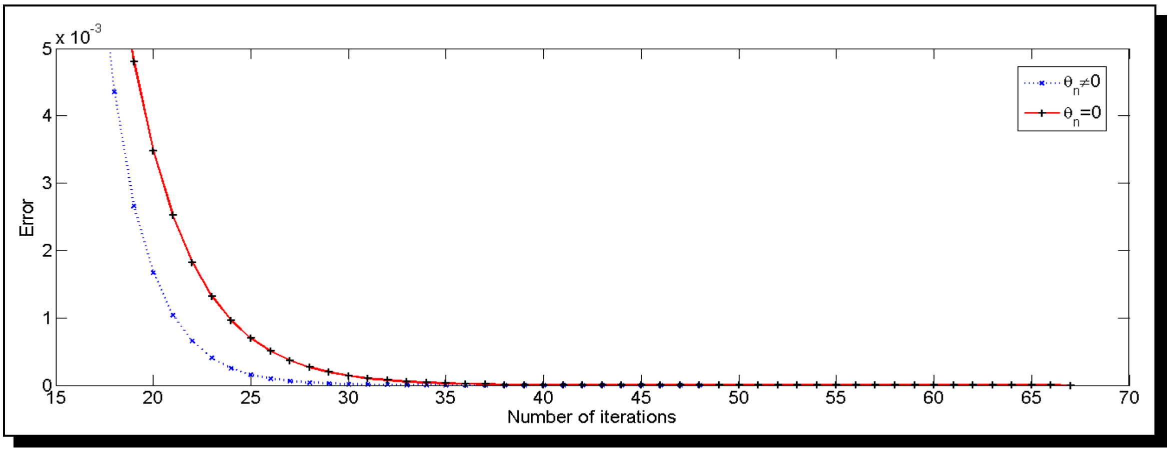

Figure 4. Comparison of iterations for Choice 1 in Examnple 4.2

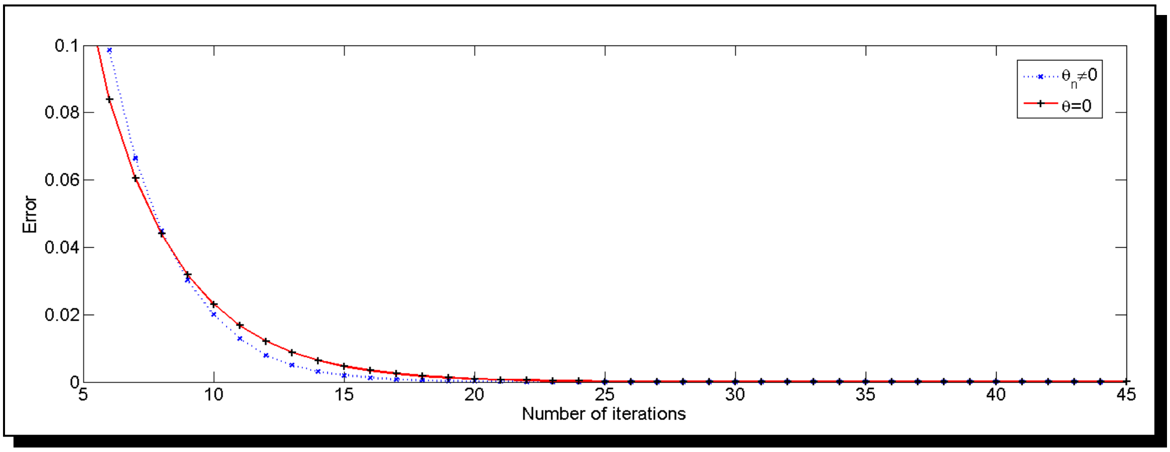

Figure 5. Comparison of iterations for Choice 2 in Examnple 4.2

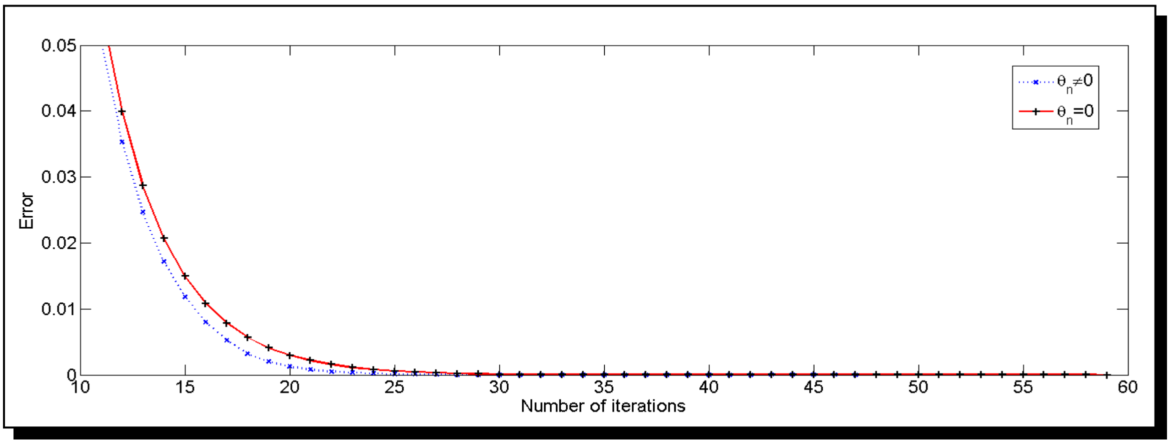

Figure 6. Comparison of iterations for Choice 3 in Examnple 4.2

4.2 Variational Inequality Problem

The variational inequality problem (VIP) is to find a point $\hat{x} \in C$ such that

$$\langle A\hat{x}, x-\hat{x}\rangle \geq 0, \quad \forall\ x \in C \tag{4.7}$$

where $A : C \to H$ is a nonlinear monotone operator. The solution set of (4.7) will be denoted by S. The extragradient method is used to solve the VIP (4.7). It is also known that the VIP is a special case of the problem of finding zeros of the sum of two monotone operators. Indeed, the resolvent of the normal cone is nothing but the projection operator. So, we obtain immediately the following results.

Theorem 4.3. *Let H_1, H_2 be two real Hilbert spaces and $C \subset H_1$, $Q \subset H_2$ be nonempty closed convex subsets of Hilbert spaces H_1 and H_2, respectively. Let $T : H_1 \to H_2$ be a bounded linear operator. Let $A : H \to H$ be an α-inverse strongly monotone operator and C be a nonempty closed convex subset of H. Assume that $S \cap SFP \neq \emptyset$. Let $\{x_n\}, \{y_n\}$ and $\{z_n\}$ be sequences generated by $x_0, x_1 \in H_1$ and*

$$\begin{cases} y_n = x_n + \theta_n(x_n - x_{n-1}) \\ z_n = \alpha_n y_n + (1-\alpha_n)P_C(y_n - \gamma T^*(I-P_Q)Ty_n), \\ w_n = \beta_n z_n + (1-\beta_n)P_C(z_n - s_nAz_n), \\ C_{n+1} = \{z \in C_n : \|w_n - z\|^2 \leq \|x_n - z\|^2 + 2\theta_n^2\|x_n - x_{n-1}\|^2 - 2\theta_n\langle x_n - z, x_{n-1} - x_n\rangle\} \\ x_{n+1} = P_{C_{n+1}}x_1, \quad n \geq 1 \end{cases} \tag{4.8}$$

where $\{s_n\} \subset (0, 2\alpha)$, $\{\theta_n\} \subset [0, \theta]$, $\theta \in [0, 1)$ with $\gamma \in (0, 1/\|T\|^2)$ and $\{\alpha_n\}$, $\{\beta_n\}$ are sequences in $[0,1]$. *Assume that the following conditions hold:*

(i) $\sum_{n=1}^{\infty} \theta_n\|x_n - x_{n-1}\| < \infty$;

(ii) $\limsup_{n\to\infty} \alpha_n < 1$;

(iii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;

(iv) $0 < \liminf_{n\to\infty} s_n \leq \limsup_{n\to\infty} s_n < 2\alpha$.

Then the sequence $\{x_n\}$ converges strongly to $q = P_{S\cap SFP}x_1$.

Example 4.4. Let $H_1 = H_2 = \mathbb{R}^3$, $C = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : \sqrt{x_1^2 + x_2^2 + x_3^2} \leq 1\}$ and $Q = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : \langle a, x\rangle \geq b\}$, where $a = (2,4,5)$ and $b = 5$. Let $T = \begin{pmatrix} 1 & 2 & 1 \\ -1 & 0 & 1 \\ 0 & -2 & 2 \end{pmatrix}$ and $A = \begin{pmatrix} 1 & -1 & 5 \\ 0 & 1 & 3 \\ 0 & 0 & 2 \end{pmatrix}$.

In this example, we choose $\gamma = 0.1$. Let α_n, β_n, s_n and θ_n be as in Example 3.4. The stoping criterion is defined by $\|x_{n+1} - x_n\| < 10^{-9}$. The different choices of x_0 and x_1 are given as follows:
Choice 1: $x_0 = (3,-7,1)^T$ and $x_1 = (-9,6,2)^T$;
Choice 2: $x_0 = (7,1,-3)^T$ and $x_1 = (4,9,-5)^T$;
Choice 3: $x_0 = (-1.5,4.1,-0.5)^T$ and $x_1 = (3.2,-7.4,-1.5)^T$.

Table 3. Comparison of $\theta_n \neq 0$ and $\theta_n = 0$ in Example 4.2

	No. of Iter.		cpu (Time)	
	$\theta_n \neq 0$	$\theta_n = 0$	$\theta_n \neq 0$	$\theta_n = 0$
Choice 1: $x_0 = (3,-7,1)^T$ $x_1 = (-9,6,2)^T$	42	47	0.012358	0.022709
Choice 2: $x_0 = (7,1,-3)^T$, $x_1 = (4,9,-5)^T$	32	60	0.006002	0.018696
Choice 3: $x_0 = (-1.5,4.1,-0.5)^T$, $x_1 = (3.2,-7.4,-1.5)^T$	36	43	0.013849	0.021260

The error plotting E_n of $\theta_n \neq 0$ and $\theta_n = 0$ for each of the choices in Table 3 is shown in Figures 7-9, respectively.

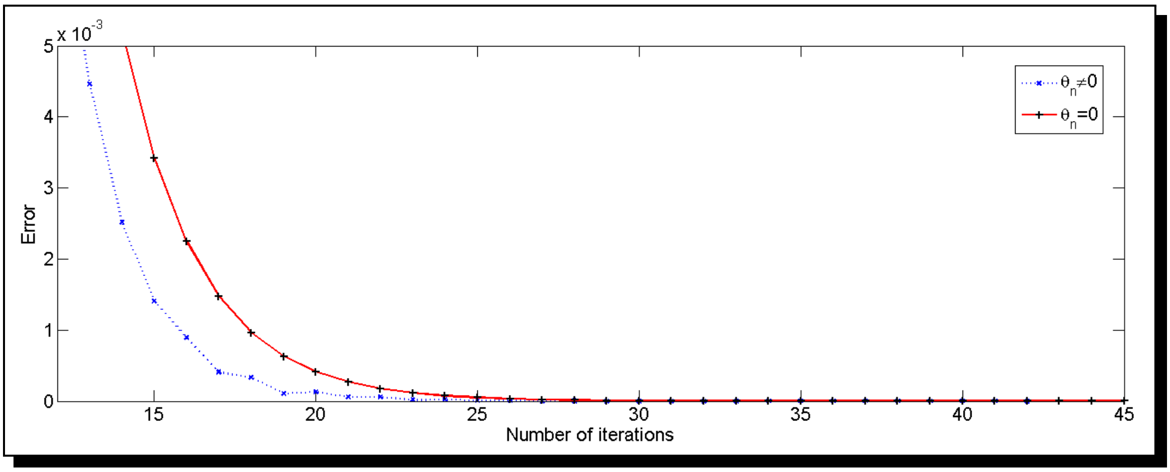

Figure 7. Comparison of iterations for Choice 1 in Example 4.4

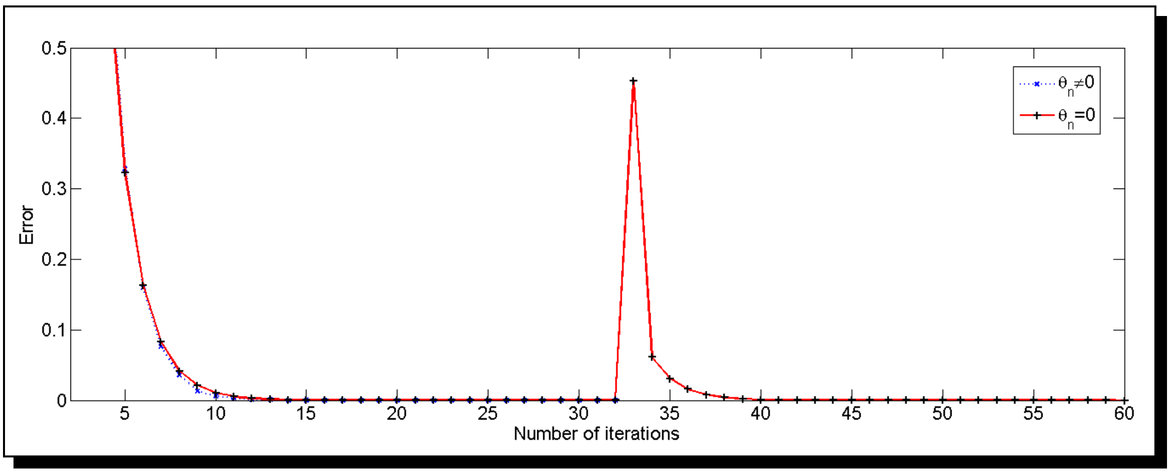

Figure 8. Comparison of iterations for Choice 2 in Example 4.4

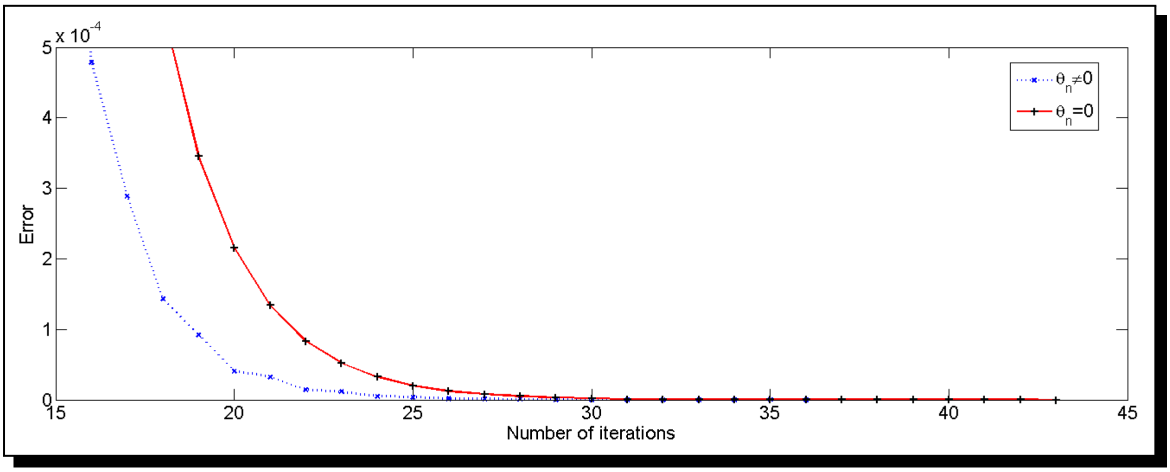

Figure 9. Comparison of iterations for Choice 3 in Example 4.4

5. Conclusion

In this paper, we present a new modified inertial forward-backward splitting method combining the SP-iteration for solving the split equilibrium problem and the inclusion problem. The weak convergence theorem is established under some suitable conditions in Hilbert spaces. We then use the shrinking projection method for obtaining the strong convergence theorem and apply our result to find the common elements of the set of solutions of split feasibility problems and variational inequality problems and also find the common elements of the set of solutions of split feasibility problems and the minimization problems. Some numerical experiments show that our inertial forward-backward method have a competitive advantage over the standard forward-backward method (see in Tables 1-3 and Figures 1-9).

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Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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