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Research Article

On Inertial Relaxation CQ Algorithm for Split Feasibility Problems

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Abstract. In this work, we introduce an inertial relaxation CQ algorithm for the split feasibility problem in Hilbert spaces. We prove weak convergence theorem under suitable conditions. Numerical examples illustrating our methods's efficiency are presented for comparing some known methods.

Keywords. Split feasibility problem; CQ algorithm; Hilbert space; Projection; Inertial

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1. Introduction

Let C and Q be nonempty closed convex sets in real Hilbert spaces H_1 and H_2, respectively, and $A : H_1 \to H_2$ a bounded linear operator. The problem in solving

$$x^* \in C \text{ with } Ax^* \in Q, \tag{1.1}$$

is called the split feasibility problem (SFP) which was introduced by Censor and Elfving [7]. We denote by S the solution set of (1.1). This problem has a variety of specific applications in real world, such as medical care, image reconstruction, and signal processing [1, 10, 13, 16, 17].

Byrne [5,6] suggested a projection method called the CQ algorithm for solving the SFP that does not involve matrix inverses as follows:

$$x_{n+1} = P_C(x_n - \alpha_n A^*(I - P_Q)Ax_n), \tag{1.2}$$

where P_C and P_Q denote the metric projection onto C and Q, respectively and A^* is the adjoint operator of A. However, in some cases it is impossible or needs too much work to exactly compute the orthogonal projection. In [22], by using the relaxed projection technology, Yang presented a relaxed CQ algorithm of solving the SFP. This method employs two half spaces C_n and Q_n instead of C and Q, respectively. It is very convenient in practice because the formula can be easily computed.

In 2008, Mainge [14] introduced Mann type algorithm involving the inertial term for the fixed point problem of nonexpansive mappings in Hilbert spaces.

In 2015, Bot *et al.* [4] proposed an inertial Douglas-Rachford splitting algorithm for finding the set of zeros of the sum of two maximally monotone operators in Hilbert spaces and investigate its convergence properties.

Qu and Xiu [15] introduced the stepsize self-adaptively by adopting Armijo-line searches as follows:

Algorithm 1. For any $\gamma > 0$, $\ell \in (0,1)$ and $\mu \in (0,1)$, take arbitrarily $x_1 \in \mathbb{R}^N$ and calculate

$$x_{n+1} = P_C(x_n - \alpha_n A^T(I - P_Q)Ax_n), \tag{1.3}$$

where $\alpha_n = \gamma\ell^{m_n}$ and m_n is the smallest nonnegative integer m such that

$$\begin{aligned} &f(P_C(x_n - \sigma\rho^m A^T(I-P_Q)Ax_n)) \\ &\quad \le f(x_n) - \mu\langle A^T(I-P_Q)Ax_n, x_n - P_C(x_n - \sigma\rho^m A^T(I-P_Q)Ax_n)\rangle, \end{aligned} \tag{1.4}$$

where $f(x) = \frac{1}{2}\|Ax - P_Q Ax\|^2$.

Recently, Gibali *et al.* [12] modified relaxation CQ algorithm with the Armijo-linesearch of Qu and Xiu [15] in real Hilbert spaces as follows:

Algorithm 2. Given constants $\gamma > 0$, $\ell \in (0,1)$ and $\mu \in (0,1)$. Let x_1 be arbitrary. For $n \ge 1$, calculate

$$y_n = P_{C_n}(x_n - \alpha_n F_n(x_n)), \tag{1.5}$$

where $\alpha_n = \gamma\ell^{m_n}$ and m_n is the smallest nonnegative integer such that

$$\alpha_n\|F_n(x_n) - F_n(y_n)\| \le \mu\|x_n - y_n\|. \tag{1.6}$$

Construct the next iterative step x_{n+1} by

$$x_{n+1} = P_{C_n}(x_n - \alpha_n f_n(y_n)). \tag{1.7}$$

In this work, we propose an inertial relaxation CQ algorithm for the split feasibility problem. We prove weakly convergence of our algorithm and present numerical examples comparing algorithm of Gibali *et al.* [12]. There have been some iterative methods invented for solving the SFP in the literature (see, e.g. [8,18–20]).

2. Preliminaries

In this section, we give some definitions and lemma, which will be used in the main results. Throughout this paper, we recall the following definitions:

A mapping $T: H_1 \to H_1$ is said to be *firmly nonexpansive* if, for all $x, y \in H_1$,

$$\langle x-y, Tx-Ty\rangle \geq \|Tx-Ty\|^2. \tag{2.1}$$

F is said to be monotone on C if

$$\langle F(x)-F(y), x-y\rangle \geq 0,\ \forall x, y \in C; \tag{2.2}$$

In a real Hilbert space H, we have the following equality:

$$\langle x, y\rangle = \frac{1}{2}\|x\|^2 + \frac{1}{2}\|y\|^2 - \frac{1}{2}\|x-y\|^2. \tag{2.3}$$

A differentiable function f is convex if and only if there holds the inequality:

$$f(z) \geq f(x) + \langle \nabla f(x), z-x\rangle \tag{2.4}$$

for all $z \in H_1$.

An element $g \in H_1$ is called a *subgradient* of $f: H_1 \to \mathbb{R}$ at x if

$$f(z) \geq f(x) + \langle g, z-x\rangle \tag{2.5}$$

for all $z \in H_1$, which is called the *subdifferentiable inequality*.

A function $f: H_1 \to \mathbb{R}$ is said to be *subdifferentiable* at x if it has at least one subgradient at x.

The set of subgradients of f at the point x is called the *subdifferentiable* of f at x, which is denoted by $\partial f(x)$.

A function f is said to be *subdifferentiable* if it is subdifferentiable at all $x \in H_1$. If a function f is differentiable and convex, then its gradient and subgradient coincide.

A function $f: H_1 \to \mathbb{R}$ is said to be *weakly lower semi-continuous* (shortly, w-lsc) at x if $x_n \rightharpoonup x$ implies

$$f(x) \leq \liminf_{n\to\infty} f(x_n). \tag{2.6}$$

We know that the orthogonal projection P_C from H_1 onto a nonempty closed convex subset $C \subset H_1$ is a typical example of a firmly nonexpansive mapping, which is defined by

$$P_C x := \arg\min_{y\in C} \|x-y\|^2 \tag{2.7}$$

for all $x \in H_1$.

Lemma 1 ([2]). *Let C be a nonempty closed convex subset of a real Hilbert space H_1. Then, for any $x \in H_1$, the following assertions hold:*

(1) $\langle x-P_Cx, z-P_Cx\rangle \leq 0$ *for all* $z \in C$;

(2) $\|P_Cx-P_Cy\|^2 \leq \langle P_Cx-P_Cy, x-y\rangle$ *for all* $x, y \in H_1$;

(3) $\|P_Cx-z\|^2 \leq \|x-z\|^2 - \|P_Cx-x\|^2$ *for all* $z \in C$.

From Lemma 1, the operator $I-P_C$ is also firmly nonexpansive, where I denotes the identity operator, i.e., for any $x, y \in H_1$,

$$\langle (I-P_C)x-(I-P_C)y, x-y\rangle \geq \|(I-P_C)x-(I-P_C)y\|^2. \tag{2.8}$$

Lemma 2 ([14]). *Assume $x_n \in [0,\infty)$ and $\delta_n \in [0,\infty)$ satisfy:*

(i) $x_{n+1} - x_n \le \theta_n(x_n - x_{n-1}) + \delta_n$,

(ii) $\sum_{n=1}^{+\infty} \delta_n < \infty$,

(iii) $\{\theta_n\} \subset [0,\theta]$, *where* $\theta \in [0,1)$. *Then the sequence* $\{x_n\}$ *is convergent with* $\sum_{n=1}^{+\infty} [x_{n+1} - x_n]_+ < \infty$, *where* $[t]_+ := \max\{t,0\}$ *(for any* $t \in \mathbb{R}$*).*

Lemma 3 ([3]). *Let S be a nonempty subset of a real Hilbert space H_1 and $\{x_n\}$ be a sequence in H_1 that satisfies the following properties:*

(i) $\lim_{n\to\infty} \|x_n - x\|$ *exists for each* $x \in S$;

(ii) *every sequential weak limit point of* $\{x_n\}$ *is in* S.

Then $\{x_n\}$ converges weakly to a point in S.

3. Inertial Relaxation CQ Algorithm with Armijo-line Search

In this section, we introduce an inertial relaxation CQ algorithm with Armijo-line search, in which the closed convex subsets C and Q have particular structure.

For the SFP, we assume that the convex sets C and Q satisfy the following conditions:

(A1) The set C is given by

$$C = \{x \in H_1 : c(x) \le 0\}, \tag{3.1}$$

where $c : H_1 \to \mathbb{R}$ is a convex function and C is a nonempty set. The set Q is given by

$$Q = \{y \in H_2 : q(y) \le 0\}, \tag{3.2}$$

where $q : H_2 \to \mathbb{R}$ is a convex function and Q is a nonempty set. Assume that c and q are subdifferentiable on C and Q, respectively, and c and q are bounded on bounded sets. Note that this condition is automatically satisfied in finite dimensional spaces.

For any $x \in H_1$, at least one subgradient $\xi \in \partial c(x)$ can be calculated, where$\partial c(x)$ is defined as follows:

$$\partial c(x) = \{z \in H_1 : c(u) \ge c(x) + \langle u - x, z\rangle, \forall\, u \in H_1\}. \tag{3.3}$$

For any $y \in H_2$, at least one subgradient $\eta \in \partial q(y)$ can be calculated, where

$$\partial q(x) = \{w \in H_2 : q(v) \ge q(y) + \langle v - y, w\rangle, \forall\, v \in H_2\}. \tag{3.4}$$

Define the sets C_n and Q_n by the following half-spaces:

$$C_n = \{x \in H_1 : c(x_n) + \langle \xi_n, x - x_n\rangle \le 0\}, \tag{3.5}$$

where $\xi_n \in \partial c(x_n)$, and

$$Q_n = \{y \in H_2 : q(Ax_n) + \langle \eta_n, y - Ax_n\rangle \le 0\}, \tag{3.6}$$

where $\eta_n \in \partial q(Ax_n)$.

By the definition of the subgradient, it is clear that $C \subseteq C_n$ and $Q \subseteq Q_n$. The projections onto C_n and Q_n are easy to compute since C_n and Q_n are two half-spaces.

We define the function $F_n : H_1 \to H_1$ by

$$F_n(x) = A^*(I - P_{Q_n})A(x). \tag{3.7}$$

where $A : H_1 \to H_2$ a bounded linear operator, and A^* is the adjoint operator of A.

Algorithm 3. Given constant $\gamma > 0$, $\ell \in (0,1)$, $\mu \in (0,1)$ and $\{\theta_n\} \subset [0,\theta)$, where $\theta \in [0,1)$. Let x_1 be arbitrary and let

$$w_n = x_n + \theta_n(x_n - x_{n-1}), \tag{3.8}$$

$$y_n = P_{C_n}(w_n - \alpha_n F_n(w_n)), \tag{3.9}$$

where $\alpha_n = \gamma \ell^{m_n}$ and m_n is the smallest nonnegative integer such that

$$\alpha_n \|F_n(w_n) - F_n(y_n)\| \le \mu \|w_n - y_n\|. \tag{3.10}$$

Construct the next iterative step x_{n+1} by

$$x_{n+1} = P_{C_n}(w_n - \alpha_n F_n(y_n)). \tag{3.11}$$

Lemma 4 ([12])**.** *The Armijo-line search* (3.10) *terminates after a finite number of steps. In addition,*

$$\frac{\mu\ell}{L} < \alpha_n \le \gamma, \quad \text{for all } n \ge 1 \tag{3.12}$$

where $L = \|A\|^2$.

In what follows, we denote S by the solution set of the problem (SFP) and also assume that S is nonempty.

Theorem 1. *Assume that the solution set S is nonempty and $\sum_{n=1}^{\infty} \theta_n \|x_n - x_{n-1}\|^2 < \infty$. Then any sequence $\{x_n\}$ generated by Algorithm* 3 *converges weakly to a solution in S.*

Proof. Let $z \in S$. Since $C \subseteq C_n$, $Q \subseteq Q_n$, then $z = P_C(z) = P_{C_n}(z)$ and $Az = P_Q(Az) = P_{Q_n}(Az)$. This implies that $F_n(z) = 0$. By Lemma 1 (iii), we obtain

$$\begin{aligned}
\|x_{n+1} - z\|^2 &= \|P_{C_n}(w_n - \alpha_n F_n(y_n)) - z\|^2 \\
&\le \|w_n - \alpha_n F_n(y_n) - z\|^2 - \|x_{n+1} - w_n + \alpha_n F_n(y_n)\|^2 \\
&= \|w_n - z\|^2 - 2\alpha_n \langle w_n - z, F_n(y_n)\rangle + \|\alpha_n F_n(y_n)\|^2 - \|x_{n+1} - w_n\|^2 \\
&\quad - 2\alpha_n \langle x_{n+1} - w_n, F_n(y_n)\rangle - \|\alpha_n F_n(y_n)\|^2 \\
&= \|w_n - z\|^2 - 2\alpha_n \langle w_n - y_n + y_n - z, F_n(y_n)\rangle - \|x_{n+1} - w_n\|^2 \\
&\quad - 2\alpha_n \langle x_{n+1} - w_n, F_n(y_n)\rangle \\
&= \|w_n - z\|^2 - 2\alpha_n \langle w_n - y_n, F_n(y_n)\rangle - 2\alpha_n \langle y_n - z, F_n(y_n)\rangle - \|x_{n+1} - w_n\|^2 \\
&\quad - 2\alpha_n \langle x_{n+1} - w_n, F_n(y_n)\rangle \\
&= \|w_n - z\|^2 - 2\alpha_n \langle y_n - z, F_n(y_n)\rangle - 2\alpha_n \langle x_{n+1} - y_n, F_n(y_n)\rangle \\
&\quad - \|x_{n+1} - y_n + y_n - w_n\|^2 \\
&= \|w_n - z\|^2 - 2\alpha_n \langle y_n - z, F_n(y_n)\rangle - 2\alpha_n \langle x_{n+1} - y_n, F_n(y_n)\rangle \\
&\quad - \|x_{n+1} - y_n\|^2 - 2\langle x_{n+1} - y_n, y_n - w_n\rangle - \|y_n - w_n\|^2 \\
&= \|w_n - z\|^2 - 2\alpha_n \langle y_n - z, F_n(y_n)\rangle - \|x_{n+1} - y_n\|^2 - \|y_n - w_n\|^2 \\
&\quad - 2\langle x_{n+1} - y_n, y_n - w_n + \alpha_n F_n(y_n)\rangle.
\end{aligned} \tag{3.13}$$

On the other hand, we have

$$\begin{aligned}\|w_n - z\|^2 &= \|x_n + \theta_n(x_n - x_{n-1}) - z\|^2 \\ &= \|x_n - z\|^2 + 2\langle x_n - z, \theta_n(x_n - x_{n-1})\rangle + \|\theta_n(x_n - x_{n-1})\|^2 \\ &= \|x_n - z\|^2 + 2\theta_n\langle x_n - z, x_n - x_{n-1}\rangle + \theta_n^2\|x_n - x_{n-1}\|^2. \end{aligned} \tag{3.14}$$

By (2.3), we obtain

$$\langle x_n - z, x_n - x_{n-1}\rangle = \frac{1}{2}\|x_n - z\|^2 + \frac{1}{2}\|x_n - x_{n-1}\|^2 - \frac{1}{2}\|x_{n-1} - z\|^2. \tag{3.15}$$

Combining (3.14) and (3.15), we have

$$\begin{aligned}\|w_n - z\|^2 &= \|x_n - z\|^2 + \theta_n\|x_n - z\|^2 + \theta_n\|x_n - x_{n-1}\|^2 - \theta_n\|x_{n-1} - z\|^2 + \theta_n^2\|x_n - x_{n-1}\|^2 \\ &\le \|x_n - z\|^2 + \theta_n(\|x_n - z\|^2 - \|x_{n-1} - z\|^2) + 2\theta_n\|x_n - x_{n-1}\|^2. \end{aligned} \tag{3.16}$$

From (2.8) and $F_n(z) = 0$, we have

$$\begin{aligned}2\alpha_n\langle y_n - z, F_n(y_n)\rangle &= 2\alpha_n\langle y_n - z, F_n(y_n) - F_n(z)\rangle \\ &= 2\alpha_n\langle y_n - z, A^*(I - P_{Q_n})Ay_n - A^*(I - P_{Q_n})Az\rangle \\ &= 2\alpha_n\langle Ay_n - Az, (I - P_{Q_n})Ay_n - (I - P_{Q_n})Az\rangle \\ &\ge 2\frac{\mu\ell}{L}\|(I - P_{Q_n})Ay_n\|^2. \end{aligned} \tag{3.17}$$

Using Lemma 1 (i) and the definition of $y_n (y_n \in C_n)$, we obtain

$$\langle x_{n+1} - y_n, y_n - w_n + \alpha_n F_n(w_n)\rangle \ge 0. \tag{3.18}$$

We see that

$$\begin{aligned}-2\langle x_{n+1} - y_n, y_n - w_n + \alpha_n F_n(y_n)\rangle &\le 2\langle x_{n+1} - y_n, w_n - y_n - \alpha_n F_n(y_n)\rangle \\ &\quad + 2\langle x_{n+1} - y_n, y_n - w_n + \alpha_n F_n(w_n)\rangle \\ &= 2\alpha_n\langle x_{n+1} - y_n, F_n(w_n) - F_n(y_n)\rangle \\ &\le 2\alpha_n\|x_{n+1} - y_n\|\|F_n(w_n) - F_n(y_n)\| \\ &\le \alpha_n^2\|F_n(w_n) - F_n(y_n)\|^2 + \|x_{n+1} - y_n\|^2 \\ &\le \mu^2\|w_n - y_n\|^2 + \|x_{n+1} - y_n\|^2. \end{aligned} \tag{3.19}$$

Combining (3.13), (3.16), (3.17) and (3.19) we have

$$\begin{aligned}\|x_{n+1} - z\|^2 \le{}& \|x_n - z\|^2 + \theta_n(\|x_n - z\|^2 - \|x_{n-1} - z\|^2) + 2\theta_n\|x_n - x_{n-1}\|^2 \\ & - 2\frac{\mu\ell}{L}\|(I - P_{Q_n})Ay_n\|^2 - \|x_{n+1} - y_n\|^2 - \|y_n - w_n\|^2 \\ & + \mu^2\|w_n - y_n\|^2 + \|x_{n+1} - y_n\|^2 \\ ={}& \|x_n - z\|^2 + \theta_n(\|x_n - z\|^2 - \|x_{n-1} - z\|^2) + 2\theta_n\|x_n - x_{n-1}\|^2 \\ & - 2\frac{\mu\ell}{L}\|(I - P_{Q_n})Ay_n\|^2 - (1 - \mu^2)\|y_n - w_n\|^2. \end{aligned} \tag{3.20}$$

Lemma 2 gives $\lim_{n\to\infty}\|x_n - z\|$ exists and $\{x_n\}$ is bounded.

From (3.20), it follows that

$$\lim_{n\to\infty}\|y_n - w_n\| = 0, \tag{3.21}$$

$$\lim_{n\to\infty} \|(I-P_{Q_n})Ay_n\| = 0. \tag{3.22}$$

From (3.8), we have

$$\lim_{n\to\infty} \|w_n - x_n\| = 0. \tag{3.23}$$

Using (3.21) and (3.23), we get

$$\begin{aligned}\|x_n - y_n\| &= \|x_n - w_n + w_n - y_n\| \\ &\le \|x_n - w_n\| + \|w_n - y_n\| \\ &\to 0 \text{ as } n \to \infty. \end{aligned} \tag{3.24}$$

From (3.9), (3.10), (3.11) and (3.24), we obtain

$$\begin{aligned}\|x_{n+1} - x_n\| &\le \|x_{n+1} - y_n\| + \|y_n - x_n\| \\ &= \|P_{C_n}(w_n - \alpha_n F_n(y_n)) - P_{C_n}(w_n - \alpha_n F_n(w_n))\| + \|y_n - x_n\| \\ &\le \alpha_n \|F_n(y_n) - F_n(w_n)\| + \|y_n - x_n\| \\ &= \mu\|y_n - w_n\| + \|y_n - x_n\| \\ &\to 0 \text{ as } n \to \infty. \end{aligned} \tag{3.25}$$

By (3.23) and (3.25), we get

$$\begin{aligned}\|x_{n+1} - w_n\| &\le \|x_{n+1} - x_n\| + \|x_n - w_n\| \\ &\to 0 \text{ as } n \to \infty. \end{aligned} \tag{3.26}$$

Since $\{x_n\}$ is bounded, set $\omega_w(x_n)$ is nonempty. Let $x^* \in \omega_w(x_n)$, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightharpoonup x^*$. Now, we show that x^* is a solution of the SFP, which will show that $\omega_w(x_n) \subseteq S$. Since $x_{n_k+1} \in C_{n_k}$, then by the definition of C_{n_k}, we obtain

$$c(x_{n_k}) + \langle \xi_{n_k}, x_{n_k+1} - x_{n_k}\rangle \le 0 \tag{3.27}$$

where $\xi_{n_k} \in \partial c(x_{n_k})$. Since ∂c is bounded and (3.25), we have

$$\begin{aligned}c(x_{n_k}) &\le \langle \xi_{n_k}, x_{n_k} - x_{n_k+1}\rangle \\ &\le \|\xi_{n_k}\|\|x_{n_k} - x_{n_k+1}\| \\ &\to 0 \text{ as } k \to \infty. \end{aligned} \tag{3.28}$$

From the w-lsc and since $x_{n_k} \rightharpoonup x^*$, it implies that

$$c(x^*) \le \liminf_{k\to\infty} c(x_{n_k}) \le 0, \tag{3.29}$$

hence, $x^* \in C$. Since $P_{Q_{n_k}}(Ay_{n_k}) \in Q_{n_k}$, we have

$$q(Ay_{n_k}) + \langle \eta_{n_k}, P_{Q_{n_k}}(Ay_{n_k}) - Ay_{n_k}\rangle \le 0, \tag{3.30}$$

where $\eta_{n_k} \in \partial q(Ay_{n_k})$. Since ∂q is bounded and (3.22), we obtain

$$\begin{aligned}q(Ay_{n_k}) &\le \langle \eta_{n_k}, Ay_{n_k} - P_{Q_{n_k}}(Ay_{n_k})\rangle \\ &\le \|\eta_{n_k}\|\|Ay_{n_k} - P_{Q_{n_k}}(Ay_{n_k})\| \\ &\to 0 \text{ as } k \to \infty. \end{aligned} \tag{3.31}$$

We note that $x_{n_k} \rightharpoonup x^*$ and $\|x_{n_k} - y_{n_k}\| \to 0$. So $y_{n_k} \rightharpoonup x^*$ and thus $Ay_{n_k} \rightharpoonup Ax^*$ as $k \to \infty$. By the w-lsc, we get

$$q(Ax^*) \le \liminf_{k\to\infty} q(Ay_{n_k}) \le 0. \tag{3.32}$$

So, $Ax^* \in Q$. Using Lemma 3, we conclude that the sequence $\{x_n\}$ converges weakly to a solution of the SFP. This completes the proof. □

4. Numerical Experiments

Example 1. Consider the following LASSO problem [21]:

$$\min\left\{\frac{1}{2}\|Ax-b\|^2 : x \in \mathbb{R}^3, \|x\|_1 \le \tau\right\}, \tag{4.1}$$

where $A = \begin{pmatrix} 3 & 3 & -1 \\ 5 & 4 & 0 \\ 2 & -5 & 1 \end{pmatrix}$, $b = (0,2,0)$. We define $C = \{x \in \mathbb{R}^3 : \|x\|_1 \le \tau\}$ and $Q = \{b\}$. Since the projection onto the closed convex C does not have a closed form solution and so we make use of the subgradient projection. Define a convex function $c(x) = \|x\|_1 - \tau$ and denote the level set C_n by:

$$C_n = \{x \in \mathbb{R}^3 : c(x_n) + \langle \xi_n, x - x_n\rangle \le 0\}, \tag{4.2}$$

where $\xi_n \in \partial c(x_n)$. Then the orthogonal projection onto C_n can be calculated by the following:

$$P_{C_n}(x) = \begin{cases} x, & \text{if } c(x_n) + \langle \xi_n, x - x_n\rangle \le 0, \\ x - \frac{c(x_n) + \langle \xi_n, x - x_n\rangle}{\|\xi_n\|^2}\xi_n, & \text{otherwise.} \end{cases} \tag{4.3}$$

It is worth noting that the subdifferential ∂c at x_n is

$$\partial c(x_n) = \begin{cases} 1, & \text{if } x_n > 0, \\ [-1,1], & \text{if } x_n = 0, \\ -1, & \text{if } x_n < 0. \end{cases} \tag{4.4}$$

The iteration process is stopped when the following criteria satisfied $\|x_{n+1} - x_n\| < 10^{-4}$.

We let $\theta = 0.5$ and $\theta_n = \min\left\{\theta, \frac{1}{n^2\|x_{n+1} - x_n\|^2}\right\}$.

We consider four cases as follows:

Case 1: $x_1 = (-1,2,0)$, $x_0 = (-2,0,-9)$, $\gamma = \frac{1}{\|A\|^2}$, $\ell = 0.4$ and $\mu = 0.8$.

Case 2: $x_1 = (1,-9,4)$, $x_0 = (-5,2,1)$, $\gamma = \frac{1.6}{\|A\|}$, $\ell = 0.9$ and $\mu = 0.9$.

Case 3: $x_1 = (7,9,-4)$, $x_0 = (4,6,-3)$, $\gamma = \frac{2}{\|A\|}$, $\ell = 0.3$ and $\mu = 0.1$.

Case 4: $x_1 = (5,4,0)$, $x_0 = (3,5,-2)$, $\gamma = \frac{3}{\|A\|^2}$, $\ell = 0.2$ and $\mu = 0.5$.

Table 1. Algorithm 3 with different cases

		Algorithm 2	Algorithm 3
Case 1	No. of Iter.	215	187
	cpu (Time)	0.0153	0.0145
Case 2	No. of Iter.	140	122
	cpu (Time)	0.1657	0.1248
Case 3	No. of Iter.	403	362
	cpu (Time)	0.0983	0.0759
Case 4	No. of Iter.	253	208
	cpu (Time)	0.0637	0.0357

The convergence behavior of the error E_n for each Cases is shown in Figure 1-4, respectively.

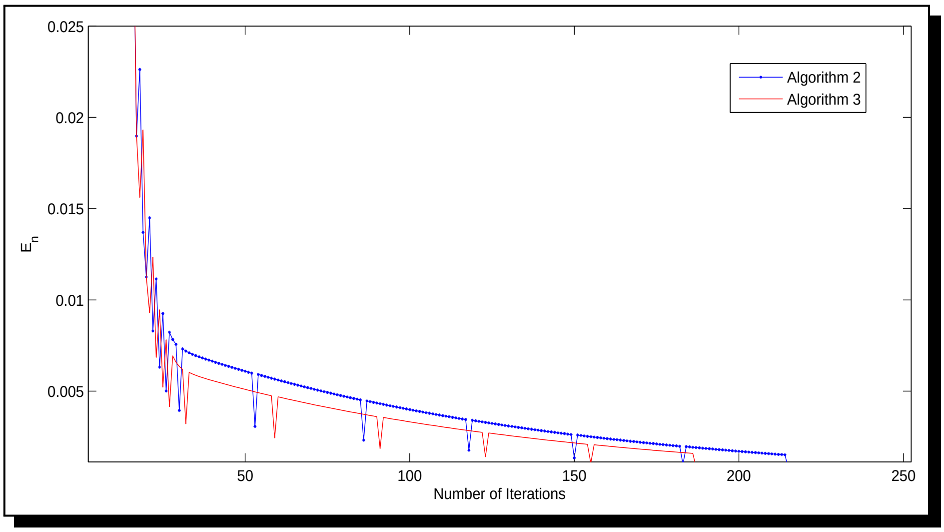

Figure 1. Error plotting E_n for Case 1 in Example 1

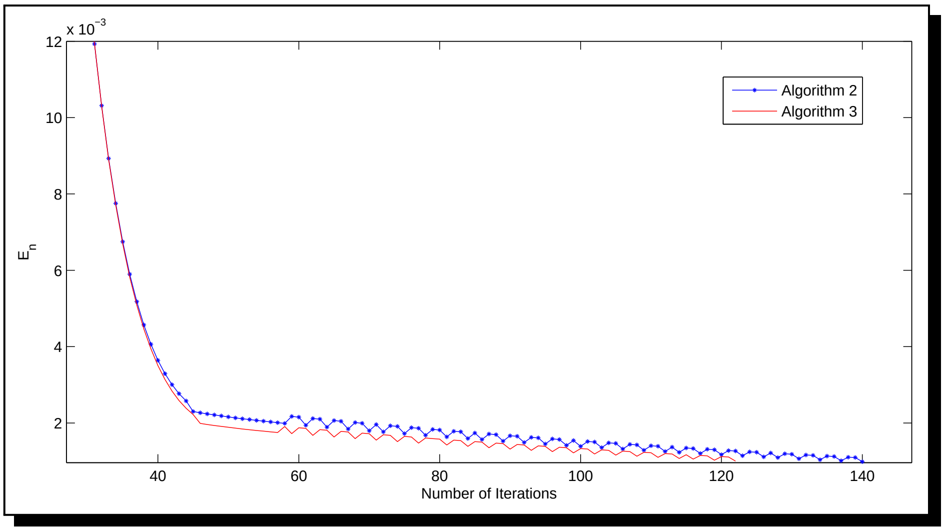

Figure 2. Error plotting E_n for Case 2 in Example 1

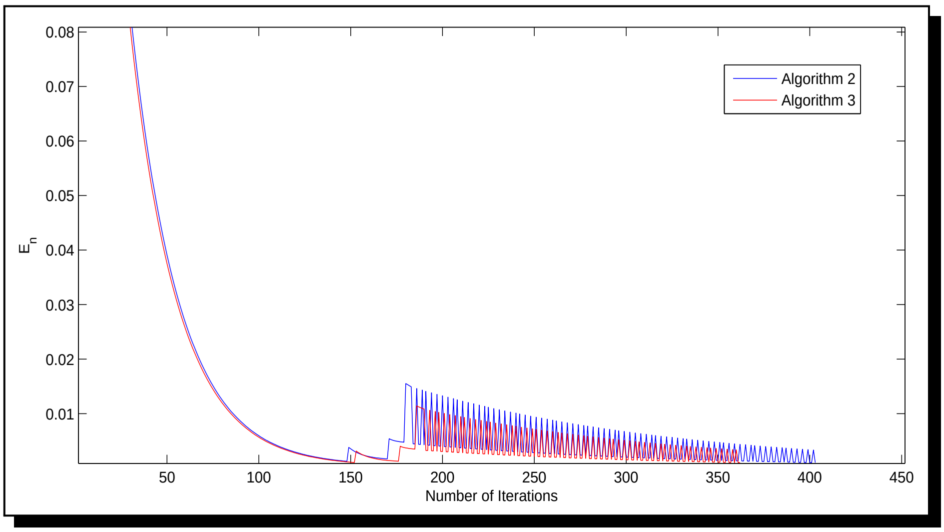

Figure 3. Error plotting E_n for Case 3 in Example 1

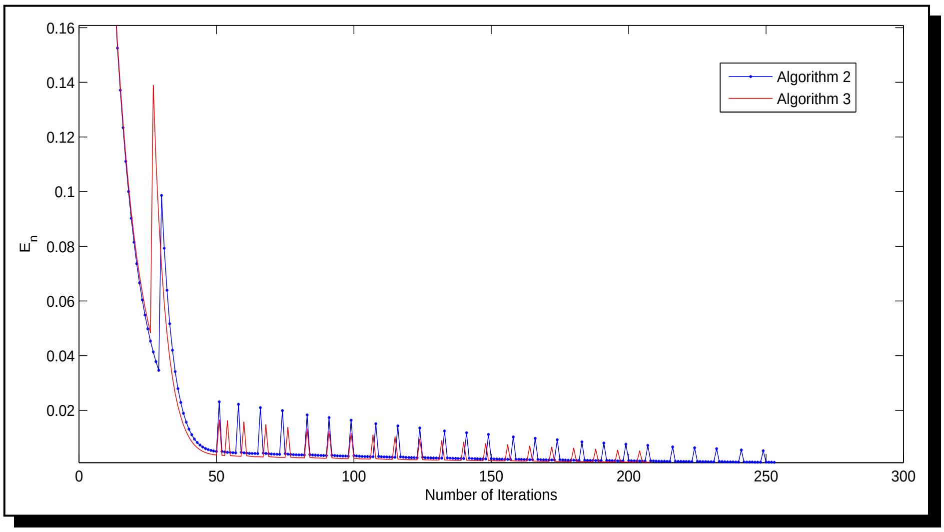

Figure 4. Error plotting E_n for Case 4 in Example 1

5. Conclusions

In this paper, we introduce the inertial relaxation CQ algorithm to solve the split feasibility problem in real Hilbert spaces. We focus on the relaxation CQ algorithm with the Armijo-linesearch for the SFP. The numerical experiments show that our algorithm converges faster than that of Gibali *et al*. [12].

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Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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