Communications in Mathematics and Applications
Vol. 9, No. 4, pp. 705–711, 2018
ISSN 0975-8607 (online); 0976-5905 (print)
Published by RGN Publications

http://www.rgnpublications.com

DOI: 10.26713/cma.v9i4.1044

Research Article

Properties of a Composition of Exponential and Ordinary Generating Functions

Dmitry V. Kruchinin*, Yuriy V. Shablya, Vladimir V. Kruchinin and Alexander A. Shelupanov

Faculty of Security, Tomsk State University of Control Systems and Radioelectronics, Tomsk, Russia
*__Corresponding author:__ kruchinindm@gmail.com

Abstract. In this paper, we consider the composition of ordinary and exponential generating functions. The obtained property of the composition of ordinary and exponential generating functions can be used for distinguishing prime numbers from composite numbers. For example, it can be applied for constructing new probabilistic primality criteria. Using the obtain property, we get several congruence relations for the Uppuluri-Carpenter, Euler, and Fubini numbers.

Keywords. Generating function; Composition; Composita; Primality criterion; Euler number

MSC. 11Y11; 11Y55; 05A15

Received: August 3, 2018 **Accepted:** December 28, 2018

This article was submitted to "*International Conference on Mathematics*" organized by Fatih Sultan Mehmet Vakıf University, Topkapı (Cevizlibağ) Campus, Istanbul, Turkey during July 3-6, 2018.

1. Introduction

Generating functions are a powerful tool for solving problems in number theory, combinatorics, algebra, probability theory, and other fields of mathematics. One of the advantages of generating functions is that an infinite number sequence can be represented in a form of a single expression. Many authors have studied generating functions and their properties and found applications for them (for instance, Comtet [1], Flajolet [2], Graham [3], Robert [12], Stanley [15], and Wilf [17]).

There are different kinds of generating functions, for example, ordinary, exponential, Lambert, Bell, Dirichlet, multivariate and other generating functions. In this paper, we consider ordinary and exponential generating functions.

According to Stanley [16], ordinary generating functions are defined as follows:

Definition 1. An ordinary generating function of the sequence $(a_n)_{n\geq 0}$ is the formal power series

$$A(x)=a_0+a_1x+a_2x^2+\ldots=\sum_{n\geq 0}a_nx^n. \tag{1}$$

According to Flajolet [2], exponential generating functions are defined as follows:

Definition 2. A exponential generating function is an ordinary generating function in the form

$$E(x)=\sum_{n\geq 0}a_nx^n=\sum_{n\geq 0}\frac{e_n}{n!}x^n. \tag{2}$$

Kruchinin [7–9] introduced the mathematical notion of the *composita* of a given generating function, which can be used for calculating the coefficients of a composition of generating functions.

Definition 3. The composita of the generating function $F(x)=\sum\limits_{n>0}f_nx^n$ is the function with two variables

$$F^{\Delta}(n,k)=\sum_{\pi_k\in C_n}f_{\lambda_1}f_{\lambda_2}\cdots f_{\lambda_k},$$

where C_n is the set of all compositions of an integer n, π_k is the composition n into k parts such that $\sum\limits_{i=1}^{k}\lambda_i=n$.

The expression of a composita for an exponential generating function (2) takes the following form:

$$E^{\Delta}(n,k)=\sum_{\pi_k\in C_n}a_{\lambda_1}a_{\lambda_2}\cdots a_{\lambda_k}=\sum_{\pi_k\in C_n}\frac{e_{\lambda_1}e_{\lambda_2}\cdots e_{\lambda_k}}{\lambda_1!\lambda_1!\cdots\lambda_k!}.$$

Using the expression of the composita of a given generating function $F^{\Delta}(n,k)$, we can get powers of the generating function $F(x)$:

$$(F(x))^k=\sum_{n\geq k}F^{\Delta}(n,k)x^n.$$

In this paper, we continue the research given in [10] and study the following composition of generating functions:

$$G(x)=B(E(x))=\sum_{n\geq 0}\frac{g_n}{n!}x^n, \tag{3}$$

where $B(x)=\sum\limits_{n\geq 0}b_nx^n$ is an ordinary generating function with $b_n\in\mathbb{Z}$ and $E(x)=\sum\limits_{n>0}\frac{e_n}{n!}x^n$ is an exponential generating function with $e_n\in\mathbb{Z}$.

2. Main Results

In the paper [10] we proved that for the given composition of exponential generating functions $G(x) = B(E(x)) = \sum\limits_{n\geq 0} \frac{g_n}{n!}x^n$, where $A(x) = \sum\limits_{n\geq 0} \frac{a_n}{n!}x^n$ and $E(x) = \sum\limits_{n>0} \frac{e_n}{n!}x^n$ are exponential generating functions with $a_n, e_n \in \mathbb{Z}$, the value of the expression

$$\frac{g_n - e_n a_1 - e_1^n a_n}{n} \tag{4}$$

is an integer for every prime n.

Example 1. Let us consider the generating function for the Uppuluri-Carpenter [6] or complementary Bell numbers $\tilde{B}_n$ (the integer sequence A000587 from the on-line encyclopedia of integer sequences [13])

$$G(x) = e^{1-e^x} = \frac{1}{0!} + \frac{-1}{1!}x + \frac{0}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \frac{-2}{5!}x^5 + \frac{-9}{6!}x^6 + \ldots = \sum_{n\geq 0} \frac{\tilde{B}_n}{n!}x^n$$

as the composition $G(x) = A(E(x))$ of the exponential generating functions

$$A(x) = e^x = \frac{1}{0!} + \frac{1}{1!}x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \frac{1}{5!}x^5 + \frac{1}{6!}x^6 + \ldots = \sum_{n\geq 0} \frac{1}{n!}x^n$$

and

$$E(x) = 1 - e^x = \frac{-1}{1!}x + \frac{-1}{2!}x^2 + \frac{-1}{3!}x^3 + \frac{-1}{4!}x^4 + \frac{-1}{5!}x^5 + \frac{-1}{6!}x^6 + \ldots = \sum_{n>0} \frac{-1}{n!}x^n.$$

Substituting the values in (4), we get that the value of the expression

$$\frac{g_n - e_n a_1 - e_1^n a_n}{n} = \frac{\tilde{B}_n - (-1)\cdot 1 - (-1)^n \cdot 1}{n} = \frac{\tilde{B}_n + 1 - (-1)^n}{n}$$

is an integer for every prime n.

Hence, we can get the following primality criterion: if n is a prime, then $\tilde{B}_n \equiv (-1)^n - 1 \bmod n$.

Next we consider the case of a composition of generating functions when outer function is an ordinary generating function (1). We obtain a new property for such composition that is described in the following theorem:

Theorem 1. *For the composition of generating functions* $G(x) = B(E(x)) = \sum\limits_{n\geq 0} \frac{g_n}{n!}x^n$, *where* $B(x) = \sum\limits_{n\geq 0} b_n x^n$ *is an ordinary generating function with* $b_n \in \mathbb{Z}$ *and* $E(x) = \sum\limits_{n>0} \frac{e_n}{n!}x^n$ *is an exponential generating function with* $e_n \in \mathbb{Z}$, *the value of the expression*

$$\frac{g_n - e_n b_1}{n} \tag{5}$$

is an integer for every prime n.

Proof. If we consider the composition $A(x) = R(F(x)) = \sum\limits_{n\geq 0} a_n x^n$ of generating functions $R(x) = \sum\limits_{n\geq 0} r_n x^n$ and $F(x) = \sum\limits_{n>0} f_n x^n$, then we can get the values of the coefficients a_n by using the following formula [9]:

$$a_n = \begin{cases} r_0, & \text{for } n = 0; \\ \sum\limits_{k=1}^{n} F^{\Delta}(n,k) r_k, & \text{otherwise.} \end{cases} \tag{6}$$

Applying (6) for the composition of ordinary and exponential generating functions (3), we get

$$\frac{g_n}{n!} = \sum_{k=1}^{n} E^{\Delta}(n,k) b_k$$

or

$$g_n = n! \sum_{k=1}^{n} E^{\Delta}(n,k) b_k. \tag{7}$$

According to [10], if $E(x) = \sum_{n>0} \frac{e_n}{n!} x^n$ is an exponential generating function with integer e_n, then the expression

$$\frac{n!}{k!} E^{\Delta}(n,k)$$

is integer for $k, n \in \mathbb{Z}$ and $k \le n$. Hence, the value of g_n in (7) is an integer for $n \in \mathbb{N}$.
Let us consider the special case for the sum in (7) with $k = 1$:

$$n! E^{\Delta}(n,k) b_k = n! E^{\Delta}(n,1) b_1 = n! b_1 \sum_{\pi_1 \in C_n} \frac{e_{\lambda_1}}{\lambda_1!} = n! b_1 \frac{e_n}{n!} = e_n b_1.$$

Since $e_i \in \mathbb{Z}$ and $b_i \in \mathbb{Z}$, then $e_n b_1$ is integer for $n \in \mathbb{N}$ and the expression

$$g_n - e_n b_1 = n! \sum_{k=2}^{n} E^{\Delta}(n,k) b_k = n! \sum_{k=2}^{n} b_k \sum_{\pi_1 \in C_n} \frac{e_{\lambda_1} e_{\lambda_2} \cdots e_{\lambda_k}}{\lambda_1! \lambda_1! \cdots \lambda_k!} \tag{8}$$

is integer for $n \in \mathbb{Z}$.
If $k > 1$, then $\lambda_i < n$ $(i = \overline{1,k})$. For prime n we get

$$\gcd(n, \lambda_1! \lambda_1! \cdots \lambda_k!) = 1.$$

Therefore, for prime n we can divide (8) by n and the value of the obtain expression

$$\frac{g_n - e_n b_1}{n}$$

will still be an integer. □

3. Application

The obtained property can be used for distinguishing prime numbers from composite numbers [11]. The problem of distinguishing prime numbers from composite numbers is one of the most important and useful in arithmetic. The solution of this problem is interesting both theoretically and practically, for example, in cryptography it can be applied for constructing new probabilistic primality criteria [14, 18].

Corollary 1. *Suppose the composition of generating functions* $G(x) = B(E(x)) = \sum_{n\ge 0} \frac{g_n}{n!} x^n$, *where* $B(x) = \sum_{n\ge 0} b_n x^n$ *is an ordinary generating function with* $b_n \in \mathbb{Z}$ *and* $E(x) = \sum_{n>0} \frac{e_n}{n!} x^n$ *is an exponential generating function with* $e_n \in \mathbb{Z}$. *If* n *is a prime, then*

$$g_n - e_n b_1 \equiv 0 \mod n.$$

Example 2. Let us consider the generating function for the Euler [4] or zigzag numbers A_n (the integer sequence A000111 from the on-line encyclopedia of integer sequences [13])

$$G(x) = \frac{1}{1-\sin x} = \frac{1}{0!} + \frac{1}{1!}x + \frac{2}{2!}x^2 + \frac{5}{3!}x^3 + \frac{16}{4!}x^4 + \frac{61}{5!}x^5 + \frac{272}{6!}x^6 + \ldots = \sum_{n\ge 0} \frac{A_{n+1}}{n!} x^n$$

as the composition $G(x) = B(E(x))$ of the ordinary generating function

$$B(x) = \frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + \ldots = \sum_{n\geq 0} x^n$$

and the exponential generating function

$$E(x) = \sin x = \frac{1}{1!}x + \frac{0}{2!}x^2 + \frac{-1}{3!}x^3 + \frac{0}{4!}x^4 + \frac{1}{5!}x^5 + \frac{0}{6!}x^6 + \ldots = \sum_{n>0} \frac{((-1)^{n-1}+1)(-1)^{\frac{n-1}{2}}}{2n!}x^n.$$

Substituting the values in (5), we get

$$\frac{g_n - e_n b_1}{n} = \frac{A_{n+1} - \frac{((-1)^{n-1}+1)(-1)^{\frac{n-1}{2}}}{2}}{n}.$$

For odd n, we obtain

$$\frac{A_{n+1} - \frac{((-1)^{n-1}+1)(-1)^{\frac{n-1}{2}}}{2}}{n} = \frac{A_{n+1} - (-1)^{\frac{n-1}{2}}}{n}.$$

By Theorem 1, the value of the expression

$$\frac{A_{n+1} - (-1)^{\frac{n-1}{2}}}{n}$$

is an integer for every odd prime n.

Hence, we can get the following primality criterion: if n is an odd prime, then

$$A_{n+1} \equiv (-1)^{\frac{n-1}{2}} \mod n.$$

Example 3. Let us consider the generating function for the Fubini [5] or ordered Bell numbers F_n with $F_0 = 0$ (the integer sequence A000670 from the on-line encyclopedia of integer sequences [13])

$$G(x) = \frac{1}{2-e^x} - 1 = \frac{1}{1!}x + \frac{3}{2!}x^2 + \frac{13}{3!}x^3 + \frac{75}{4!}x^4 + \frac{541}{5!}x^5 + \frac{4683}{6!}x^6 + \ldots = \sum_{n>0} \frac{F_n}{n!}x^n$$

as the composition $G(x) = B(E(x))$ of the ordinary generating function

$$B(x) = \frac{x}{1-x} = x + x^2 + x^3 + x^4 + x^5 + x^6 + \ldots = \sum_{n>0} x^n$$

and the exponential generating function

$$E(x) = e^x - 1 = \frac{1}{1!}x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \frac{1}{5!}x^5 + \frac{1}{6!}x^6 + \ldots = \sum_{n>0} \frac{1}{n!}x^n.$$

Then, we get

$$G(x) = B(E(x)) = \frac{e^x - 1}{1-(e^x-1)} = \frac{e^x-1}{2-e^x} = \frac{1}{2-e^x} - 1.$$

Substituting the values in (5), we get that the value of the expression

$$\frac{g_n - e_n b_1}{n} = \frac{F_n - 1\cdot 1}{n} = \frac{F_n - 1}{n}$$

is an integer for every prime n.

Hence, we can get the following primality criterion: if n is a prime, then

$$F_n \equiv 1 \mod n.$$

4. Conclusion

Using the notion of a composita, we get a new property for a composition of ordinary and exponential generating functions. The obtained property of the composition of ordinary and exponential generating functions can be used for distinguishing prime numbers from composite numbers. On the basis of the obtain property, we have got several congruence relations for the Uppuluri-Carpenter, Euler, and Fubini numbers.

Acknowledgement

This work was funded by Russian Foundation for Basic Research and the government of the Tomsk region of Russian Federation (grants no. 18-41-703006 and no. 18-31-00201) and the Ministry of Education and Science of Russian Federation (government order no. 2.8172.2017/8.9, TUSUR).

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

References

[1] L. Comtet, *Advanced Combinatorics*, D. Reidel Publishing Company (1974).

[2] P. Flajolet and R. Sedgewick, *Analytic Combinatorics*, Cambridge University Press (2009).

[3] R.L. Graham, D.E. Knuth and O. Patashnik, *Concrete Mathematics*, Addison-Wesley (1989).

[4] K.-W. Hwang, D.V. Dolgy, D.S. Kim, T. Kim and S.H. Lee, Some theorems on Bernoulli and Euler numbers, *Ars Combin.* **109** (2013), 285 – 297.

[5] N. Kilar and Y. Simsek, A new family of Fubini type numbers and polynomials associated with Apostol-Bernoulli numbers and polynomials, *J. Korean Math. Soc.* **54**(5) (2017), 1605 – 1621.

[6] M. Klazar, Bell numbers, their relatives, and algebraic differential equations, *J. Combin. Theory Ser. A* **102**(1) (2003), 63 – 87.

[7] D.V. Kruchinin and V.V. Kruchinin, A method for obtaining generating functions for central coefficients of triangles, *J. Integer Seq.* **15** (2012), 1 – 10.

[8] D.V. Kruchinin and V.V. Kruchinin, Application of a composition of generating functions for obtaining explicit formulas of polynomials, *J. Math. Anal. Appl.* **404**(1) (2013), 161 – 171.

[9] V.V. Kruchinin and D.V. Kruchinin, Composita and its properties, *Journal of Analysis and Number Theory* **2** (2014), 37 – 44.

[10] D.V. Kruchinin, Y.V. Shablya, O.O. Evsutin and A.A. Shelupanov, Integer properties of a composition of exponential generating functions, in *Proc. Intl. Conf. of Numerical Analysis and Applied Mathematics*, eds. T. Simos and C. Tsitouras, American Institute of Physics Inc., 2017, pp. 1–4.

[11] P. Ribenboim, *The Little Book of Bigger Primes*, Springer-Verlag (2004).

[12] A.M. Robert, *A Course in p-adic Analysis*, Springer (2000).

[13] N.J.A. Sloane, The on-line encyclopedia of integer sequences, URL: `http://oeis.org`.

[14] N.P. Smart, *Cryptography Made Simple*, Springer International Publishing (2016).

[15] R.P. Stanley, Generating functions, in *Studies in Combinatorics*, ed. G.-C. Rota, Mathematical Association of America, 1978, pp. 100 – 141.

[16] R.P. Stanley, *Enumerative Combinatorics*, Vol. 2, Cambridge University Press (1999).

[17] H.S. Wilf, *Generating Functionology*, Academic Press, New York (1994).

[18] S.Y. Yan, *Primality Testing and Integer Factorization in Public-key Cryptography*, Springer (2009).