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Research Article

An Optimal Harvesting of a Discrete Time System With Ricker Growth

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Abstract. In this study, the dynamic behavior of a sporadic model (prey-predator) is investigated with Ricker function growth in prey species. We also found the system has four fixed points. We set the conditions that required to achieve local stability of all fixed points. The rate of harvest in the case of being a fixed quantity in the community and the existence of the bionomic equilibrium in the absence of predator are discussed, then the system is extended to an optimal harvesting policy. The Pontryagin's maximum principle is used to solve the optimality problem. Numerical simulations have been applied to enhance the results of mathematical analysis of the system.

Keywords. Ricker function; Local stability; Discrete system; Optimal harvesting

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1. Introduction

Mathematical modeling and computational simulations have a very useful tools to understand the biology society. In the real world species do not exist a lone so that the interaction, mutualism and competitive mechanisms are formed by using a set of differential equations. They have been investigated extensively in the recent years by researchers [4–6]. After the earliest work of Lotka-Volterra prey-predator model, many authors have been given a modification of the system using nonlinear difference equations or partial differential equations [3, 8, 9]. It is very interesting to investigate the behavior of solutions of a system of nonlinear difference equation and to discuss the local asymptotic stability of their equilibrium points. In [9] the author has reviewed a class of models describing populations experiencing which is helpful in describing dramatic fluctuations in abundance. In [10, 11, 14], authors have been carried out studying the chaotic dynamics that occurs in multi-species in continues time as well as discrete time prey-predator models. Optimal control theory has a long history of being applied to problems in exploitation of renewable resources and it gives deep insight in many problem biology for more details see [2, 7, 12, 13]. This paper is organized as follows: In Section 2 the stability analysis of all fixed points is discussed. In Section 3 a constant harvesting in case of absence of predator is considered. In Section 4, the model is extended to an optimal control problem. The extension version of Pontryagin's maximum principle is used to find the optimal solution with corresponding state solution. In Section 5, numerical simulations have been applied to enhance the results of mathematical analysis of the system.

We will investigate a nonlinear discrete time prey-predator system which is modeled through the following two dimensional system

$$\left.\begin{aligned} x_{t+1} &= r x_t e^{-a x_t} - b x_t y_t \\ y_{t+1} &= c y_t + d x_t y_t \end{aligned}\right\} \tag{1.1}$$

where x_t, y_t are total population density of the prey and the predator, respectively $x_t \geq 0$, $y_t \geq 0$ for all $t = 1, 2, \cdots$. The r, b, c, d are all positive real parameters.

In the absence of predator the prey population density grows by Ricker function, which is well known and widely used in marine fish as well as it is used to describe the growth of other animals other biological species.

2. The Stability Analysis of the Fixed Points

In this section the existence of fixed points of system (1.1) are determined and then their local stability at each fixed points are studied. For determining the fixed points,one has to solve the following nonlinear algebraic system.

$$\left.\begin{aligned} x &= r x e^{-a x} - b x y \\ y &= c y + d x y \end{aligned}\right\}$$

By simple computations we get the following lemma.

Lemma 2.1. *For all r, b, c and d the system (1.1) has four fixed points namely E_1, E_2, E_3 and E_4. They are given by:*

(1) *The extinction point, $E_1 = (0, 0)$ is always exist.*

(2) *The first boundary fixed point, $E_2 = (0, 1)$ is exist if $c = 1$.*

(3) *The second boundary fixed point, $E_3 = \left(\frac{\ln r}{a}, 0\right)$ is always exist when $r > 1$.*

(4) *The unique positive fixed point, $E_4 = (x^*, y^*) = \left(\frac{1-c}{d}, \frac{r e^{-a x^*} - 1}{b}\right)$ is exist if $0 < x^* < \frac{\ln r}{a}$ and $r > 1$.*

To study the local stability of the model we have to compute the Jacobian matrix of system (1.1). This is given by

$$J(x, y) = \begin{pmatrix} r e^{-a x}(1 - a x) - b y & -b x \\ d y & c + d x \end{pmatrix}.$$

So that the characteristic equation of Jacobian matrix is then

$$F(\lambda) = \lambda^2 + p\lambda + q\,, \tag{2.1}$$

where $p = -\operatorname{trac}(J)$ *and* $q = \det(J)$.

2.1 Stability Analysis of the E_1,E_2 and E_3

For the extinction fixed point $E_1 = (0,0)$, the Jacobian matrix is given by

$$J_{E_1} = \begin{pmatrix} r & 0 \\ 0 & c \end{pmatrix}.$$

where the eigenvalues are $\lambda_1 = r$ and $\lambda_2 = c$, therefore, we have the following lemma.

Lemma 2.2. *For the fixed point E_1, we have*

(1) *E_1 is sink if and only if $r < 1$ and $c < 1$.*

(2) *E_1 is source if and only if $r > 1$ and $c > 1$.*

(3) *E_1 is saddle point either $r > 1$ with $c < 1$ or $r < 1$ with $c > 1$.*

(4) *E_1 is non-hyperbolic if $r = 1$ or $c = 1$.*

The proof is clear, hence it is omitted.

For the first boundary fixed point $E_2 = (0,1)$, the Jacobian matrix is $J_{E_2} = \begin{bmatrix} r-b & 0 \\ d & c \end{bmatrix}$, then the eigenvalues are $\lambda_1 = r - b$ and $\lambda_2 = c = 1$.

Therefore, the E_2 is always non-hyperbolic point and for all parameters it is never to be sink or source or saddle point.

Now in order of discuss the behavior of the second boundary fixed point $E_3 = \left(\frac{\ln r}{a}, 0\right)$, we are also have to compute the Jacobian matrix at E_3, this is given by:

$$J_{E_1} = \begin{pmatrix} 1-\ln r & -\frac{b}{a}\ln r \\ 0 & c+\frac{d}{a}\ln r \end{pmatrix}.$$

Therefore, the eigenvalues are $\lambda_1 = 1 - \ln r$ and $\lambda_2 = c + \frac{d}{a}\ln r$, we have the following lemma.

Lemma 2.3. *For the fixed point E_3, we have*

(1) *E_3 is sink if $\ln r \in I_1 \cap I_2$, where $I_1 = (0,2)$ and $I_2 = \left(-\frac{a}{d}(c+1), \frac{a}{d}(1-c)\right)$.*

(2) *E_3 is source if $\ln r \in I_3$, where $I_3 = \left(\max\left\{2, \frac{a}{d}(1-c)\right\}, \infty\right)$.*

(3) *E_3 is saddle point if $\ln r \in I_4$, where $I_4 = \left(\min\left\{2, \frac{a}{d}(1-c)\right\}, \max\left\{2, \frac{a}{d}(1-c)\right\}\right)$.*

(4) *E_3 is non-hyperbolic point if one of the following hold:*

 (i) $r = e^2$.

 (ii) $r = e^{-\frac{a}{d}(c+1)}$.

 (iii) $r = e^{\frac{a}{d}(1-c)}$.

Proof. (1) It is clear that $|\lambda_1| < 1$ if and only if $-1 < 1 - \ln r < 1 \Leftrightarrow -2 < -\ln r < 0$.
$\Leftrightarrow 0 < \ln r < 2$ therefore $|\lambda_1| < 1$ if and only if $\ln r \in I_1$.

Now, $|\lambda_2| < 1$ if and only if $-1 < c + \frac{d}{a}\ln r < 1 \Leftrightarrow \frac{a}{d}(-c-1) < \ln r < \frac{a}{d}(1-c)$ hence $|\lambda_2| < 1$ if and only if $\ln r \in I_2$ and the fixed pint E_3 is sink.

(2) It follows from (1.1) if $\ln r \in I_3$ or $\ln r \in I_4$ the fixed point E_3 is source.

(3) It is clear from (1.1) and (2.1) the fixed point E_3 is saddle point.

(4) If one the condition is satisfied then either $|\lambda_1| = 1$ or $|\lambda_2| = 1$, hence the results can be easily obtained. □

2.2 Stability Analysis of the Positive Fixed Point

To study the behavior stability of the unique positive fixed point E_4, we need the following lemma which is appeared in [15].

Lemma 2.4. *Let $F(\lambda) = \lambda^2 + p\lambda + q$ suppose that $F(1) > 0$, and λ_1, λ_2 are the roots of F then:*

(1) $|\lambda_1| < 1$ *and* $|\lambda_2| < 1$ *if and only if* $F(-1) > 0$ *and* $q < 1$.

(2) $|\lambda_1| > 1$ *and* $|\lambda_2| > 1$ *if and only if* $F(-1) > 0$ *and* $q > 1$.

(3) $|\lambda_1| > 1$ *and* $|\lambda_2| < 1$ *(or* $|\lambda_1| < 1$ *and* $|\lambda_2| > 1$*), if and only if* $F(-1) < 0$. $\lambda_1 = -1$ *and* $|\lambda_2| \neq 1$ *if and only if* $F(-1) = 0$ *and* $P \neq 0, 2$.

The Jacobain's matrix at E_4 *is:*

$$J_{E_4} = \begin{pmatrix} re^{-ax^*} - arx^*e^{-ax^*} - by^* & -bx^* \\ dy^* & 1 \end{pmatrix}$$

by simple computations the p *and* q *in equation* (2.1) *are*

$$p = arx^*e^{-ax^*} - 2 \quad \text{and} \quad q = 1 - arx^*e^{-ax^*} + rdx^*e^{-ax^*} - dx^*.$$

We have the following lemma which gives the behavior local stability of the positive fixed point of the system (1.1).

Lemma 2.5. (1) *The positive fixed point* E_4 *is sink if this condition holds:*

$r \in \left(e^{ax^*}, \min\left\{\frac{(dx^*-4)e^{ax^*}}{(dx^*-2ax^*)}, \frac{de^{ax^*}}{d-a}\right\}\right)$ *and* $a < d < 2a$ *with* $c < 1$.

(2) *The positive fixed point* E_4 *is source if*

(i) $\frac{a(5-c)}{4} < d < 2a$

(ii) $r \in \left(\frac{de^{ax^*}}{d-a}, \frac{(dx^*-4)e^{ax^*}}{(dx^*-2ax^*)}\right)$

(3) *The positive fixed point* E_4 *is saddle point if* $r \in \left(\frac{(dx^*-4)e^{ax^*}}{(dx^*-2ax^*)}, \infty\right)$ *and* $a < d < 2a$ *with* $c < 1$

(4) *The positive fixed point* E_4 *is non-hyperbolic point if*

(i) $r = \frac{(dx^*-4)e^{ax^*}}{(dx^*-2ax^*)}$

(ii) $r \neq \frac{2e^{ax^*}}{ax^*}$ *or* $r \neq \frac{4e^{ax^*}}{ax^*}$

Proof. (1) If $r > e^{ax^*}$ then one can easy get $F(1) > 0$.

Now if $r < \min\left\{\frac{(dx^*-4)e^{ax^*}}{(dx^*-2ax^*)}, \frac{de^{ax^*}}{d-a}\right\}$ with $a < d < 2a$, and $c < 1$ we get $F(-1) > 0$ and $q < 1$, hence by Lemma 2.4, the positive fixed point is sink.

(2) Suppose that $\frac{a(5-c)}{4} < d < 2a$ with $c < 1$. Let $r > \frac{de^{ax^*}}{d-a} \Leftrightarrow r > \frac{dxe^{ax^*}}{(dx^*-ax^*)}$ since $\frac{a(5-c)}{4} < d$ this gives $d > a$ and $r(dx^* - ax^*) > dxe^{ax^*}$, therefore, $r(dx^* - ax^*)e^{-ax^*} > dx^*$ and $-arx^*e^{-ax^*} + rdx^*e^{-ax^*} - dx^* > 0$ hence $q > 1$.

Now it is clear if $r < \frac{(dx^*-4)e^{ax^*}}{(dx^*-2ax^*)}$ then $F(-1) > 0$ so that the fixed point is source.

(3) According to (3.1) in Lemma 2.4, it is enough to show that $F(-1) < 0$. Let $r \in \left(\frac{(dx^*-4)e^{ax^*}}{(dx^*-2ax^*)}, \infty\right)$ if and only if $r(dx^* - 2ax^*)e^{-ax^*} < (dx^* - 4)$, Therefore, $F(-1) < 0$. It is clear that if $a < d$ then $d > \frac{a(1-c)}{2}$, with $d < 2a$ one can easy to check that $\frac{dx^*-4}{dx^*-2ax^*} > 1$. Therefore, $r > e^{ax^*}$ and $F(1) > 0$, then the fixed point $E_4$is saddle point.

(4) If $r = \frac{(dx^*-4)e^{ax^*}}{(dx^*-2ax^*)}$ then $F(-1) = 0$ when $r \neq \frac{2e^{ax^*}}{ax^*}$ and $r \neq \frac{4e^{ax^*}}{ax^*}$ therefore E_4 is non-hyperbolic point. □

3. The Constant Harvesting

In this section, harvesting or fishing on a single population will be presented, that means the removal of a constant number of population during each time period t. People needs to know how killing a certain number of animals will affect the population at size. The system (1.1) in which the absent of predator including the harvesting will be as following.

$$x_{t+1} = rx_t e^{-ax_t} - Ex_t = f(x_t) \tag{3.1}$$

All the parameters are defined as same as earlier. The E stands for the harvesting effort. To discuss the equilibria analysis of system (3.1) one can easy to cheek that the system (3.1) has two fixed points $e_0 = 0$ which is always exist, and the unique positive fixed point $e_1 = x = \frac{1}{a}\ln\left(\frac{r}{1+qE}\right)$ is exist when $r > 1 + qE$. The next lemma gives the behavior stability of e_0 and e_1.

Lemma 3.1. (1) *For the fixed point e_0 we have*

(i) *e_0 is sink if $qE - 1 < r < qE + 1$*

(ii) *e_0 is source if $r > 1 + qE$*

(iii) *e_0 is nonhyperbolic fixed point if either $r = qE + 1$ or $r = qE - 1$*

(2) *For the fixed point e_1 we have*

(i) *e_1 is sink if $r < (1 + qE)e^{\frac{2}{a(1+qE)}}$*

(ii) *e_1 is source if $r > (1 + qE)e^{\frac{2}{a(1+qE)}}$*

(iii) *e_1 is nonhyperbolic fixed point if $r = (1 + qE)e^{\frac{2}{a(1+qE)}}$*

Proof. (1): Let $f(x_k) = rx_k e^{-ax_k} - qEx_k$ then $f'(x_k) = re^{-ax_k} - rax_k e^{-ax_k} - qE$. Therefore, $|f'(e_0)| < 1 \Leftrightarrow qE - 1 < r < qE + 1$ and the results can be obtained.

(2) It clear that $|f'(e_1)| < 1 \Leftrightarrow \left|1 - a\ln\left(\frac{r}{1+qE}\right)(1+qE)\right| < 1$ this gives $|f'(e_1)| < 1$ $\Leftrightarrow r < (1 + qE)e^{\frac{2}{a(1+qE)}}$. The proof (i),(ii) and (iii) can be get. □

4. The Optimal Harvesting

In this section we extend the system (1.1) to an optimal control problem. The aim of the problem is to ensure the survival of the population with a sustainable development, and to get an optimal net revenue which is given by

$$N(h_t) = c_1 h_t x_t - c_2 {h_t}^2 \quad \text{for all } t = 1,2,\dots$$

where $h(t)$ stands for the harvesting effort, which represents the control variable. c_1 is the catchability coefficient which is positive constant and the c_2 is the cost parameter of the harvesting effort. Therefore, the goal of the control problem is to maximize the objective functional

$$J(h_t) = \sum_{t=1}^{T-1} N(h_t)$$

subject to the state equations

$$\left.\begin{aligned} x_{t+1} &= r x_t e^{-a x_t} - b x_t y_t - h_t x_t \\ y_{t+1} &= c y_t + d x_t y_t \end{aligned}\right\} \tag{4.1}$$

with control constraints $0 \le h_t \le h_{\max} < 1$, for solving the problem we use the Pontryagin's maximum principle (for more details see [1,7,12]). The adjoints variables λ_1 and λ_1 are exist as well as the Hamiltonian function is given as follows:

$$H_t = c_1 h_t x_t - c_2 h^2 + \lambda_{1,t+1}\left(r x_t e^{-a x_t} - b x_t y_t - h_t x_t\right) + \lambda_{2,t+1}(c y_t + d x_t y_t).$$

According to the Pontryngin's maximum principle we have the necessary conditions. They are given by

$$\lambda_{1,t} = \frac{\partial H}{\partial x_t} = c_1 h_t + \lambda_{1,t+1}\left(r e^{-a x_t} - a r x_t e^{-a x_t} - b y_t - h_t\right) + \lambda_{2,t+1}(d y_t),$$

$$\lambda_{2,t} = \frac{\partial H}{\partial y_t} = \lambda_{1,t+1}(-b x_t) + \lambda_{2,t+1}(c + d x_t)$$

and the optimality condition which is

$$\frac{\partial H}{\partial h_t} = c_1 x_t - 2 c_2 h_t - \lambda_{1,t+1} x_t = 0.$$

Then the characterization of the optimal harvesting policy is

$$h_t^* = \begin{cases} 0 & \text{if } \frac{(c_1 - \lambda_{1,t+1}) x_t}{2 c_2} \le 0 \\ \frac{(c_1 - \lambda_{1,t+1}) x_t}{2 c_2} & \text{if } 0 < \frac{(c_1 - \lambda_{1,t+1}) x_t}{2 c_2} < h_{\max} \\ h_{\max} & \text{if } h_{\max} < \frac{(c_1 - \lambda_{1,t+1}) x_t}{2 c_2} \end{cases}$$

The optimal solution h^* at time t will be determined numerically by maximizing the Hamiltonian function at that t.

5. Numerical Simulations

In order to confirm the theoretical analysis of the system (1.1) we give numerical simulations at different set of parameters which shows that the local stability of the fixed points. For the fixed point E_1 we choose the values of parameters as follows: $r = 0.7$, $a = 0.7$, $c = 0.5$, $d = 1.1$ and $b = 0.5$, with the initial condition $(0.5, 0.4)$. According to the condition in Lemma 2.2

Figure 1 shows that the fixed point $E_1 = (0,0)$ is locally stable. For the fixed point E_3 we choose the set of values of parameters $r = 1.1$, $a = 0.5$, $c = 0.5$, $d = 1$ and $b = 0.5$, with the initial condition (0.6,0.6) then $E_2 = (0.1906,0)$ and $I_2 = (-0.75,0.25)$ and the condition1 in Lemma 2.3 is satisfied.

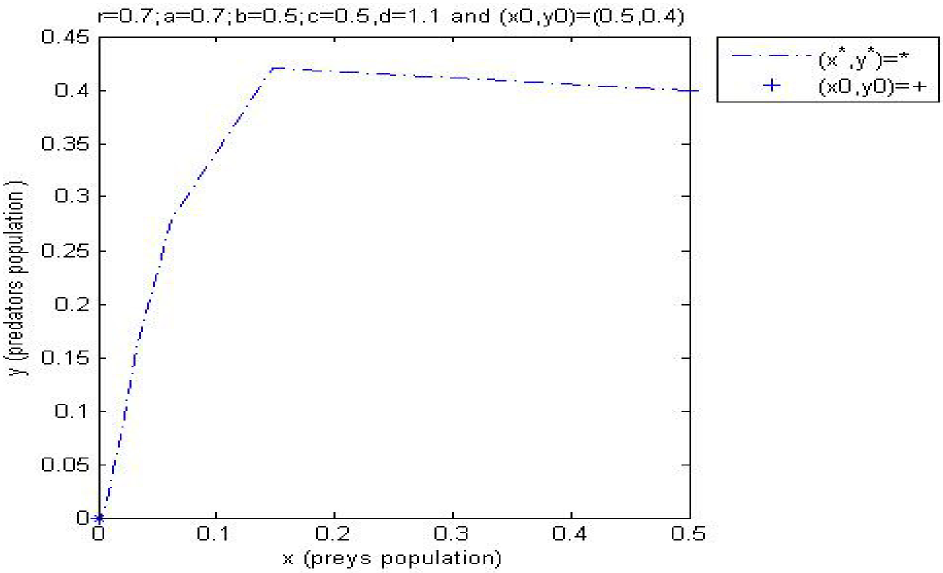

Figure 1. This figure shows the local stability of the fixed point fixed point $E_1 = (0,0)$.

Figure 2 illustrates the local stability of E_3. For the positive fixed point we choose $r = 1.6$, $a = 0.7$, $c = 0.5$, $d = 1.1$ and $b = 0.5$, with these values of parameters the condition (1) in Lemma 2.5 is satisfied and $E_4 = (0.4545, 0.3279)$ is locally stable.

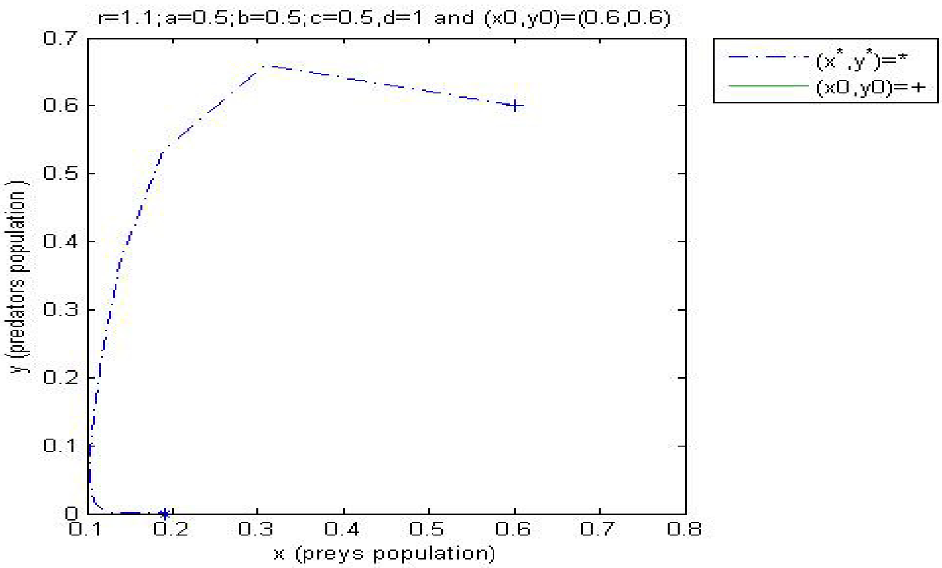

Figure 2. This figure shows the local stability of the fixed point E_3

Figure 3 shows the local stability of E_4. Other set of values of parameters may be given. In order to solve the optimal harvesting resources we follow the procedure which is found in [7].

For that we choose the set of values of parameters as follows: $r = 1.6$, $a = 0.7$, $c = 0.5$, $d = 1.1$, $b = 0.5$, $c_1 = 0.01$, $c_2 = 0.001$, and $T = 80$, we get the total optimal harvesting $J = 0.048$.

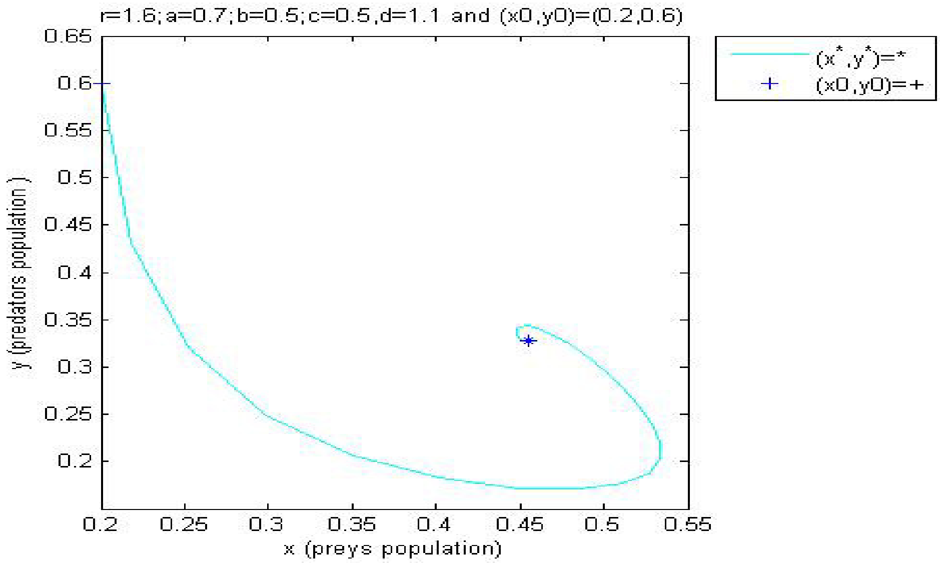

Figure 3. This figure shows the local stability of the unique positive fixed point E_4

In Figure 4 the prey population is plotted according to the system (4.1). The dotted line shows the prey species without harvesting i.e. $h_t = 0$ for all t, while the solid line represents typical harvesting profiles. One can see that all solutions appear with this model are of this three type, the first phase is a time of recovering to the population from low levels. This phase depends on the initial value of population, then the harvesting at optimal rate, and the final phase, the unrestricted harvesting sets in.

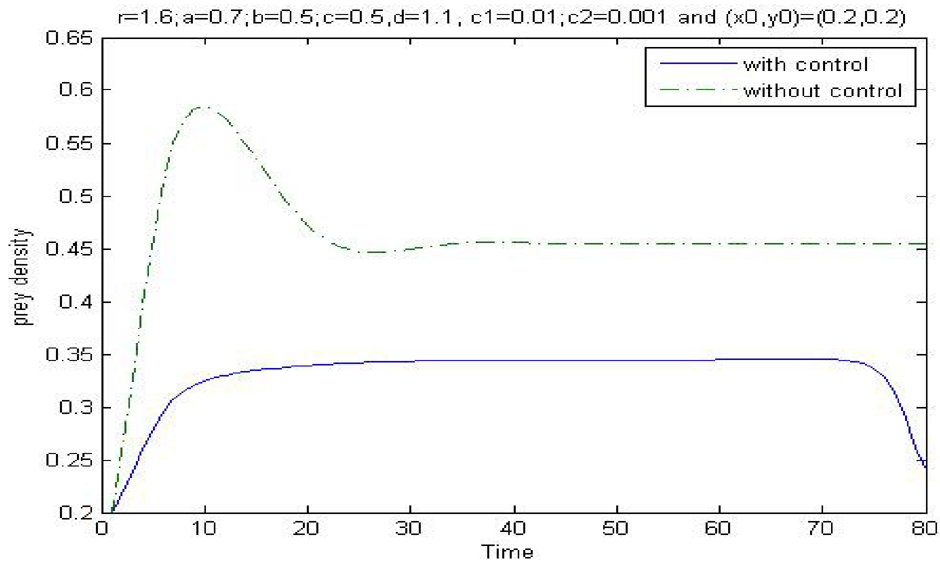

Figure 4. This shows illustrates the prey density population with and without harvesting

Figure 5 shows the effect of the harvesting on the predator species according to the system (4.1).

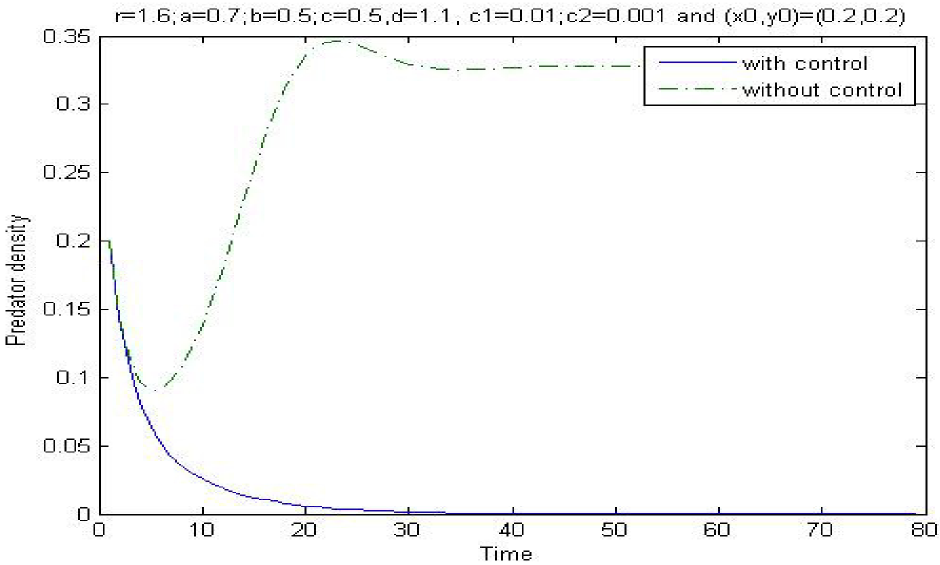

Figure 5. This shows illustrates the affect of harvesting on the predator population with and without harvesting

Figure 6 illustrates the optimal harvesting strategy as a function of time.

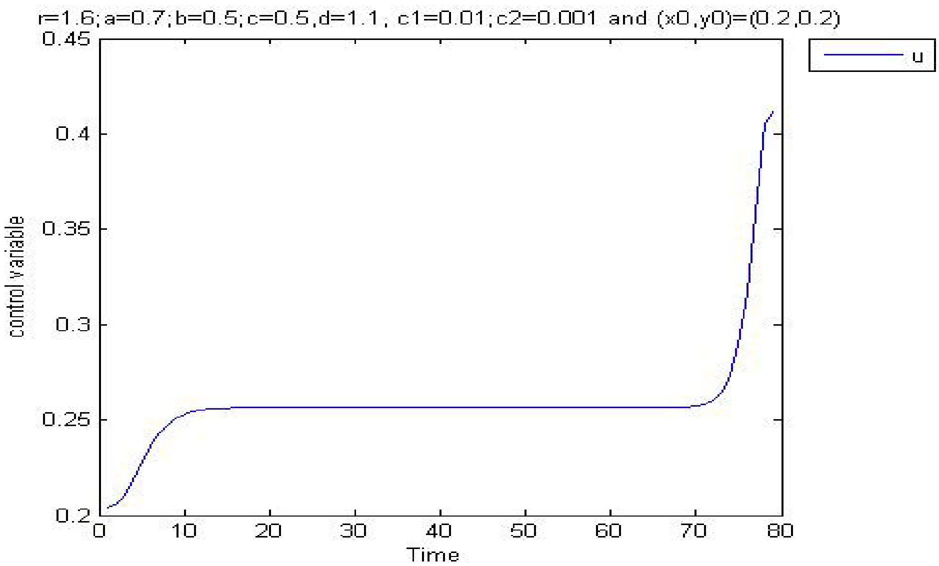

Figure 6. The optimal harvesting is plotted as a function of time

The Table 1 compares the optimal harvesting results with other harvesting strategies by using the same values of parameters and the same initial condition.

Table 1. The results of optimal harvesting with other strategies. All values of parameters are the same

The control variable	Total net harvesting (J)
$h_t = h^*$	$J = 0.0476$
$h_t = 0.27$	$J = 0.0463$
$h_t = 0.25$	$J = 0.0471$
$h_t = 0.22$	$J = 0.0470$
$h_t = 0.4$	$J = 0.0275$

6. Conclusions

In this paper a discrete time prey-predator model with Ricker function growth has been investigated. The model has four fixed points. The trivial fixed point is always exist and the other fixed points are exist for some values of parameters. Moreover, we give and derive the conditions for the local stability of all fixed points. An optimal harvesting policy is applied to the model. The Ponryagins maximum principle is used to determine the optimal strategy. Numerical analysis indicates and confirms the analytic results for various parameters.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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