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Research Article

# The Dynamics and Analysis of Stage-Structured Predator-Prey Model With Prey Refuge and Harvesting Involving Disease in Prey Population

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**Abstract.** In this paper, a mathematical model consisting of the prey-predator model with SI infectious disease in prey is proposed and analyzed. The model includes harvesting on the infected prey population, it is assume that the disease is not transmitted from prey to predator. In addition, the disease spread by contact between susceptible individuals and infected individuals, the mature predator only can predate the susceptible and infected prey which are outside refuge according to Lotka-Volterra type of functional response. While, the immature predator depends completely in it's feeding on the mature predator. The existence, uniqueness and boundedness of the solution are discussed. The stability analysis of all possible equilibrium points is studied. Also, Lyapunove function is used to study the global dynamics of the model. Further, the effect of the disease, refuge and harvest on the dynamical of the system is discussed using numerical simulation.

**Keywords.** Eco-epidemiological model; SI epidemic disease; Prey-predator model; Refuge; Stage structure; Harvest; Lyapunove function

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## 1. Introduction

The explanation of stage structure of people in the period life history of an alluring of the population dynamics, since in real world, there are many species whose individual members exhibit excessive diversity. Stage structure models have received much interest in recent years.

There has been case amount of previous work on modeling with various ages of life history using continuous and discrete models for example [3, 10, 24].

The relation between densities of a predator and its prey is an essential topic in ecology. The mixture of disease into this predator-prey relationship is a fundamental factor to be investigated in the relatively new field of eco-epidemiology, which comprises aspects of both epidemiology and ecology, to study of how diseases communicate [6].

Many studies have examined how diseases grow with variations of the classic SIR model (including populations of susceptible, infected and recovered or resistant individuals) first considered by Kermack and Mckendrick (1927). Some of these studies have been dedicated to the interactions between prey, predator, and parasite or pathogen, such as the models proposed by Roy and Holt [18] and Xiao and Chen [23].

So in the last years, mathematical models have become extremely fundamental tools in understanding and analyzing the communicate and control of infectious disease. Through the study the various types from disease for example SI, SIS, SIR, SEIR, and SEIRS. Whereas many diseases are transmitted in the species not only through contact, but also directly from environment, Majeed and Shawka [14] studied prey-predator model involving SI and SIS infectious disease in prey population and the disease transmitted within the same species by contact and external source. In addition to Khalaf, Majeed and Naji [13] proposed and analyzed a prey-predator model involving SIS infectious disease in prey population this disease passed from a prey to predator through attacking of predator to prey and the disease transmitted within the same species by external source and contact, while Naji and Mustafa [16] proposed a prey-predator model involving SI infectious disease in prey and the disease transmitted within the same species by connection.

On the other hand, the harvest rate has a strong effect on the dynamic advancement of the population perhaps one of the most important hunting the fish or disease removal, Bhattacharyya and Mukhopadhyay [8] studied prey-predator model with harvest and disease, and he assumed that the harvest can disease eradication, also Bairagi et al. [7] studied prey-predator model with harvest and disease, and he assumed that the harvest can remove a parasite.

Some studies that address the population contain the harvest, Brauer and Soudack [9] studied a predator-prey model under constant rate of harvesting. On other hand, there are many studies includes disease and proportional harvesting, Abdul Satar [19] studied a prey-predator model with disease SIS-type and harvesting on prey and predator, while Wuhaih [22] and Agnihotri [1] proposed a prey-predator model with disease SIS-type, SI-type and harvesting in prey only. In addition, many researchers have considered a predator-prey systems with nonlinear harvesting functions [11, 15], while some of the studies using time delay with harvest were considered by Aiello and Freedman [2], and Rosen [17].

In spite of the influence of prey refuges on the dynamical behavior of the system is very complex in the reality, but it has been considered and analysis in this paper. Many authors have considered the dynamic behaviors of prey-predator model with prey refuge for example [4, 5] .

Recently, Sujatha and Gunasekarana [21] had proposed and analyzed a non-linear mathematical model to study the dynamics of a disease transmission among the prey population, the model includes the harvesting of infected prey. While, Shashi and Vivek [20] proposed a stage structured eco-epidemiological model with linear functional response, in this model the stages for prey and predator have been considered. Infection occurs in the prey population only.

In this paper, an eco-epidemiological mathematical model consisting of prey-predator model involving SI infectious disease in prey and stage structured predator species with harvest in infectious population has been proposed and analyzed. Further, in this model, linear types of functional responses for the predation of susceptible and infected prey which are outside refuge as well as linear incidence rate for describing the transition of disease are used. Our aim is to study the role of harvesting on the dynamics of disease propagation and/or eradication.

## 2. Mathematical Model

In this section, an eco-epidemiological model is proposed for study. The model consists of a prey, whose total population density at time $T$ is denoted by $N(T)$, interacting with stag-structured predator. It is assumed that the prey population is infected by infectious disease. Now, the following assumptions are adopted in formulating the basic eco-epidemiology model:

1. There is an $SI$ epidemic disease in the prey population divides the prey population into two classes namely $S(T)$ that represents the density of susceptible prey at time $T$ and $I(T)$ which represents the density of infected prey at time $T$. Therefore, at any time $T$ we have $N(T) = S(T) + I(T)$. The predator population is divided into two classes namely $X(T)$ that represents the density of immature predator at time $T$ and $Y(T)$ which represents the density of mature predator at time $T$.
2. It is assumed that only susceptible prey $S$ is capable of reproducing in logistic growth with carrying capacity $K > 0$ and intrinsic growth rate constant $r > 0$, the infected prey $I$ is removed before having the possibility of reproducing. However, the infected prey population $I$ still contribute with $S$ to population growth toward the carrying capacity.
3. The disease is transmitted within the same species by contact with an infected individual at infection rates $l > 0$ for the prey.
4. The mature predator consumes the susceptible and infected prey which are outside refuge according to Lotka-Volterra of functional response with maximum attack rates $c_1 > 0$ and $c_2 > 0$ for susceptible and infected respectively.
5. There is type of protection of prey species from facing predation by the mature predator with refuge rates constants $m_1 \in (0.1)$ and $m_2 \in (0,1)$ for susceptible and infected prey respectively.
6. The immature predator depends completely in its feeding on his parents, so that it feeds on the portion of up taken food by mature predator from the susceptible and infected prey with portion rates $0 < n_1 < 1$ and $0 < n_2 < 1$ with uptake rates $0 < e_1 < 1$ and $0 < e_2 < 1$ respectively. The immature predator individuals grown up and become mature predator individuals with grown up rate $g > 0$.
7. In the absence of the prey the immature and mature predator decay exponentially with natural death rates $d_1 > 0$ and $d_2 > 0$, respectively.

8. The disease may causes mortality with a constant mortality rate $\alpha > 0$ for the prey.
9. Finally, the infected population is harvest with constant rate $h > 0$ for the prey.

According to the above assumptions, the proposed mathematical model can be represented mathematically by the following set of first order non-linear differential equations, while the block diagram of this model can be illustrated in Figure 2.1.

$$\left.\begin{aligned}
\frac{dS}{dT} &= rS\left(1-\frac{S+I}{K}\right) - lSI - c_1(1-m_1)SY, \\
\frac{dl}{dT} &= lSI - c_2(1-m_2)IY - \alpha I - hI, \\
\frac{dX}{dT} &= n_1e_1c_1(1-m_1)SY + n_2e_2c_2(1-m_2)IY - gX - d_1X, \\
\frac{dY}{dT} &= gX + (1-n_1)e_1c_1(1-m_1)SY + (1-n_2)e_2c_2(1-m_2)IY - d_2Y.
\end{aligned}\right\} \tag{2.1}$$

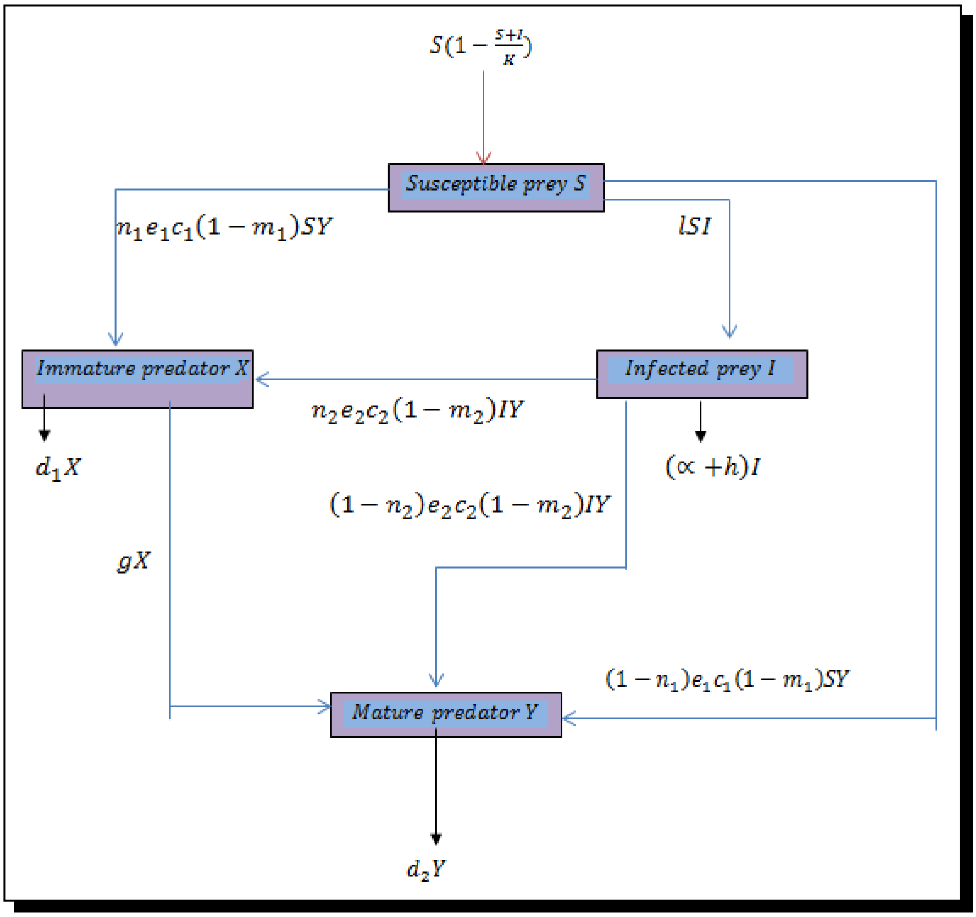


**Figure 2.1.** Block diagram for prey-predator model given by system (2.1)

Note that the above proposed model has 16 parameters which makes the mathematical analysis of the system difficult. So in order to reduce the number of parameters and determine which parameter represents the control parameter, the following dimensionless variables are used:

$$t = rT, \quad s = \frac{S}{k}, \quad i = \frac{I}{k}, \quad x = \frac{X}{k}, \quad y = \frac{Y}{k}.$$

Then system (2.1) can be written in the following dimensionless form:

$$\left.\begin{aligned} \frac{ds}{dt} &= s(1-(s+i)) - a_1 si - a_2 sy = f_1(s,i,x,y), \\ \frac{di}{dt} &= a_1 si - a_3 iy - a_4 i - a_5 i = f_2(s,i,x,y), \\ \frac{dx}{dt} &= a_6 sy + a_7 iy - (a_8 + a_9)x = f_3(s,i,x,y), \\ \frac{dy}{dt} &= a_8 x + a_{10} sy + a_{11} iy - a_{12} y = f_4(s,i,x,y) \end{aligned}\right\} \tag{2.2}$$

where

$$a_1 = \frac{lk}{r}, \ a_2 = \frac{c_1(1-m_1)K}{r}, \ a_3 = \frac{c_2(1-m_2)K}{r}, \ a_4 = \frac{h}{r}, \ a_5 = \frac{\alpha}{r},$$
$$a_6 = \frac{e_1 c_1 n_1 (1-m_1)K}{r}, \ a_7 = \frac{e_2 c_2 n_2 (1-m_2)K}{r}, \ a_8 = \frac{g}{r}, \ a_9 = \frac{d_1}{r},$$
$$a_{10} = \frac{e_1 c_1 (1-n_1)(1-m_1)K}{r}, \ a_{11} = \frac{e_2 c_2 (1-n_2)(1-m_2)K}{r}, \ a_{12} = \frac{d_2}{r}.$$

Represent the dimensionless parameter of system (2.2). It is observed that the number of parameters have been reduced from sixteen in the system (2.1) to twelve in the system (2.2).

Since the density of any species cannot be negative, therefore we will solve system (2.2) with the following initial condition $s(0) \geq 0$, $i(0) \geq 0$, $x(0) \geq 0$ and $y(0) \geq 0$.

It is easy to verify that all the interaction functions $f_1$, $f_2$, $f_3$ and $f_4$ on the right hand side of system (2.2) are continuous and have continuous partial derivatives on $R^4_+$ with respect to dependent variables $s$, $i$, $x$ and $y$. Accordingly they are Lipschitzian functions and hence system (2.2) has a unique solution for each non-negative initial condition. Further, the boundedness of the system is shown in the following theorem:

**Theorem 2.1.** *All the solutions of system* (2.2) *which initiate in $R^4_+$ are uniformly bounded.*

*Proof.* Let $(s(t), i(t), x(t), y(t))$ be any solution of the system (2.2) with non-negative initial condition $(s(0), i(0), x(0), y(0))$. According to the first equation of system (2.2), we have:

$$\frac{ds}{dt} \leq s(1-s).$$

Clearly according to the theory of differential inequality, we get:

$$\lim_{t\to\infty} \sup s(t) \leq 1.$$

Define the function $Z(t) = s(t) + i(t) + x(t) + y(t)$. Therefore,

$$\frac{dZ}{dt} < 2s - (a_2 - (a_6 + a_{10}))sy - (a_3 - (a_7 + a_{11}))iy - (s + (a_4 + a_5)i + a_9 x + a_{12} y).$$

Now, since the conversion rate constant from prey population to predator population can not be exceeding the maximum predation rate constant of predator population to prey population, hence from the biological point of view, always $a_6 + a_{10} < a_2$, and $a_7 + a_{11} < a_3$, hence it is obtained that:

$$\frac{dZ}{dt} \le 2 - mZ,$$

where $m = \min\{1, a_4 + a_5, a_9, a_{12}\}$.

Now, by using the comparison theorem [12] on the above differential inequality, we get that:

$$Z(t) \le \frac{2}{m} + \left(Z(0) - \frac{2}{m}\right)e^{-mt}.$$

Thus $0 \le Z(t) \le 2/m$ as $t \to \infty$. Hence all the solutions of system (2.2) are uniformly bounded and the proof is complete □

## 3. Existence of Equilibrium Points

In this section, the conditions for the existence of all possible equilibrium points of the system (2.2) are discussed. System (2.2) results in the following six equilibrium points.

1. The trivial equilibrium point $E_0 = (0,0,0,0)$ always exist.
2. The axial equilibrium point $E_1 = (1,0,0,0)$. This disease and predator free equilibrium also exists unconditionally.
3. The predator-free equilibrium point $E_2 = (\bar{s}, \bar{\imath}, 0, 0)$ exists if and only if there is a positive solution to the following set of equations:

$$s(1-(s+i)) - a_1 si = 0, \tag{3.1a}$$

$$a_1 si - (a_4 + a_5) = 0. \tag{3.1b}$$

From equation (3.1b) we have

$$\bar{s} = \frac{a_4 + a_5}{a_1} \tag{3.1c}$$

Now, by substituting equation (3.1c) in equation (3.1a) we get:

$$\bar{\imath} = \frac{a_1 - (a_4 + a_5)}{a_1(1-a_1)} \tag{3.1d}$$

Note that equation (3.1d) is positive, provided that:

$$a_4 + a_5 < a_1 < 1 \tag{3.1e}$$

4. The disease-free equilibrium point exists if and only if there is a positive solution to the following set of equations:

$$s(1-s) - a_2 sy = 0, \tag{3.2a}$$

$$a_6 sy - (a_8 + a_9)x = 0, \tag{3.2b}$$

$$a_8 x + a_{10} sy - a_{12} y = 0, \tag{3.2c}$$

From equation (3.2a) we have,

$$y = \frac{1-s}{a_2}. \tag{3.2d}$$

Note that equation (3.2d) is a positive, provided that:

$$s < 1. \tag{3.2e}$$

By substitute equations (3.2d) in (3.2b) we get,

$$x = \frac{a_6 s(1-s)}{a_2(a_8+a_9)}. \tag{3.2f}$$

Note that $x$ is positive under condition (3.2e).

Now, by substituting equations (3.2d) and (3.2f) in eq. (3.2c) we get:

$$-[a_6a_8 + a_{10}(a_8+a_9)]s^2 + [a_6a_8 + (a_8+a_9)(a_{10}+a_{12})]s - a_{12}(a_8+a_9) = 0. \tag{3.2g}$$

Clearly, due to discard rule equation (3.2g) has either two positive roots or else there are no positive roots depending on the following condition whether it hold or violate respectively,

$$[a_6a_8 + (a_8+a_9)(a_{10}+a_{12})]^2 > 4a_{12}(a_6a_8 + a_{10}(a_8+a_9))(a_8+a_9). \tag{3.2h}$$

That is there are two disease free equilibrium points $E_3 = (s_1, 0, x_1, y_1)$ and $E_4 = (s_2, 0, x_2, y_2)$ provided conditions (3.2e) and (3.2h) are hold.

5. The coexistence equilibrium point $E_5 = (s^*, i^*, x^*, y^*)$ exists if and only if there is a positive solution to the following set of equations:

$$1-(s+i) - a_1 i - a_2 y = 0, \tag{3.3a}$$

$$a_1 s - a_3 y - (a_4+a_5) = 0, \tag{3.3b}$$

$$a_6 sy + a_7 iy - (a_8+a_9)x = 0, \tag{3.3c}$$

$$a_8 x + a_{10} sy + a_{11} iy - a_{12} y = 0. \tag{3.3d}$$

From equation (3.3b) we have,

$$y = \frac{a_1 s - (a_4+a_5)}{a_3}. \tag{3.3e}$$

By substituting equations (3.3e) in equation (3.3a) we get:

$$i = \frac{a_3 + a_2(a_4+a_5) - (a_3+a_2a_1)s}{a_3 + a_1a_2} \tag{3.3f}$$

Also, by substituting equations (3.3e) and (3.3f) in equation (3.3c) we get:

$$x = \frac{1}{(a_8+a_9)}\left\{\Big(a_1 s - (a_4+a_5)\Big)\left(\frac{a_6 s}{a_3} + \frac{a_7}{a_3^2(1+a_1)}\Big((a_3 + a_2(a_4+a_5)\Big) - (a_3+a_1a_2)s\right)\right\} \tag{3.3g}$$

Now, by substituting equations (3.3e), (3.3f) and (3.2g in equation (3.3d) we get:

$$M_1 s^2 + M_2 s + M_3 = 0, \tag{3.3h}$$

where:

$$M_1 = \frac{a_1[a_3a_6a_8 + (a_8+a_9)((a_3a_{10}(1+a_1) - a_{11}(a_3+a_1a_2))]}{a_3^2(1+a_1)(a_8+a_9)},$$

$$M_2 = a_8(a_4+a_5)(a_7(a_1a_2+a_3) - a_3a_6) - a_1a_7a_8$$
$$+ (a_8+a_9)(a_{11}(a_1a_3 + (a_4+a_5)(a_3+2a_1a_2)) - a_3(1+a_1)(a_{10}(a_4+a_5) + a_1a_{12}),$$

$$M_3 = (a_4+a_5)\left(\frac{a_{12}}{a_3}\right) - (a_3 + a_2(a_4+a_5))\left(a_7 + \frac{a_{11}}{a_3^2(1+a_1)}\right).$$

Note that eq. (3.3h) has a unique positive root, namely $s^*$ provided that $M_1 > 0$, $M_2 < 0$ and $M_3 < 0$.

That is the following conditions are hold:

$$(a_3+a_1a_2) < \frac{a_3a_{10}(1+a_1)}{a_{11}}, \tag{3.3i}$$

$$a_{11}(a_1a_3+(a_4+a_5)(a_3+2a_1a_2) < a_3(1+a_1a_2)(a_{10}(a_4+a_5)+a_1a_{12}), \tag{3.3j}$$

$$\frac{a_{12}}{a_3} < (a_3+a_2(a_4+a_5))\left(a_7+\frac{a_{11}}{a_3^2(1+a_1)}\right). \tag{3.3k}$$

Substituting the value of $s^*$ in (3.3e), (3.3f) and (3.3g) yield that $y(s^*)=y^*$, $i(s^*)=i^*$ and $x(s^*)=x^*$ which are positive if the following condition hold:

$$\frac{a_4+a_5}{a_1} < s^* < \frac{a_3+a_2(a_4+a_5)}{a_3+a_1a_2}. \tag{3.3l}$$

## 4. Local Stability Analysis

In this section, we analyzed the local stability of the system (2.2) around each equilibrium point and discussed through computing the Jacobian matrix $J(s,i,x,y)$ and determined the eigenvalues of system (2.2) at each of them. The Jacobian matrix $J(s,i,x,y)$ of the system (2.2) at each of them can be written:

$$J=\begin{bmatrix} \frac{\partial f_1}{\partial s} & \frac{\partial f_1}{\partial i} & \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial s} & \frac{\partial f_2}{\partial i} & \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \\ \frac{\partial f_3}{\partial s} & \frac{\partial f_3}{\partial i} & \frac{\partial f_3}{\partial x} & \frac{\partial f_3}{\partial y} \\ \frac{\partial f_4}{\partial s} & \frac{\partial f_4}{\partial i} & \frac{\partial f_4}{\partial x} & \frac{\partial f_4}{\partial y} \end{bmatrix} \tag{4.1}$$

where $f_i$, $i=1,2,3,4$ are given in system (2.2) and

$$\frac{\partial f_1}{\partial s}=1-2s-i-a_1i-a_2y,\quad \frac{\partial f_1}{\partial i}=-(1+a_1)s,\quad \frac{\partial f_1}{\partial x}=0,$$
$$\frac{\partial f_1}{\partial y}=-a_2s,\quad \frac{\partial f_2}{\partial s}=a_1i,\quad \frac{\partial f_2}{\partial i}=a_1s-a_3y-(a_4+a_5),\quad \frac{\partial f_2}{\partial x}=0,$$
$$\frac{\partial f_2}{\partial y}=-a_3i,\quad \frac{\partial f_3}{\partial s}=a_6y,\quad \frac{\partial f_3}{\partial i}=a_7y,\quad \frac{\partial f_3}{\partial x}=-(a_8+a_9),$$
$$\frac{\partial f_3}{\partial y}=a_6s+a_7i,\quad \frac{\partial f_4}{\partial s}=a_{10}y,\quad \frac{\partial f_4}{\partial i}=a_{11}y,\quad \frac{\partial f_4}{\partial x}=a_8,\quad \frac{\partial f_4}{\partial y}=a_{10}s+a_{11}i-a_{12}.$$

### 4.1 Local stability of equilibrium point $E_0=(0,0,0,0)$

At $E_0$ the Jacobian matrix become

$$J(E_0)=\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -(a_4+a_5) & 0 & 0 \\ 0 & 0 & -(a_8+a_9) & 0 \\ 0 & 0 & a_8 & -a_{12} \end{bmatrix}. \tag{4.1a}$$

Then the eigenvalues of $J(E_0)$ are $\lambda_{0s}=1$, $\lambda_{0i}=-(a_4+a_5)$, $\lambda_{0x}=-(a_8+a_9)$ and $\lambda_{0y}=-a_{12}$. Thus, the equilibrium point $E_0$ is unstable.

### 4.2 Local stability of equilibrium point $E_1 = (1,0,0,0)$

At $E_1$ the Jacobian matrix become

$$J(E_1) = \begin{bmatrix} -1 & -(1+a_1) & 0 & -a_2 \\ 0 & a_1-(a_4+a_5) & 0 & 0 \\ 0 & 0 & -(a_8+a_9) & a_6 \\ 0 & 0 & a_8 & a_{10}-a_{12} \end{bmatrix} \tag{4.2a}$$

Then the characteristic equation of $J(E_1)$ is given by

$$(-1-\lambda)(a_1-(a_4+a_5)-\lambda)(\lambda^2-((a_8+a_9)+(a_{10}-a_{12}))\lambda-((a_8+a_9)(a_{10}-a_{12})+a_6a_8)=0.$$

Therefore, the eigenvalues $J(E_1)$ are $\lambda_{1s}=-1$, $\lambda_{1i}=a_1-(a_4+a_5)$, $\lambda_{1x}+\lambda_{1y}=(a_8+a_9)+(a_{10}-a_{12})$ and $\lambda_{1x}\lambda_{1y}=-((a_8+a_9)(a_{10}-a_{12})+a_6a_8)$.

Thus, the equilibrium point $E_1$ becomes stable, provided that:

$$a_4+a_5 > a_1, \tag{4.2b}$$

$$a_{12} > a_{10}, \tag{4.2c}$$

$$a_{12}-a_{10} > \max\left\{a_8+a_9, \frac{(a_6a_8)}{(a_8+a_9)}\right\} \tag{4.2d}$$

otherwise, $E_1$ is unstable.

### 4.3 Local stability of equilibrium point $E_2 = (\bar{s},\bar{\imath},0,0)$

At $E_2$ the Jacobian matrix become:

$$J(E_2) = [k_{ij}]_{4\times 4} \tag{4.3a}$$

Here

$$k_{11} = 1-2\bar{s}-\bar{\imath}(1+a_1),\ \ k_{12}=-\bar{s}-a_1\bar{s},\ \ k_{13}=0,\ \ k_{14}=-a_2\bar{s},$$

$$k_{21}=a_1\bar{\imath},\ \ k_{22}=a_1\bar{s}-(a_4+a_5),\ \ k_{23}=0,\ \ k_{24}=-a_3\bar{\imath},$$

$$k_{31}=0,\ \ k_{32}=0,\ \ k_{33}=-(a_8+a_9),\ \ k_{34}=a_6\bar{s}+a_7\bar{\imath},$$

$$k_{41}=0,\ \ k_{42}=0,\ \ k_{43}=a_8,\ \ k_{44}=a_{10}\bar{s}+a_{11}\bar{\imath}-a_{12}.$$

Then the characteristic equation of $J(E_2)$ is given by:

$$[\lambda^2+A_1\lambda+A_2][\lambda^2+B_1\lambda+B_2]=0$$

where:

$$A_1=-(k_{11}+k_{22})>0,\ \ A_2=k_{11}k_{22}-k_{12}k_{21}>0,$$

$$B_1=-(k_{33}+k_{44})\ \text{ and }\ B_2=k_{33}k_{44}-k_{34}k_{43}.$$

So, either

$$\lambda_{2s}+\lambda_{2i}=-A_1=(1+a_1\bar{s})-(2\bar{s}+(1+a_1)\bar{\imath}+a_4+a_5 \tag{4.3b}$$

and

$$\lambda_{2s}\lambda_{2i}=A_2=\bar{s}(a_1+2(a_4+a_5))+\bar{\imath}(1+a_1)(a_4+a_5)-[(a_4+a_5)+2a_1\bar{s}^2] \tag{4.3c}$$

or

$$\lambda_2x+\lambda_{2y}=-B_1=-(a_8+a_9)+a_{10}\bar{s}+a_{11}\bar{\imath}-a_{12}, \tag{4.3d}$$

$$\lambda_{2x}\lambda_{2y} = B_2 = -(a_8+a_9)(a_{10}\bar{s}+a_{11}\bar{\imath}-a_{12}) - (a_6\bar{s}+a_7\bar{\imath})a_8. \tag{4.3e}$$

Hence the eigenvalues of $J(E_2)$ are negative if the following conditions hold:

$$2\bar{s}+(1+a_1)\bar{\imath}+(a_4+a_5) > 1+a_1\bar{s}, \tag{4.3f}$$

$$\bar{s}(a_1+2(a_4+a_5))+(1+a_1)(a_4+a_5) > (a_4+a_5)+2a_1\bar{s}^2, \tag{4.3g}$$

$$a_{10}\bar{s}+a_{11}\bar{\imath} < a_{12}, \tag{4.3h}$$

$$(a_8+a_9)(a_{12}-(a_{10}\bar{s}+a_{11}\bar{\imath})) > a_8(a_6\bar{s}+a_7\bar{\imath}). \tag{4.3i}$$

Therefore, $E_2$ is stable equilibrium point if conditions (4.3e)-(4.3i) are hold. However, it is unstable otherwise.

## 4.4 Local stability of equilibrium point $E_3 = (s_1, 0, x_1, y_1)$ and $E_4 = (s_2, 0, x_2, y_2)$

The Jacobian matrix of system (2.2) at the free disease equilibrium point $E_3 = (s_1, 0, x_1, y_1)$, similarly for $E_4 = (s_2, 0, x_2, y_2)$ can be written as:

$$J(E_5) = [r_{ij}]_{4\times 4}, \tag{4.4a}$$

here

$$r_{11} = 1-2s_1-a_2y_1,\quad r_{12} = -(1+a_1)s_1,\quad r_{13}=0,\quad r_{14} = -a_2s_1,$$

$$r_{21}=0,\quad r_{22} = a_1s_1-a_3y_1-(a_4+a_5),\quad r_{23}=0,\quad r_{24}=0,$$

$$r_{31} = a_6y_1,\quad r_{32} = a_7y_1,\quad r_{33} = -(a_8+a_9),\quad r_{34} = a_6s_1,$$

$$r_{41} = a_{10}y_1,\quad r_{42} = a_{11}y_1,\quad r_{43} = a_8,\quad r_{44} = a_{10}s_1-a_{12}.$$

Then the characteristic equation of $J(E_3)$ is given by:

$$[\lambda^3+M_1\lambda^2+M_2\lambda+M_3](r_{22}-\lambda) = 0 \tag{4.4b}$$

where:

$$M_1 = -(r_{11}+r_{33}+r_{44}),$$

$$M_2 = r_{11}(r_{33}+r_{44})+r_{33}r_{44}-r_{14}r_{41} > 0,$$

$$M_3 = -r_{11}r_{33}-r_{14}(r_{31}r_{43}-r_{41}r_{33}).$$

So, either

$$(r_{22}-\lambda) = 0,$$

that is

$$\lambda_{22} = r_{22} < 0, \tag{4.4c}$$

provided that:

$$s_1 > \frac{a_3y_1+a_4+a_5}{a_1} \tag{4.4d}$$

However by using Routh Hurwitz criterion all the other eigenvalues, which represent the roots of first part of (4.4b), have negative real parts if and only if $M_1 > 0, M_3 > 0$ and $M_1M_2 - M_3 > 0$. Straightforward computation shows that:

$$\begin{aligned} M_1M_2-M_3 = {} & -r_{11}^2(r_{33}+r_{44}) - r_{33}r_{44}(r_{11}+r_{33}+r_{44}) - r_{11}((r_{33}+r_{44})^2 \\ & -(r_{14}r_{41}-r_{33}) + r_{14}(r_{31}r_{43}+r_{41}r_{44}). \end{aligned} \tag{4.4e}$$

Now, $M_1 > 0$ and $M_3 > 0$ provided that:

$$\frac{1-a_2y_1}{2} < s_1 < \frac{a_{12}}{a_{10}} \text{ with } y_1 < \frac{1}{a_2}. \tag{4.4f}$$

Moreover, the first four terms of (4.4e) are positive under condition (4.4f), the last term is positive provided that:

$$a_{10}a_{12} > a_6a_8 + a_{10}^2 s_1. \tag{4.4g}$$

So, all the eigenvalues of $J(E_3)$ have negative real part under the given conditions and hence $E_3$ is locally asymptotically stable. However, it is unstable otherwise.

### 4.5 Local stability of equilibrium point $E_5 = (s^*, i^*, x^*, y^*)$

At $E_5$ the Jacobian matrix become:

$$J(E_5) = [l_{ij}]_{4\times 4}, \tag{4.5a}$$

here

$$\begin{aligned}
&l_{11} = 1-2s^* - (1+a_1)i^* - a_2y^*, \ l_{12} = -(1+a_1)s^* < 0, \ l_{13} = 0, l_{14} = -a_2s^* < 0,\\
&l_{21} = a_1i^* > 0, \ l_{22} = a_1s^* - a_3y^* - (a_4+a_5), \ l_{23} = 0, \ l_{24} = -a_3i^* < 0,\\
&l_{31} = a_6y^* > 0, \ l_{32} = a_7y^* > 0, \ l_{33} = -(a_8+a_9) < 0, \ l_{34} = a_6s^* + a_7i^* > 0,\\
&l_{41} = a_{10}y^* > 0, \ l_{42} = a_{11}y^* > 0, \ l_{43} = a_8 > 0, \ l_{44} = a_{10}s^* + a_{11}i^* - a_{12}.
\end{aligned}$$

Then the characteristic equation of $J(E_5)$ is given by:

$$\lambda^4 + C_1\lambda^3 + C_2\lambda^2 + C_3\lambda + C_4 = 0 \tag{4.5b}$$

where:

$$\begin{aligned}
C_1 &= -(\propto_0 + \propto_1)\\
C_2 &= \propto_0\propto_1 + \propto_2 + \propto_3 - (\propto_4 + \propto_5 + \propto_6 + \propto_7)\\
C_3 &= -[(\propto_0(\propto_2 - \propto_4) + \propto_1(\propto_3 - \propto_6) - l_{24}(\propto_8 - \propto_9 + \propto_{10}) + \propto_6\propto_{11} + l_{14}(\propto_{12} - \propto_{13} + \propto_{14})]\\
C_4 &= (\propto_2 - \propto_4)(\propto_3 - \propto_6) + (\propto_9 - \propto_{10})(\propto_{15} - \propto_{16}) + \propto_{17}(\propto_{14} - \propto_{18}) + \propto_{19}(\propto_{13} - \propto_{14}).
\end{aligned}$$

with

$$\begin{aligned}
&\propto_0 = l_{11} + l_{22}, \ \propto_1 = l_{33} + l_{44}, \ \propto_2 = l_{33}l_{44}, \ \propto_3 = l_{11}l_{22},\\
&\propto_4 = l_{34}l_{43} > 0, \ \propto_5 = l_{24}l_{42} < 0, \ \propto_6 = l_{12}l_{21} < 0, \ \propto_7 = l_{14}l_{41} < 0, \ \propto_8 = l_{11}l_{42},\\
&\propto_9 = l_{32}l_{43} > 0, \ \propto_{10} = l_{42}l_{33} < 0, \ \propto_{11} = l_{41} + l_{14}, \ \propto_{12} = l_{22}l_{41} < 0, \ \propto_{13} = l_{31}l_{42} > 0,\\
&\propto_{14} = l_{41}l_{33} < 0, \ \propto_{15} = l_{11}l_{24}, \ \propto_{16} = l_{14}l_{21} < 0, \ \propto_{17} = l_{12}l_{24} > 0, \ \propto_{18} = l_{31}l_{43} > 0\\
&\propto_{19} = l_{14}l_{22}
\end{aligned}$$

Now by using Routh Hurwitz criterion all the eigenvalues, which represent the roots of (4.5b), have negative real parts if and only if $C_1 > 0$, $C_3 > 0$, $C_4 > 0$ and $\Delta = (C_1C_2 - C_3)C_3 - C_1^2C_4 > 0$.

Now, $C_i > 0$, $i = 1, 3$ and 4, provided that

$$\max\left\{\frac{1-(1+a_1)i^* - a_2y^*}{2}, R\right\} < s^* < \min\left\{\frac{a_3y^* + a_4 + a_5}{a_1}, \frac{a_{12} - a_{11}i^*}{a_{10}}, \frac{a_{10}y^*}{a_2}\right\}, \tag{4.5c}$$

$$(a_8+a_9)(a_{12} - (a_{10}s^* + a_{11}i^*)) > (a_6s^* + a_7i^*)a_8, \tag{4.5d}$$

$$a_3(1+a_1)s^*i^*(a_6a_8y^* + a_{10}(a_8+a_9)y^*)$$

$$< -a_2 s^*(a_1 s^* - a_3 y^* - (a_4+a_5))(a_6 a_{11} {y^*}^2 + a_{10}(a_8+a_6)y^*) \quad (4.5e)$$

with

$$i^* < \min\left\{\frac{a_{12}}{a_{11}}, \frac{1-a_2 y^*}{1+a_1}, \frac{(1+a_3+a_9+a_{12})-((a_4+a_5)+(a_2+a_3)y^*)}{1+a_1+a_{11}}\right\}, \quad (4.5f)$$

$$y^* < \frac{1}{a_2}, \quad (4.5g)$$

$$1+a_8+a_9+a_{12} > (a_4+a_5)+(a_2+a_3)y^*, \quad (4.5h)$$

$$a_1 < a_{10}+2 \quad (4.5i)$$

where:

$$R = \frac{1+a_8+a_9+a_{12}-(a_4+a_5)-(a_2+a_3)y^*-(1+a_1+a_{11})i^*}{a_1-(a_{10}+2)} \quad (4.5j)$$

Further, it is easy to check that:

$\Delta = N_1 - N_2$, where

$$N_1 = (\propto_0\propto_1 - \propto_5 - \propto_7)(\propto_0 + \propto_1)[\propto_0(\propto_2 - \propto_4) + \propto_1(\propto_3 - \propto_6) - l_{24}(\propto_8 - \propto_9 + \propto_{10})$$
$$+ \propto_6\propto_{11} - l_{14}(\propto_{12} - \propto_{13} + \propto_{14})] + \propto_0\propto_1(\propto_2 - \propto_4)^2 + l_{24}(\propto_0 - \propto_1)(\propto_8 - \propto_9 + \propto_{10})$$
$$(\propto_2 - \propto_3 - \propto_4 + \propto_6) + \propto_6\propto_{11}(\propto_0 - \propto_1)(\propto_3 - \propto_2 - \propto_6 + \propto_4) + l_{14}(\propto_2 - \propto_4)$$
$$(\propto_{12} - \propto_{13} + \propto_{14})[\propto_2(\propto_{12} - \propto_{13} + \propto_{14}) - \propto_1] + \propto_0\propto_1(\propto_3 - \propto_6)^2 - l_{14}(\propto_0 - \propto_1)$$
$$(\propto_3 - \propto_6)(\propto_{12} - \propto_{13} + \propto_{14}),$$

$$N_2 = l_{24}^2(\propto_8 - \propto_9 + \propto_{10})^2 + l_{14}^2(\propto_{12} - \propto_{13} + \propto_{14})^2 + \propto_6^2\propto_{11}^2 + (\propto_{12} - \propto_{13} + \propto_{14})[2l_{14}l_{24}$$
$$(\propto_8 - \propto_9 + \propto_{10}) - 2l_{14}\propto_6\propto_{11}] - 2l_{24}\propto_6\propto_{11}(\propto_8 - \propto_9 + \propto_{10})$$
$$+ 2\propto_0\propto_1(\propto_2 - \propto_4)(\propto_3 - \propto_6) + (\propto_0 + \propto_1)^2[(\propto_9 - \propto_{10})(\propto_{15} - \propto_{16})$$
$$+ \propto_{17}(\propto_{14} - \propto_{18}) + \propto_{19}(\propto_{13} - \propto_{14})].$$

Clearly, $N_i$, for $i = 1,2$ are positive under conditions (4.5c)-(4.5i), with the following condition

$$N_1 > N_2. \quad (4.5k)$$

Hence $\Delta = (C_1C_2 - C_3)C_3 - C_1^2C_4 > 0$. So, all the eigenvalues of $J(E_5)$ have negative real part under the given conditions and hence $E_5$ is locally asymptotically stable. However, it is unstable otherwise.

## 5. Global Stability Analysis

In this section, the global stability analysis for the equilibrium points, which are locally asymptotically stable of system (2.2) is studied analytically by use the suitable of Lyapunov method as shown in the following theorems.

**Theorem 5.1.** *Assume that the disease and predator free equilibrium point* $E_1 = (1,0,0,0)$ *of system* (2.2) *is locally asymptotically stable in the* $R_+^4$. *Then* $E_1$ *is globally asymptotically stable provided that the following condition hold:*

$$a_4 + a_5 > 1 + a_1, \quad (5.1a)$$

$$a_{12} > a_2. \quad (5.1b)$$

*Proof.* Consider the following function

$$G_1(s,i,x,y) = (s-1-\ln s) + i + x + y.$$

It is easy to see that $G_1(s,i,x,y) \in C^1(R_+^4, R)$, $G_1(E_1) = 0$ and $G_1(s,i,x,y) > 0$; for all $(s,i,x,y) \neq E_1$. Now by differentiating $G_1$ with respect to time $t$ and going some algebraic handling, given that:

$$\frac{dG_1}{dt} = -(s-1)^2 - si + (1 + a_1 - (a_4 + a_5))i + (a_2 - a_{12})y - (a_2 - (a_5 + a_9))sy - (a_3 - (a_6 + a_{10}))iy.$$

Now, according to the conditions in theory (2.1) and (5.1a)-(5.1b) we obtain that $\frac{dG_1}{dt} < 0$. Hence $E_1$ is a globally asymptotically stable and then the proof is complete. □

**Theorem 5.2.** *Assume that the predator free equilibrium point* $E_2 = (\bar{s}, \bar{\imath}, 0, 0)$ *of system* (2.2) *is locally asymptotically stable in the* $R_+^4$. *Then* $E_2$ *is globally asymptotically stable on the region* $\omega_1$ *in the* Int $R_4^+$ *that satisfies the following conditions:*

$$a_{11} > a_2\bar{s} + a_3\bar{\imath}. \tag{5.2}$$

*Proof.* Consider the following function

$$G_2(s,i,x,y) = \left(s - \bar{s} - \bar{s}\ln\frac{s}{\bar{s}}\right) + \left(i - \bar{\imath} - \bar{\imath}\ln\frac{i}{\bar{\imath}}\right) + x + y.$$

It is easy to see that $G_2(s,i,x,y) \in C^1(R_+^4, R)$, $G_2(E_2) = 0$, and $G_2(s,i,x,y) > 0$, for all $(s,i,x,y) \neq E_2$. Now by differentiating $G_2$ with respect to time $t$ and going some algebraic handling, given that:

$$\frac{dG_2}{dt} < -(s-\bar{s})^2 - (s-\bar{s})(i-\bar{\imath}) - (a_2 - (a_6 + a_{10}))sy - (a_3 - (a_7 + a_{11}))iy + (a_2\bar{s} + a_3\bar{\imath} - a_{12})y.$$

Then $\frac{dG_2}{dt} < 0$ under the conditions in theory (2.1) and (5.2). Hence, $E_2$ is a globally asymptotically stable and then the proof is complete. □

Furthermore since there are two free disease equilibrium points $E_3(s_1, 0, x_1, y_1)$ and $E_4 = (s_2, 0, x_2, y_2)$ in the interior of $R_4^+$ having the same local stability conditions but with different neighborhood of starting points then its not possible to studying the global stability of them using Lyapunove function. Therefore, we will study it numerically instead of analytically as shown in last section.

**Theorem 5.3.** *Assume that the positive equilibrium point* $E_5 = (x^*, y^*, z^*, w^*)$ *of system (2.2) is locally asymptotically stable. Then* $E_5$ *is globally asymptotically stable in the sub region of* $R_+^4$. *That satisfies the following conditions:*

$$s^* > s, \tag{5.3a}$$

$$i^* > i, \tag{5.3b}$$

$$x^* < x, \tag{5.3c}$$

$$y^* > y. \tag{5.3d}$$

*Proof.* Consider the following function

$$G_3(s,i,x,y) = \left(s - s^* - s^*\ln\frac{s}{s^*}\right) + \left(i - i^* - i^*\ln\frac{i}{i^*}\right) + \left(x - x^* - x^*\ln\frac{x}{x^*}\right) + \left(y - y^* - y^*\ln\frac{y}{y^*}\right).$$

It is easy to see that $G_3(s,i,x,y) \in C^1(R_+^4, R)$, $G_3(E_5) = 0$, and $G_3(s,i,x,y) > 0$, for all $(s,i,x,y) \neq E_5$. Now by differentiating $G_3$ with respect to time $t$ and going some algebraic handling, given that:

$$\begin{aligned}\frac{dG_3}{dt} &< -(s-s^*)^2 - (s-s^*)(i-i^*) - (a_2-a_{10})(s-s^*)(y-y^*) - (a_3-a_{11})(i-i^*)(y-y^*) \\ &\quad -(x-x^*)\left(\frac{a_6sy+a_7iy}{xx^*}\right) + (y-y^*)\left(\frac{a_6s^*+a_7i^*}{x^*}\right) + \frac{a_6y}{x^*}(s-s^*) + \frac{a_7y}{x^*}(i-i^*) \\ &\quad -\frac{a_8x}{yy^*}(y-y^*)^2 + \frac{a_9}{y^*}(x-x^*)(y-y^*).\end{aligned}$$

Then $\frac{dG_3}{dt}$ is negative definite under conditions (5.3a)-(5.3d), with the biological facts $a_{10} < a_2$ and $a_{11} < a_3$. Hence, $E_5$ is a globally asymptotically stable and then the proof is complete. □

## 6. Numerical Simulation

In this section, we confirm our obtained results in the previous sections numerically by using Runge-Kutta method along with predictor-corrector method. Note that, we use turbo C++ in programming and MATLAB for plotting and then discuss our obtained results. The system (2.2) is studied numerically for different sets of parameters and initial points. The objectives of this study are; first investigate the effect of varying the value of each parameter on the dynamical behaviour of system (2.2) and second confirm our obtained analytical results. It is observed that, for the following set of hypothetical parameters that satisfies stability conditions of the positive equilibrium point, system (2.2) has a globally asymptotically stable positive equilibrium point as shown in Figure 6.1(a-d).

$$\left.\begin{aligned}&a_1 = 2,\ a_2 = 2,\ a_3 = 0.9,\ a_4 = 0.1,\ a_5 = 0.1,\ a_6 = 0.6,\ a_7 = 0.4,\\ &a_8 = 0.5,\ a_9 = 0.1,\ a_{10} = 0.6,\ a_{11} = 0.4,\ a_{12} = 0.3.\end{aligned}\right\} \tag{6.1}$$

Clearly, Figure 6.1 shows that system (2.2) has a globally asymptotically stable as the solution of system (2.2) approaches asymptotically to the positive equilibrium point $E_5 = (0.223, 0.064, 0.078, 0.288)$ starting from three different initial points and this is confirming our obtained analytical results.

Now, in order to discuss the effect of the parameters values of system (2.2) on the dynamical behavior of the system. The system is solved numerically for the data given in (6.1) with varying one parameter at each time and sometime two parameters the obtained results are given below.

The effect of varying the infection rate of prey in the range $0.1 < a_1 \le 1.93$ keeping other parameters as data given in (6.1) is studied, causes extinction in the infected prey and the system will approach to the infected prey free equilibrium point as shown in Figure 6.2. However, for $1.93 < a_1 < 2$ it is observed that system (2.2) still approach asymptotically to the positive equilibrium point.

The effect of varying the predation rate on susceptible prey in the range $0.1 < a_2 \le 1.88$ keeping other parameters as data given in (6.1) is studied, it is observed that system (2.2) still approach asymptotically to the free infected prey equilibrium point, while increasing this parameter further $1.88 < a_2 \le 2$ the solution of the system will approach to the positive equilibrium point as shown in Figure 6.3 for typical value $a_2 = 1.88$.

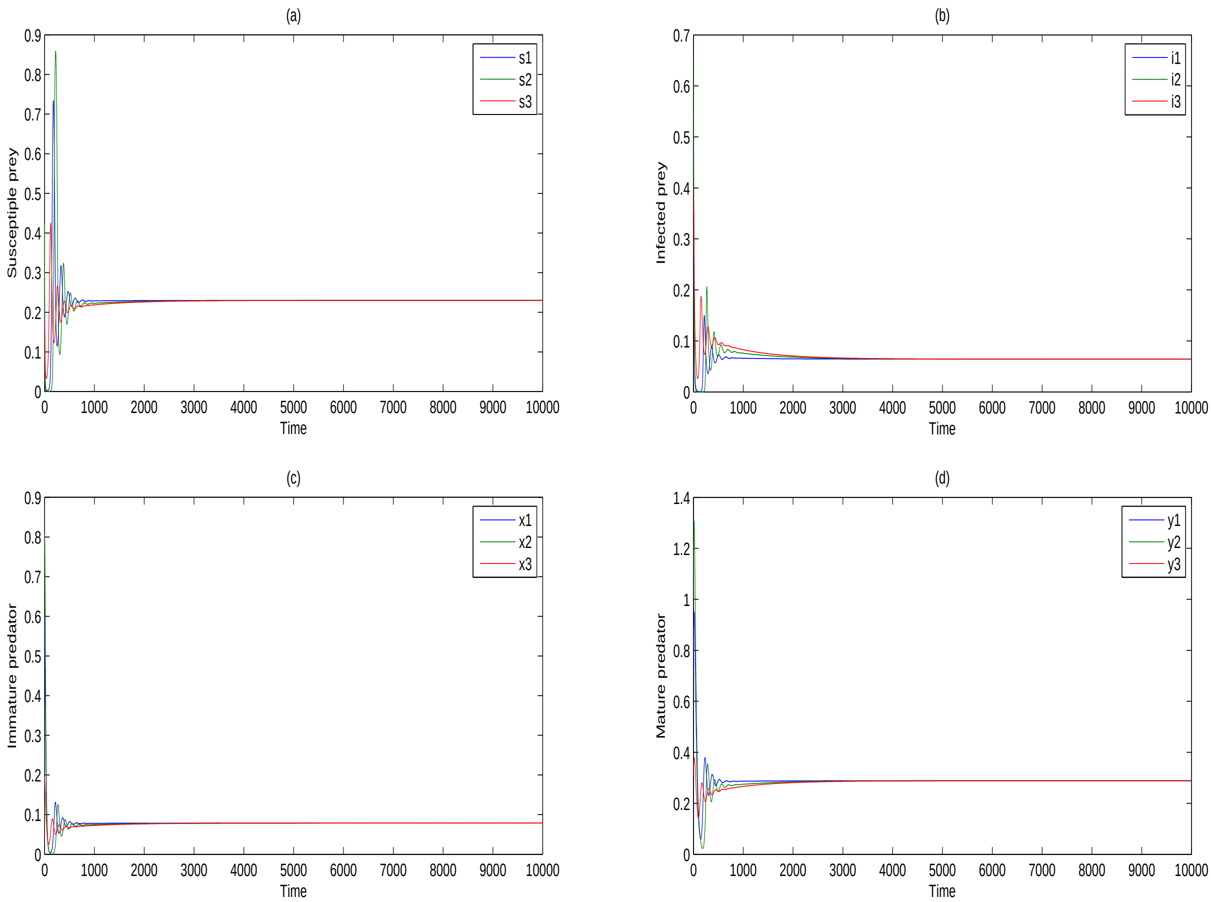


**Figure 6.1.** The time series of the solution of system (2.2) started from the three different initial points (0.4, 0.5, 0.6, 0.7), (0.6, 0.7, 0.8, 0.9) and (0.2, 0.4, 0.2, 0.3) for the data given by (6.1) (a) the trajectories of s as a function of time, (b) the trajectories of $i$ as a function of time, (c) trajectories of $x$ as a function of time and (d) the trajectories of $y$ as a function of time

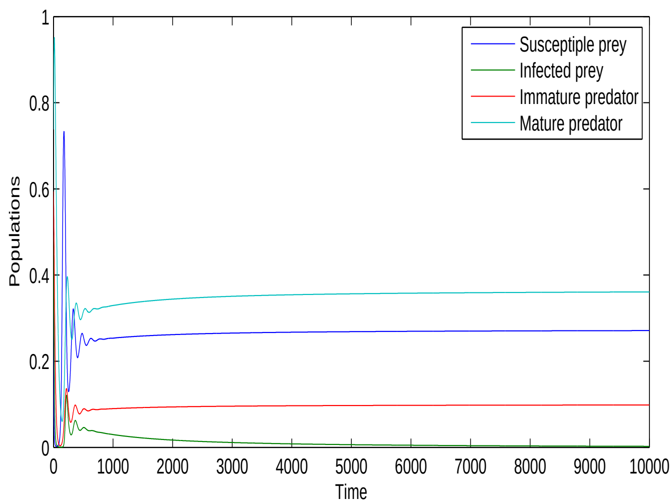


**Figure 6.2.** Time series of the solution of system (2.2) approaches asymptotically to the infected prey free equilibrium point $E_2 = (0.271, 0, 0.098, 0.360)$ for the data given in (6.1) with $a_1 = 1.93$

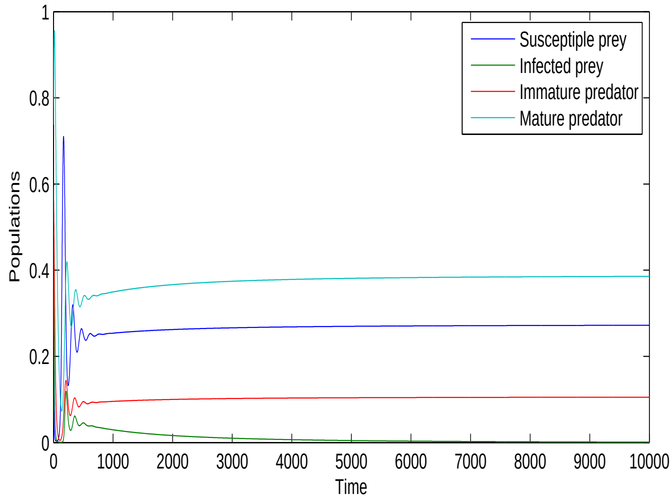


**Figure 6.3.** Time series of the solution of system (2.2) approaches asymptotically to the infected prey free equilibrium point $E_3 = (0.272, 0, 0.105, 0.385)$ for the data given in (6.1) with $a_2 = 1.88$

On other hand, varying the predation rate on infected prey in the range $0.1 < a_3 \le 0.95$ keeping other parameters as data given in (6.1) is studied, it is observed that system (2.2) still approach asymptotically to the positive equilibrium point. While increasing this parameter further $0.95 < a_3 < 1$ causes extinction in the infected prey and the system will approach to the free infected prey equilibrium point $E_3$ as shown in Figure 6.4 for typical value $a_3 = 0.98$.

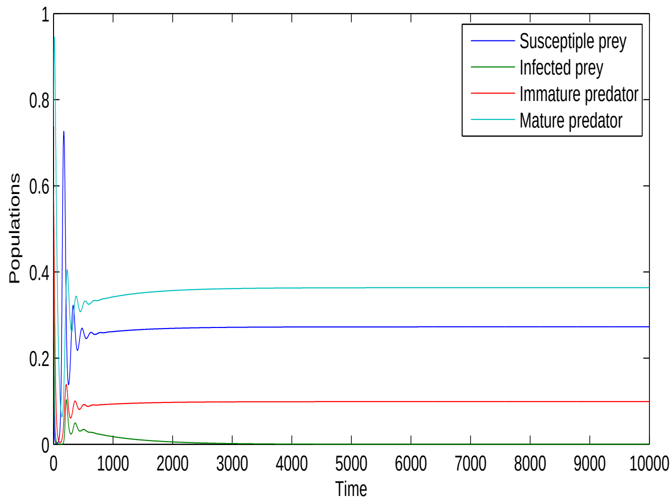


**Figure 6.4.** Time series of the solution of system (2.2) approaches asymptotically to the infected prey free equilibrium point $E_3 = (0.272, 0, 0.099, 0.363)$ for the data given in (6.1) with $a_3 = 0.98$

The effect of varying the harvesting rate of the infected prey, in the range $0 < a_4 \le 0.02$ keeping other parameters as data given in (6.1) causes extinction in the predator, and that system (2.2) still approach asymptotically to the free predator equilibrium point as shown in Figure 6.5(a) for typical value $a_4 = 0.011$, however increasing this parameter further

$0.02 < a_4 \leq 0.11$ it is observed that the system (2.2) still approach to the positive equilibrium point as shown in Figure 6.5(b) for typical value $a_4 = 0.05$ additional for $0.11 < a_4 < 1$ causes extinction in the infected prey and the system will approach to the infected prey free equilibrium point as shown in Figure 6.5(c) for typical value $a_4 = 0.2$.

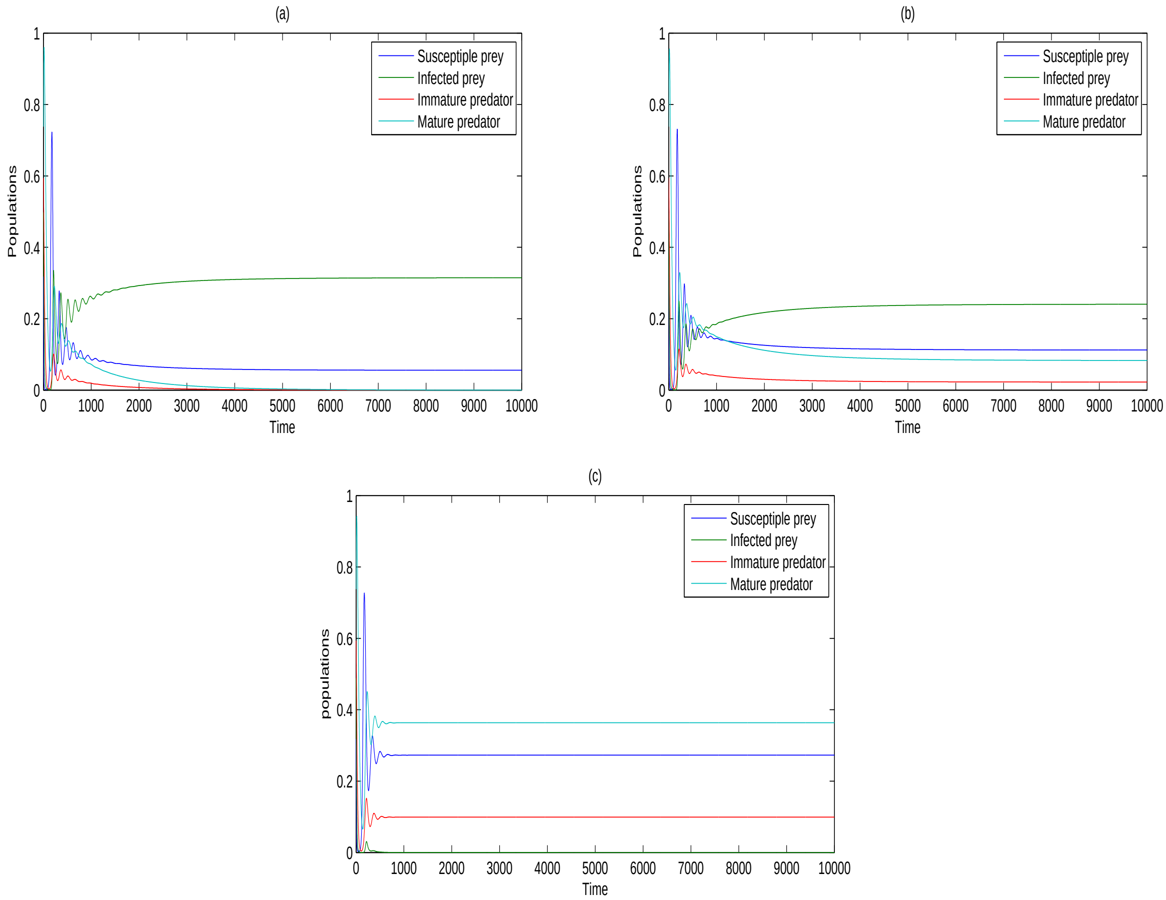


**Figure 6.5.** (a) Time series of the solution of system (2.2) for the data given by (6.1) with $a_4 = 0.011$, which approaches to $E_2 = (0.055, 0.315, 0, 0)$, (b) Time series of the solution of system (2.2) for the data given by (6.1) with $a_4 = 0.05$, which approaches to $E_5 = (0.112, 0.240, 0.022, 0.083)$, (c) Time series of the solution of system (2.2) for the data given by (6.1) with $a_4 = 0.2$, which approaches to $E_3 = (0.272, 0, 0.099, 0.363)$.

Moreover, varying the mortality death rate of the infected prey due to disease, in the range $0 < a_5 \leq 0.02$ keeping other parameters as data given in (6.1) causes extinction in the predator, and that system (2.2) still approach asymptotically to the free predator equilibrium point $E_2$, however increasing this parameter further $0.02 < a_5 \leq 0.11$ it is observed that the system (2.2) still approach to the positive equilibrium point $E_5$. In additional, for $0.11 < a_5 < 1$ causes extinction in the infected prey and the system will approach the infected prey free equilibrium point $E_3$.

The varying of the parameter $a_6$ which represents the conversion rate from the susceptible prey to the immature predator in the range $0.1 \leq a_6 \leq 0.2$, and keeping the rest of parameters values as data given in (6.1) , causes extinction in the predator, and that system (2.2) still approach asymptotically to the free predator equilibrium point $E_2$, and increasing further

$0.2 < a_6 \leq 0.6$ it is observed that for the solution of system (2.2) still approaches asymptotically to a positive equilibrium point $E_5$.

The varying of the parameter $a_7$which represents the conversion rate from the infected prey to the immature predator in the range $0.1 \leq a_7 \leq 0.28$, and keeping the rest of parameters values as data given in (6.1), causes extinction in the predator, and that system (2.2) still approach asymptotically to the free predator equilibrium point $E_2$, and increasing further $0.28 < a_7 \leq 0.4$ it is observed that the solution of system (2.2) still approaches asymptotically to a positive equilibrium point $E_5$.

Now, varying the growth rate parameter of immature predator $a_8$ and keeping the rest of parameters values as data given in (6.1), it is observed that for $0.1 < a_8 \leq 0.19$ causes extinction in the predator, and that system (2.2) still approach asymptotically to the free predator equilibrium point $E_2$ and increasing further $0.19 < a_8 \leq 0.74$ it is observed that for the solution of system (2.2) still approaches asymptotically to a positive equilibrium point $E_5$. Increasing further $0.74 < a_8 < 1$ cause's extinction in the infected prey and the system will approach the infected prey free equilibrium point.

The effect of varying the natural death rate of immature predator in the range $0.01 \leq a_9 \leq 0.06$ keeping other parameters as data given in (6.1) causes extinction in the infected prey and the system will approach asymptotically to the infected prey free equilibrium point $E_3$, increasing this parameter further in the range $0.06 < a_9 \leq 0.24$ is studied; it is observed that system (2.2) approach asymptotically to the positive equilibrium point $E_5$, however for $0.24 < a_9 \leq 0.99$ causes extinction in the predator and the system will approach the free predator equilibrium point $E_2$.

For varying the conversion rate of food from susceptible prey onto mature predator, in the range $0.1 < a_{10} < 0.32$ keeping other parameters as data given in (6.1) causes extinction in the predator and the system will approach asymptotically to the free predator equilibrium point $E_2$. increasing this parameter further in the range $0.32 < a_{10} \leq 0.63$ is studied; it is observed that system (2.2) approach asymptotically to the positive equilibrium point $E_5$, while for the values $0.63 < a_{10} < 1$ causes extinction in the infected prey and the system will approach asymptotically to the infected prey free equilibrium point $E_3$.

On the other hand, the effect of varying the conversion rate of food from infected prey onto mature predator, in the range $0.1 \leq a_{11} \leq 0.28$ keeping other parameters as data given in (6.1) causes extinction in the predator and the system will approach asymptotically to the free predator equilibrium point $E_2$. Increasing this parameter further in the range $0.28 < a_{11} \leq 0.4$ is studied; it is observed that system (2.2) approach asymptotically to the positive equilibrium point $E_5$.

Moreover, varying the natural death rate of mature predator, in the range $0.1 < a_{12} \leq 0.29$ keeping other parameters as data given in (6.1) causes extinction in the infected prey and the system will approach the infected prey free equilibrium point $E_3$, in additional for $0.29 < a_{12} \leq 0.33$ it is observed that system (2.2) approach asymptotically to the positive equilibrium point $E_5$, increasing this parameter further in the range $0.33 < a_{12} < 1$ causes extinction in the predator and the system will approach the free predator equilibrium point $E_2$ .

Finally, varying the parameters $a_1$, $a_4$, $a_5$, $a_6$, $a_8$, $a_9$, $a_{10}$ and $a_{13}$ into the following values which satisfies conditions (4.2b-4.2d) and (5.1a) and (5.1b),

$$\left.\begin{array}{l} a_1 = 0.5,\ a_2 = 0.8,\ a_3 = 0.3,\ a_4 = 0.9,\ a_5 = 0.9,\ a_6 = 0.6,\ a_7 = 0.4, \\ a_8 = 0.3,\ a_9 = 0.1,\ a_{10} = 0.4,\ a_{11} = 0.4,\ a_{12} = 0.9. \end{array}\right\} \tag{6.2}$$

It is observed that system (2.2) approach asymptotically to the axial equilibrium point $E_1 = (1,0,0,0)$, as shown in Figure 6.6 and this is confirming our obtained analytical results.

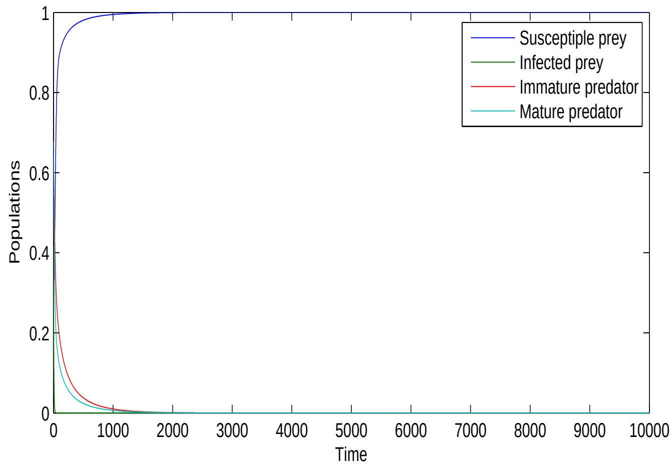


**Figure 6.6.** Time series of the solution of system (2.2) approaches asymptotically to the axial equilibrium point $E_1(1,0,0,0)$ for the data given in (6.2)

## 7. Conclusions and Discussion

In this paper, an eco-epidemiological mathematical model consisting of prey-predator model involving SI infectious disease in prey and stage structured predator species with harvest in infectious population has been proposed and analysed. Further, in this model, linear types of functional responses for the predation of susceptible and infected prey which are outside refuge as well as linear incidence rate for describing the transition of disease are used. Our aim is to study the role of harvesting on the dynamics of disease propagation and/or eradication. Therefore, system (2.2) has been solved numerically for different sets of initial points and different sets of parameters starting with the hypothetical set of data given by (6.1) and the following observations are obtained.

1. System (2.2) has one type of attractor in Int $R_+^4$ for the hypothetical set of parameters value given in eq. (6.1).
2. For the set of hypothetical parameters value given in eq. (6.1), the system (2.2) approaches asymptotically to globally stable positive point $E_5 = (0.223, 0.064, 0.078, 0.288)$.
3. As the infection rate of prey $a_1$ increasing to 1.93 keeping the rest of parameters as in eq. (6.1), the solution of system (2.2) approaches to infected free equilibrium point $E_3$. However, if $1.93 < a_1 \le 2$, then the infected prey will grow again and then the trajectory transferred from infected prey free equilibrium point to the positive equilibrium point $E_5$, thus the parameter $a_1 = 1.93$ is a bifurcation point.

4. As the attack rate of mature predator on susceptible prey $a_2$ increasing to 1.88 keeping the rest of parameters as in eq. (6.1), the solution of system (2.2) approaches to infected free equilibrium point $E_3$, however if $1.88 < a_2 \le 2$, then the infected prey will grow again and then the trajectory transferred from infected prey free equilibrium point to the positive equilibrium point $E_5$, thus the parameter $a_2 = 1.88$ is a bifurcation point.
5. As the attack rate of mature predator on infected prey $a_3$ increasing to 0.95 keeping the rest of parameters as in eq. (6.1), the solution of system (2.2) approaches to positive equilibrium point $E_5$, however if $0.95 < a_3 < 1$, then the infected prey faced extinction and the trajectory transferred from the positive equilibrium point $E_5$ to the infected prey free equilibrium point $E_3$, thus, the parameter $a_3 = 0.95$ is a bifurcation point.
6. As the harvesting rate of infected prey $a_4$ increasing to 0.02 keeping the rest of parameters as in eq. (6.1), the solution of system (2.2) approaches to free predator equilibrium point $E_2$, however if $0.02 < a_4 \le 0.11$, then the predator grown up and the trajectory transferred from the free predator equilibrium point $E_2$ to the positive equilibrium point $E_5$, while for $0.11 < a_4 < 1$ the infected prey faced extinction and the trajectory transferred from the positive equilibrium point $E_5$ to the infected prey free equilibrium point $E_3$ thus, the parameters $a_4 = 0.02$ and 0.11 are bifurcation points for system (2.2). Similarly result satisfied for varying the death rate of infectious prey due to the disease and hence the parameters $a_5 = 0.02$ and 0.11 are bifurcation points for system (2.2).
7. As the conversion rate of food from susceptible prey to immature predator $a_6$ increasing to 0.2 keeping the rest of parameters as in eq. (6.1), the solution of system (2.2) approaches to free predator equilibrium point $E_2$, however if $0.2 < a_6 \le 0.6$, then the predator grown up and the trajectory transferred from the free predator equilibrium point $E_2$ to the positive equilibrium point $E_5$, thus, the parameter $a_6 = 0.2$ is a bifurcation point for system (2.2).
8. As the conversion rate of food from infected prey to immature predator $a_7$ increasing to 0.28 keeping the rest of parameters as in eq. (6.1), the solution of system (2.2) approaches to free predator equilibrium point $E_2$, however, if $0.28 < a_7 \le 0.4$ then the predator grown up and the trajectory transferred from the free predator equilibrium point $E_2$ to the positive equilibrium point $E_5$, thus the parameter $a_7 = 0.28$ is a bifurcation point for system (2.2).
9. As the growth rate of immature predator onto mature predator $a_8$ increasing to 0.19 keeping the rest of parameters as in eq. (6.1), the solution of system (2.2) approaches to free predator equilibrium point $E_2$, however if $0.19 < a_8 \le 0.74$ then the predator grown up and the trajectory transferred from the free predator equilibrium point $E_2$ to the positive equilibrium point $E_5$, while for $0.74 < a_8 < 1$ the infected prey faced extinction and the trajectory transferred from the positive equilibrium point $E_5$ to the infected prey free equilibrium point $E_3$ thus, the parameters $a_8 = 0.19$ and 0.74 are bifurcation points for system (2.2).
10. As the natural death rate of immature predator $a_9$ increasing to 0.06 keeping the rest of parameters as in eq. (6.1), the solution of system (2.2) approaches to free infected prey equilibrium point $E_3$, however if $0.06 < a_9 \le 0.24$, then the infected prey grown up

and the trajectory transferred from the free infected prey equilibrium point $E_3$ to the positive equilibrium point $E_5$, while for $0.24 < a_9 \leq 0.99$ the predator faced extinction and the trajectory transferred from the positive equilibrium point $E_5$ to the predator free equilibrium point $E_2$ thus, the parameters $a_9 = 0.06$ and 0.24 are bifurcation points for system (2.2).

11. As the conversion rate of food from susceptible prey onto mature predator $a_{10}$ increasing to 0.32 keeping the rest of parameters as in eq. (6.1), the solution of system (2.2) approaches to free predator equilibrium point $E_2$, however if $0.32 < a_{10} \leq 0.63$, then the predator grown up and the trajectory transferred from the free predator equilibrium point $E_2$ to the positive equilibrium point $E_5$, while for $0.63 < a_{10} < 1$ the infected prey faced extinction and the trajectory transferred from the positive equilibrium point $E_5$ to the infected prey free equilibrium point $E_3$ thus, the parameters $a_{10} = 0.32$ and 0.63 are bifurcation points for system (2.2).
12. As the conversion rate of food from infected prey to mature predator $a_{11}$ increasing to 0.28 keeping the rest of parameters as in eq. (6.1), the solution of system (2.2) approaches to free predator equilibrium point $E_2$, however if $0.28 < a_{11} \leq 0.4$, then the predator grown up and the trajectory transferred from the free predator equilibrium point $E_2$ to the positive equilibrium point $E_5$, thus, the parameter $a_{11} = 0.28$ is a bifurcation point for system (2.2).
13. As the natural death rate of mature predator $a_{12}$ increasing to 0.29 keeping the rest of parameters as in eq. (6.1), the solution of system (2.2) approaches to free infected prey equilibrium point $E_3$, however if $0.29 < a_{12} \leq 0.33$, then the infected prey grown up and the trajectory transferred from the free infected prey equilibrium point $E_3$ to the positive equilibrium point $E_5$, while for $0.33 < a_{12} < 1$ the predator faced extinction and the trajectory transferred from the positive equilibrium point $E_5$ to the predator free equilibrium point $E_2$ thus, the parameters $a_{12} = 0.29$ and 0.33 are bifurcation points for system (2.2).
14. Finally, varying the hypothetical set of parameters into the set of parameters given in eq.(6.2) which satisfies conditions (4.2b)-(4.2d) and (5.1a) and (5.1b) , then the solution of system (2.2) approach asymptotically to the axial equilibrium point $E_1 = (1,0,0,0)$.

## Competing Interests

The author declares that she has no competing interests.

## Authors' Contributions

The author wrote, read and approved the final manuscript.

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