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Construction of Wavelet and Gabor's Parseval Frames

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Abstract. A new way to build wavelet and Gabor's Parseval frames for $L^2(\mathbb{R}^d)$ is shown in this paper. In the first case the construction is done using an expansive matrix B, together with only one function $h \in L^2(\mathbb{R}^d)$. In the second one, we work with a function $g \in L^2(\mathbb{R}^d)$ and two invertible matrixes B and C, with the condition that $C^t\mathbb{Z}^d \subset \mathbb{Z}^d$. The only requirement for h and g is that they have to be supported in a set Q, such that the measure of Q is finite and positive. Q has diameter lower than 1, and its border has null measurement. In addition, $\{B^jQ\}_{j\in\mathbb{Z}}$ ($\{T_{Bj}Q\}_{j\in\mathbb{Z}^d}$) is a covering of $\mathbb{R}^d\backslash\{0\}$ ($\mathbb{R}^d$), and $\{h(B^j)\}_{j\in\mathbb{Z}}$ ($\{T_{Bj}g\}_{j\in\mathbb{Z}^d}$) is a Riesz Partition of unity for $L^2(\mathbb{R}^d)$. Then, it is possible to obtain the Parseval frames with good localization properties, after adding conditions to $h(g)$. At the end, we show two examples of building of wavelet Parseval frames and Gabor's Parseval frames with a good decay, as required.

1. Introduction

Mathematical research focuses in developing of new theories, technologies and algorithms for representation, processing, analysis and interpretation of large volumes of data from different disciplines such as communications, geosciences, astronomy and medical sciences, among others. The usefulness of these data, is largely determined by their accessibility and transportability. Thus, the theories of representation that use Gabor and wavelet expansions are within the most accurate mathematical tools for this purpose, and they have found an extensive use in the analysis of signals, processing of images and many other areas. Besides, the frame concept has achieved relevance not only in pure mathematics but also in the applied ones. In the Hilbert's separable spaces, the frames are representation systems of the elements of the space less restrictive than the bases. This advantage is achieved without losing the remarkable condition of reconstruction obtained in the frames, from their duals. Parseval frames, duals of themselves, constitute a powerful tool, because the process of reconstruction or *synthesis* of an element of

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the space from its decomposition or *analysis* in Parseval frames, is very simple. Specifically, wavelet and Gabor's frames have similar features: they are generated from only one function, or from a finite collection of them by applying two countable families of operators in each case, dilations and translations in the first one, and modulations and translations in the second one. On the other hand, the *good localization* is very important in applications, being this property very appreciated by researchers.

The notion of frames was introduced by Duffin and Schaeffer [8], as a new tool to describe expansions of functions in $L^2(-\pi,\pi]$ using exponentials of the type $e^{i\lambda_n x}$ with $\lambda_n \neq 2\pi n$. These types of expansions are known as non harmonic series. Later, several authors have used these frames, ([8], [12], [6], [11]), [4], [5]).

In this article, the principal results are included in sections 3 and 4. In the first one we build wavelet Parseval frames; in the second one, we construct Gabor's Parseval frames. In section 5, we show two examples of this type of constructions.

1.1. *Previous Concepts*

Definition 1.1. Let H be a Hilbert space and I a set of countable indexes. $\{f_i\}_{i\in I} \subset H$ is a frame for H if there exist constants $0 < A \leq B < \infty$ such that:

$$A\|f\|^2 \leq \sum_{j\in I} |\langle f, f_i\rangle|^2 \leq B\|f\|^2 \quad \text{for all } f \in H\,.$$

$\{\langle f, f_i\rangle\}_{i\in I}$ is called *set of frame coefficients* of f respect to $\{f_i\}_{i\in I}$, and it is the result of the process known as *analysis*.
A and B are the constants *lower and upper bounds* of the frame $\{f_i\}_{i\in I}$ respectively. A frame will be called *tight*, if $A = B$, and it will be *Parseval frame*, if $A = B = 1$.

Definition 1.2 (Wavelet frames and Gabor's frames in $L^2(\mathbb{R}^d)$)**.**

(i) Let $\{\psi^l\}_{l=1,2,\dots,n} \subset L^2(\mathbb{R}^d)$, $A \in GL_d(\mathbb{R}))$ expansive, and Γ be a lattice. A wavelet frame of $L^2(\mathbb{R}^d)$ is a frame of the form:

$$\{\psi^l_{j\gamma}(x) := |\det A|^{j/2}\psi^l(A^j x - \gamma))\}_{j\in\mathbb{Z},\gamma\in\Gamma,l=1,\dots,n}\,.$$

(ii) Let $\{\phi^l\}_{l=1,2,\dots,n} \subset L^2(\mathbb{R}^d)$, B and $C \in GL_d(\mathbb{R})$, and Γ be a lattice. A Gabor's frame of $L^2(\mathbb{R}^d)$ is a frame of the form:

$$\{\phi^l_{j\gamma}(x) := e^{2\pi i\langle Bj,x\rangle}\phi^l(x - C\gamma))\}_{j\in\mathbb{Z},\gamma\in\Gamma,l=1,\dots,n}\,.$$

If dilation, modulation and translation operators of $L^2(\mathbb{R}^d)$ in $L^2(\mathbb{R}^d)$ are defined as:

- $(D_{A^j} f)(x) := |\det A|^{j/2} f(A^j x)$
- $(M_z f)(x) := e^{2\pi i\langle z,x\rangle} f(x)$
- $(T_k f)(x) := f(x - k)$

then

$$\psi^l_{j,\gamma} = D_{A^j} T_\gamma \psi^l, \quad \phi^l_{j\gamma}(x) = M_{Bj} T_{C\gamma}\phi^l(x)\,.$$

One essential fact in the frame theory is how to recover f vector from frame coefficients $\{\langle f, f_i\rangle\}_{i\in I}$.

Definition 1.3 (Dual frames). Let $\{f_j\}_{j\in I}$ be a frame for a Hilbert space H, the frame $\{g_j\}_{j\in I}$ is a dual frame for $\{f_j\}_{j\in I}$ if:

$$f = \sum_{k\in I} \langle f, f_k\rangle g_k \quad \text{for all } f \in H \tag{1.1}$$

The frame $\{f_j\}_{j\in I}$ carries out the analysis through a $\{\langle f, f_k\rangle\}_{k\in I}$, and the frame $\{g_j\}_{j\in I}$ makes the process known as synthesis represented by equation (1.1).

For tight frames, is easy to recover f from their frame coefficients $\{\langle f, f_i\rangle\}_{i\in\mathbb{Z}}$:

$$\sum_{j\in I} |\langle f, f_i\rangle|^2 = A\|f\|^2 \quad \Rightarrow^* \quad f = A^{-1}\sum_{j\in I}\langle f, f_i\rangle f_i. \tag{1.2}$$

Then a dual frame, for a tight frame $\{f_i\}_{i\in I}$ is $\{A^{-1}f_i\}_{i\in I}$, where A is the bound of $\{f_i\}_{i\in I}$. A Parseval frame ($A = B = 1$) is dual of itself.

Definition 1.4. Let $\mathscr{S} = \{S_j\}_{j\in J}$ be a covering of $\mathbb{R}^d$ by measurable subsets, with J a countable set of indexes; and $\rho_{\mathscr{S}} : \mathbb{R}^d \to \mathbb{N}\cup\{0\}$ defined as:

$$\rho_{\mathscr{S}}(x) := \#\{j \in J : x \in S_j\} = \sum_{j\in J} \chi_{S_j}(x)$$

where $\#R$ means the cardinal of the set R.

We call *covering index* of $\mathscr{S}$ to $\rho_{\mathscr{S}} := \|\rho_{\mathscr{S}}\|_\infty$.

Definition 1.5. A countable set $\mathscr{H} = \{h_j\}_{j\in J}$ of measurable functions of $\mathbb{R}^d$ is a *Riesz Partition of Unity (RPU)* with bounds p and P ($0 < p \le P < \infty$) if:

$$p \le \sum_{j\in J} |h_j(x)|^2 \le P \quad \text{a.e. } x \in \mathbb{R}^d. \tag{1.3}$$

We use two theorems for the proof of the main results in this work, due to Hern dez et al. [9], which characterize wavelet and Gabor's Parseval frames for $L^2(\mathbb{R}^d)$.

Theorem 1.6. *Let $\Psi = \{\psi_1, \psi_2, \ldots, \psi_l\} \subset L^2(\mathbb{R}^d)$, $A \in GL_d(\mathbb{R})$ be, such that $B = A^t$ expands*† *a subspace F of $\mathbb{R}^d$. Then the system:*

$$\{D_A^j T_k \psi_l : j \in \mathbb{Z},\ k \in \mathbb{Z}^d,\ l = 1\ldots, L\}$$

is a Parseval frame if and only if:

$$\sum_{l=1}^{L}\sum_{j\in P_m} \widehat{\psi}^l(B^{-j}(\xi)\overline{\widehat{\psi}^l(B^{-j}(\xi + m))} = \delta_{m,0} \quad a.e.\ \xi \in \mathbb{R}^d \tag{1.4}$$

for all $m \in \mathbb{Z}^d$, where $P_m = \{j \in \mathbb{Z} :\ B^{-j}m \in \mathbb{Z}^d\}$.

†The authors consider matrices that expand a subspace F of $\mathbb{R}^d$; in our case the subspace F of the previous theorem is all $\mathbb{R}^d$.

Theorem 1.7. *The Gabor system:*

$$\{M_{Bn}T_{Ck}g^l : m,k \in \mathbb{Z}^d,\ l = 1,2,\dots,L\}$$

generated by the finite family $\{g^1, g^2, \dots, g^L\} \subset L^2(\mathbb{R}^d)$ *and the pair of matrices* B *and* C *of* $GL_d(\mathbb{R})$ *is a Parseval frame if and only if:*

$$\sum_{l=1}^{L}\sum_{k\in\mathbb{Z}^d} \frac{1}{|\det C|}\widehat{g^l}(\xi - Bk)\overline{\widehat{g^l}(\xi - Bk + C^I m)} = \delta_{m,0} \tag{1.5}$$

for a.e $\xi \in R^d$*, all* $m \in \mathbb{Z}^d$*, where* $C^I = (C^t)^{-1}$

We will makes use of two lemmas following in our construction, the first of which was demonstrated by Aldroubi et al. [1]:

Lemma 1.8. *Let* $V \subset \mathbb{R}^d$ *be a bounded set such that* $0 \in V^0$*, and* A *a* $d \times d$ *expansive matrix. Let* $Q = AV\backslash V$*, then* $\{A^jQ : j \in \mathbb{Z}\}$ *is a covering of* $\mathbb{R}^d\backslash\{0\}$ *with finite covering index. Furthermore, if* $V \subset AV$ *then the sets* $\{A^jQ :\ j \in \mathbb{Z}\}$ *are disjoint.*

Lemma 1.9. *If* $A \subset GL_d(\mathbb{R})$ *is expansive, then there exists* $Q \subset \mathbb{R}^d$ *with* $\delta(Q) < 1$ *such that* $\{A^jQ\}_{j\in\mathbb{Z}}$ *covers* $\mathbb{R}^d\backslash\{0\}$ *with finite covering index (*$\delta(Q)$ *is the diameter of the set Q).*

In the following section we introduce the first theorem of this article related to the construction of wavelet Parseval frame. Then we analyze conditions that makes possible to built frames with good localization properties.

2. **Construction of Wavelet Parseval Frame**

Theorem 2.1. *Let* A *expansive,* $B = A^t$*, and* $Q \subset \mathbb{R}^d$ *be a measurable subset such that* $\delta(Q) < 1$*,* $\mu(\partial Q) = 0$ *and* $\{B^jQ\}_{j\in\mathbb{Z}}$ *is a covering of* $\mathbb{R}^d\backslash\{0\}$‡*. Let* h *be a measurable function with* supp $h \subset Q$*, such that* $\mathscr{H} = \{h_j := h(B^{-j})\}_{j\in\mathbb{Z}}$ *be RPU; then the system given by:*

$$\{|detA|^{\frac{j}{2}}\eta(A^jx - k) :\ j \in \mathbb{Z}, k \in \mathbb{Z}^d\} \tag{2.1}$$

is a Parseval frame for $L^2(\mathbb{R}^d)$*, where:*

$$\widehat{\eta}(\xi) := \begin{cases} \dfrac{h(\xi)}{\sqrt{\sum_{j\in\mathbb{Z}}|h(B^{-j}(\xi))|^2}} & si\ \xi \in \mathbb{R}^d\backslash S \\ 0 & si\ \xi \in S \end{cases} \tag{2.2}$$

being $S := \{\xi \in \mathbb{R}^d : \Sigma_{j\in\mathbb{Z}}|h(B^{-j}\xi)|^2 < p \vee \Sigma_{j\in\mathbb{Z}}|h(B^{-j}\xi)|^2 > P\}$*; with* p *and* P *bounds of the RPU* $\mathscr{H}$*. (The set* S *has null measure since* $\{h_j\}_{j\in\mathbb{Z}}$ *is RPU).*

‡If Q is built as in the Lemma 1.9, $\{B^jQ\}_{j\in\mathbb{Z}}$ has a finite covering index.

Note that Daubechies and Han [7] have obtained a similar result as the one presented here. They worked in $L^2(\mathbb{R})$ with dyadic dilation, but with different hypothesis from ours. It is necessary to note that the idea for the construction of the function $\widehat{\eta}$ has been taken from their work.

Proof. $\mathscr{H}$ is PRU with bounds p and P, then

$$p \leq \sum_{j\in\mathbb{Z}} |h_j(\xi)|^2 = \sum_{j\in\mathbb{Z}} |h(B^{-j}(\xi)|^2 \leq P \quad \text{a.e. } \xi\,.$$

This shows that $\widehat{\eta}$ is well defined a.e. According to the theorem 1.6, the system (2.1) is a Parseval frame for $L^2(\mathbb{R}^d)$ if and only if:

$$\sum_{j\in P_m} \widehat{\eta}(B^{-j}(\xi))\overline{\widehat{\eta}(B^{-j}(\xi+m))} = \delta_{m0} \quad \text{a.e. } \xi. \tag{2.3}$$

for all $m \in \mathbb{Z}^d$, being $P_m = \{j \in \mathbb{Z} : B^{-j}m \in \mathbb{Z}^d\}$.

We will prove (2.3) for our hypothesis:

(i) Let $m = 0$. We have $P_0 = \{j \in \mathbb{Z} : B^{-j}0 \in \mathbb{Z}^d\} = \mathbb{Z}$.
The proof of (2.3) for $m = 0$ is:

$$|\widehat{\eta}(B^{-j}(\xi))|^2 = \frac{|h(B^{-j}(\xi)|^2}{\sum_{k\in\mathbb{Z}} |h(B^{-k}B^{-j}(\xi)|^2} = \frac{|h(B^{-j}(\xi)|^2}{\sum_{k\in\mathbb{Z}} |h(B^{-k}(\xi)|^2}$$

then:

$$\sum_{j\in\mathbb{Z}} |\widehat{\eta}(B^{-j}(\xi))|^2 = \frac{\sum_{j\in\mathbb{Z}} |h(B^{-j}(\xi)|^2}{\sum_{k\in\mathbb{Z}} |h(B^{-k}(\xi)|^2} = 1 \quad \text{a.e. } \xi \tag{2.4}$$

(ii) Let $m \neq 0$. We know that $\operatorname{supp}\widehat{\eta} = \operatorname{supp} h \subset Q$.
Let $\xi \in \mathbb{R}^d$, $m \neq 0$ and $j \in P_m$ such that $B^{-j}(\xi) \in \operatorname{supp}\widehat{\eta}$.
If $\widehat{\eta}(B^{-j}(\xi) \neq 0 \Rightarrow \widehat{\eta}(B^{-j}(\xi+m)) = 0$ for all $j \in P_m$:

$$B^{-j}(m) = k \in \mathbb{Z}^d \quad \Rightarrow \quad B^{-j}(\xi+m) = B^{-j}(\xi) + k\,.$$

If we assume that

$$B^{-j}(\xi+m) \in Q \quad \Rightarrow \quad \|B^{-j}(\xi+m) - B^{-j}(\xi)\| = \|B^{-j}(m)\| = \|k\| \geq 1,$$

this is an absurd since $\delta(Q) < 1$. Then

$$\sum_{j\in P_m} \widehat{\eta}(B^{-j}(\xi))\widehat{\eta}(B^{-j}(\xi+m)) = 0\,. \qquad \square$$

Corollary 2.2. *If h is a function at real values,* $\operatorname{supp} h = Q$ *and* $\{B^jQ : j \in \mathbb{Z}\}$ *is a covering for almost disjoint of* $\mathbb{R}^d\backslash\{0\}$ $(\mu(B^jQ \cap B^kQ) = 0$ *if* $j \neq k)$, *then the system given by:*

$$\{|\det A|^{\frac{j}{2}} \chi_Q^{\vee}(A^j x - k) : j \in \mathbb{Z},\ k \in \mathbb{Z}^d\} \tag{2.5}$$

is a Parseval frame for $L^2(\mathbb{R}^d)$.

Proof. If $\{B^j Q : j \in \mathbb{Z}\}$ is a covering for almost disjoint of $\mathbb{R}^d \backslash \{0\}$, the function $\widehat{\eta}$ of the Theorem 2.1 is exactly χ_Q, except a set of null measurement $(S \cap Q)$. □

The frame obtained in Corollary 2.2 has no good decay. In the next corollary of the Theorem 2.1 we show that under certain conditions the wavelet Parseval frames with good localization properties can be built.

Corollary 2.3. *If* $p \leq \sum_{j\in\mathbb{Z}} |h(B^{-j}(\xi))|^2 \leq P$ *for all* $\xi \in \mathbb{R}^d \backslash \{0\}$*,* $h \in \mathscr{C}^r$ *and* $0 \notin \overline{Q}$*, then the function* $\widehat{\eta}$ *of the Theorem 2.1 is of the class*[§] $\mathscr{C}^r$.

In order to demonstrate this corollary, a result over matrix norms is introduced, this can be seen in other works (for example [10]).

Lemma 2.4. *Let* A *be a matrix of the order* $n \times n$ *and* $\varepsilon > 0$*, then there exists a matrix norm* $\|\cdot\|$ *such that*

$$\rho(A) \leq \|A\| \leq \rho(A) + \varepsilon \tag{2.6}$$

where $\rho(A)$ *is the spectral radius of the matrix* A*, defined as the maximum of the set of modules of eigenvalues of the matrix* A*.*

***Proof of the Corollary* 2.3.** Because $\operatorname{supp} h \subset Q$ and Q is bounded, h is a compactly supported function. Let's assume that h is a real function. We will to prove that $\widehat{\eta}$ is continuous a.e, for which we will observe that the series $\sum_{j\in\mathbb{Z}} (h(B^j(\xi))^2$ uniformly converges over each compact that does not contain the zero.

According to the hypothesis $0 \notin \overline{Q}$, then there exists $\varepsilon > 0$ such that $0 \notin Q_\varepsilon$ being $Q_\varepsilon := \{x \in \mathbb{R}^d : d(x, \overline{Q}) < \varepsilon\}$.

$Q \subset Q_\varepsilon$, then $\{B^j Q_\varepsilon : j \in \mathbb{Z}\}$ is a covering for open of $\mathbb{R}^d \backslash \{0\}$.

Let $K \subset \mathbb{R}^d$ be a compact set such that $0 \notin K$, then $K \subset \mathbb{R}^d \backslash \{0\}$, there exists a finite amount of integers $j_1, \ldots, j_t$ such that:

$$K \subset \bigcup_{i=1}^{t} B^{j_i} Q_\varepsilon \,. \tag{2.7}$$

We will prove that

$$K \cap B^j Q = \emptyset \quad \text{for all } j \notin \{j_1, \ldots, j_t\} \,. \tag{2.8}$$

- Because K is compact, then K is bounded. In addition $0 \notin K$, then there exists $r > 0$ and $R < \infty$ such that:
  - $B(0, r) \cap K = \emptyset$, and
  - $\|\xi\| \leq R$ for all $\xi \in K$,

[§]A function f is of $\mathscr{C}^r$ class if $\frac{\partial^s f}{\partial \xi_1^{i_1} \partial \xi_2^{i_2} \ldots \partial \xi_d^{i_d}}(\xi)$ exists and is continuous for all set $\{i_1, i_2, \ldots, i_d\} \subset \mathbb{N}$ such that $i_1 + i_2 + \ldots + i_d = s \leq r$ and for all $\xi \in \mathbb{R}^d$.

later:

$$r \leq \|\xi\| \leq R \quad \text{for all } \xi \in K \tag{2.9}$$

- There exist: $c_1 > 1$ and $c_2 < 1$ such that for all $\xi \in \mathbb{R}^d$:

$$\|B^j \xi\| \geq c_1^j \|\xi\| \quad \text{for all } j \geq 0 \text{ and } \|B^j \xi\| \leq c_2^{-j} \|\xi\|, \text{ for all } j < 0. \tag{2.10}$$

  The inequalities in (2.10) come from that B is an expansive matrix, $\rho(B) > 1$ and thus $\rho(B^{-1}) < 1$. According to (2.6) there exists a matrix norm $\|\cdot\|$ such that $\|B^{-1}\| \leq c_2 < 1$ and then $\|B\xi\| \geq c_1\|\xi\|$ (it is enough to consider $c_1 = 1/c_2$).
- Because $\overline{Q}$ is compact and $0 \notin \overline{Q}$, then exist $0 < \widetilde{r} \leq \widetilde{R} < \infty$, as in the previous analysis for K (see above), such that:

$$\widetilde{r} \leq \|\xi\| \leq \widetilde{R} \quad \text{for all } \xi \in \overline{Q}. \tag{2.11}$$

Using (2.9), (2.10) and (2.11) we are will prove the stated in (2.8)as follow:

Suppose that $K \cap B^j Q \neq \emptyset$ for non finite $j \in \mathbb{Z}$, then two possibilities exist:

(i) $K \cap B^j Q \neq \emptyset$ for all $j \in J$ with $J \subset \mathbb{N}$ of non finite cardinality, then for all $j \in J$ there exists $\xi_j \in K$ and $q \in Q$ such that $\xi_j = B^j q$, and:

$$R \geq \|\xi_j\| = \|B^j q\| \geq c_1^j \|q\| \geq c_1^j \widetilde{r}$$

this is an absurd due to $\widetilde{r} > 0$, $R < \infty$ and $c_1 > 1$. It is enough to consider j sufficiently large;

(ii) $K \cap B^j Q \neq \emptyset$ for $j \in J_1$, where J_1 is a subset of integers lower than 0, and the cardinal of J_1 is non finite. Then for all $j \in J_1$ there exists $\xi_j \in K$ and $q \in Q$ such that $\xi_j = B^j q$, then:

$$r \leq \|\xi_j\| = \|B^j q\| \leq c_2^{-j} \|q\| \leq c_2^{-j} \widetilde{R}$$

this is an absurd due to $r > 0$, $\widetilde{R} < \infty$ and $c_2 < 1$. It is sufficient to consider $j \in J_1$ of absolute value as large as it is required.

The expression (2.8) is valid according to (1) and (2). Then, it can be warranted that there exists $N \in \mathbb{N}$ such that:

$$h(B^{-j}(\xi)) = 0 \quad \text{for all } \xi \in K \ \wedge \ \text{for all } j : |j| \geq N \tag{2.12}$$

whereas $N = \max\{|j_1|, \ldots, |j_t|\} + 1$:

If $|j| \geq N$ then $j \notin \{j_1, \ldots j_t\}$ so $B^j Q \bigcap K = \emptyset$. Because $\operatorname{supp} h(B^j.) \subset B^{-j} Q$, the expression (2.12) is verified. Then:

$$\sum_{|j| \geq n} (h(B^j \xi))^2 = 0 \quad \xi \in K, \text{ for all } n \geq N \tag{2.13}$$

The equation (2.13) ensures the uniform convergence of the series $\sum_{j \in \mathbb{Z}} (h(B^j(\xi))^2$ over the compact K, and as functions $h(B^j.)$ are of class $\mathscr{C}^r$ over K, the function $\widehat{\eta}$ is of class $\mathscr{C}^r$ for being quotient of functions of class $\mathscr{C}^r$ for which the denominator is different from zero. □

Observation 2.5. From the previous corollaries it can be derived that:

(i) Because that the frame determined in (2.1) has good decay properties (h is smooth enough), the covering $\{B^j Q : j \in \mathbb{Z}\}$ of $\mathbb{R}^d \backslash \{0\}$ can not be for almost disjoint, i.e. the covering index must be strictly higher than 1.

(ii) If h is of class $\mathscr{C}^r$ and $\mathscr{H} = \{h_j := h(B^{-j})\}_{j\in\mathbb{Z}}$ is a Riesz partition of the unity with $p \leq \sum_{j\in\mathbb{Z}} |h(B^{-j}(\xi))|^2 \leq P$ for all $\xi \in \mathbb{R}^d \backslash \{0\}$, then the frame (2.1) determined in Theorem 2.1 has good localization properties, since their elements have polynomial decay if r is finite, and belong to the Schwartz class for $h \in \mathscr{C}^\infty$.

3. Construction of Gabor's Parseval Frames

We present now the second theorem in this work, from which we build Gabor's Parseval frames.

Theorem 3.1. *Let B and $C \in GL_d(\mathbb{R}^d)$ such that $\mathbb{Z}^d \subseteq C^t\mathbb{Z}^d$. Let $Q \subset \mathbb{R}^d$ such that $\delta(Q) < 1$ and $\mu(\partial Q) = 0$. Be $g \in L^2(\mathbb{R}^d)$ such that $\operatorname{supp} g \subseteq Q$. If $\widetilde{\mathscr{Q}} = \{Q_k := T_{Bk}Q : k \in \mathbb{Z}^d\}$ covers $\mathbb{R}^d$ and $\mathscr{G} = \{g_j := T_{Bj}g\}_{j\in\mathbb{Z}^d}$ is a Riesz partition of the unity with bounds p and P, then the Gabor system.*

$$\{M_{Bm}T_{Cn}\eta = e^{2\pi i\langle Bm,\cdot\rangle}\eta(\cdot\ - Cn) : m \in \mathbb{Z}^d, n \in \mathbb{Z}^d\} \tag{3.1}$$

is a Parseval frame for $L^2(\mathbb{R}^d)$, being

$$\widehat{\eta}(\xi) := \frac{g(\xi)\sqrt{|\det C|}}{\sqrt{\sum_{j\in\mathbb{Z}} |g(\xi - Bj)|^2}} \quad a.e.\ \xi\,. \tag{3.2}$$

Proof. Since $\mathscr{G}$ is RPU then $\widehat{\eta}$ is well defined. According to the Theorem 1.7, the system given in (3.1) will be a Parseval frame if we prove the equation (1.5) of the Theorem 1.7 for all $m \in \mathbb{Z}^d$:

(i) If $m = 0$, it must be verified that:

$$\sum_{k\in\mathbb{Z}^d} \frac{1}{|\det C|} |\widehat{\eta}(\xi - Bk)|^2 = 1\,. \tag{3.3}$$

The equation (3.3) is fulfilled due to

$$\sum_{k\in\mathbb{Z}^d} |\widehat{\eta}(\xi - Bk)|^2 = \sum_{k\in\mathbb{Z}^d} \left(\frac{g(\xi - Bk)\sqrt{|\det C|}}{\sqrt{\sum_{j\in\mathbb{Z}} |g(\xi - Bj)|^2}} \right)^2 = |\det C|\,.$$

(ii) If $m \neq 0$ we must prove that :

$$\sum_{k\in\mathbb{Z}^d} \frac{1}{|\det C|} \widehat{\eta}(\xi - Bk)\ \widehat{\eta}(\xi - Bk + C^I m) = 0 \tag{3.4}$$

being $C^I = (C^t)^{-1}$.

We observe that $\operatorname{supp}\widehat{\eta} = \operatorname{supp} g$. We consider $\xi \in R^d$ such that $\widehat{\eta}(\xi - Bk) \neq 0$. If $\widehat{\eta}(\xi - Bk + C^I m) \neq 0$, then $(\xi - Bk + C^I m)$ belongs to $\operatorname{supp}\widehat{\eta} \subset Q$, then

$$\|\xi - Bk - (\xi - Bk + C^I m)\| = \|C^I m\| = \|(C^t)^{-1} m\| \geq 1\,. \tag{3.5}$$

The expression (3.5) is an absurd due to $\delta(Q) < 1$, (note that the inequality in the previous expression is due to the hypothesis $\mathbb{Z}^d \subseteq C^t\mathbb{Z}^d$).

According to the Theorem 1.7, equations (3.3) and (3.4) confirm that the Gabor system given in (3.1) is a Parseval frame for $L^2(\mathbb{R}^d)$. □

Corollary 3.2. *If $\mathcal{Q}$ covers $\mathbb{R}^d$ with covering index equal to 1, then the function in Theorem 3.1 is $\widehat{\eta}(\xi) = \chi_Q(\xi)\sqrt{|\det C|}$ a.e. $\xi \in \mathbb{R}^d$. In this case the Gabor's frame given in (3.1) does not have a good decay.*

Corollary 3.3. *If $0 < p \leq \sum_{j\in\mathbb{Z}^d} |T_{Bj}g(\xi)|^2 \leq P < \infty$ for all $\xi \in \mathbb{R}^d$, $g \in \mathscr{C}^r$ and $0 \notin \overline{Q}$, then the function $\widehat{\eta}$ of the theorem 3.1 is of the class $\mathscr{C}^r$.*

Proof. Because $\operatorname{supp} g \subset Q$ and Q is bounded, then g has compact support. Let assume that g is a function at real values, and that $\sum_{j\in\mathbb{Z}^d} |T_{Bj}g|^2$ uniformly converges over compact sets. Let $K \subset \mathbb{R}^d$ be a compact set. Because K is compact, its diameter $(\delta(K))$ is finite.

$0 \notin \overline{Q} \Rightarrow \exists\ \varepsilon > 0 : 0 \notin Q_\varepsilon := \{x \in \mathbb{R}^d : d(x,Q) < \varepsilon\}$. Because $Q \subset Q_\varepsilon$ then $\{T_{Bk}Q_\varepsilon\}_{k\in\mathbb{Z}^d}$ is a covering by open subsets of $\mathbb{R}^d$. There is a finite amount of integers $j_1, j_2, \ldots, j_n$ such that

$$K \subset \bigcup_{i=1}^{n} T_{Bj_i}Q_\varepsilon\,. \tag{3.6}$$

Let prove that

$$K \cap T_{Bj}Q = \emptyset \quad \text{for all } j \in \mathbb{Z}^d \backslash \{j_1, j_2, \ldots, j_n\}\,. \tag{3.7}$$

(a) If there is only one integer $j_{n+1} \notin \{j_1, j_2, \ldots, j_n\}$ such that $K \cap T_{Bj_{n+1}}Q \neq \emptyset$, then $K \subset \bigcup_{i=1}^{n+1} T_{Bj_i}Q_\varepsilon$, and (3.7) is valid for all $j \in \mathbb{Z}^d\backslash\{j_1, j_2, \ldots, j_n, j_{n+1}\}$.

(b) If there is a finite set $\{j^1, \ldots j^l\} \subset \mathbb{Z}^d$ such that $K \cap T_{Bj^i}Q \neq \emptyset$, with $\{j_1, j_2, \ldots, j_n\} \cap \{j^1, \ldots j^l\} = \emptyset$, expressions (3.6) and (3.7) are verified similarly to the previous case.

(c) Let assume that there is a finite amount of integers for which $K \cap T_{Bj}Q \neq \emptyset$, then we can choose two of them j and j' such that:

$$\|B(j - j')\| \geq 1 + \delta(K)\,. \tag{3.8}$$

Because $K \cap T_{Bj}Q \neq \emptyset$ and $K \cap T_{Bj'}Q \neq \emptyset$, there are q and q' in Q, ξ and ξ' in K such that $q = \xi + Bj$ and $q' = \xi' + Bj'$. Then

$$\|q - q'\| = \|\xi - \xi' + B(j - j')\| \geq \Big|\ \|B(j - j')\| - \|\xi - \xi'\|\ \Big|\,. \tag{3.9}$$

Because $\|\xi - \xi'\| \leq \delta(K)$, due to (3.8) we obtain:

$$\Big| \, \|B(j-j')\| - \|\xi - \xi'\| \, \Big| = \|B(j-j')\| - \|\xi - \xi'\| \\ \geq \|B(j-j')\| - \delta(K) \geq 1\,. \tag{3.10}$$

From (3.9) and (3.10) we obtain $\|q - q'\| \geq 1$, which is absurd since the diameter of Q is strictly smaller than 1.

With (a), (b) and (c) we have proved (3.7); then there is $N \geq \{|j_1|, |j_2|, \dots |j_n|\}$ such that

$$\sum_{|j| \geq N} |g(\xi - Bj)|^2 = 0 \quad \text{for all } \xi \in \mathbb{R}^d\,. \tag{3.11}$$

The expression (3.11) ensures the uniform convergence of the series $\sum_{j \in \mathbb{Z}^d} |g(\xi - Bj)|^2$ over each compact set in $\mathbb{R}^d$: because $g \in \mathscr{C}^r$, the series is of $\mathscr{C}^r$ class. Then $\widehat{\eta}$ defined in Theorem 3.1 results of class $\mathscr{C}^r$. □

Observation 3.4. Similarly to the analysis for wavelet Parseval frames, it is derived for the construction of Gabor's Parseval Frames, from the previous corollaries it is deducted that:

(i) Because the frame determined in (3.1) has good decay properties (g is smooth enough), the covering $\{T_{Bj}Q\}_{j \in \mathbb{Z}}$ of $\mathbb{R}^d$ can not be by almost disjoint, i.e the covering index must be strictly larger that 1.

(ii) If g is of class $\mathscr{C}^r$ and $\mathscr{G} = \{g_j := T_{Bj}g\}_{j \in \mathbb{Z}^d}$ is a Riesz partition of the unity with $p \leq \sum_{j \in \mathbb{Z}} |T_{Bj}g(\xi))|^2 \leq P$ for all $\xi \in \mathbb{R}^d$, then the frame (3.1) determined in Theorem 3.1 has good localization properties, since their elements have polynomial decay in case of being finite r, and belong to the Schwartz class for $g \in \mathscr{C}^\infty$.

Next, we will show two examples derived from a frame construction of the paper [1].

4. **Examples**

Example 4.1. Let $Q = \{(\xi_1, \xi_2) \in \mathbb{R}^2 : 1/4^2 \leq \xi_1^2 + \xi_2^2 \leq 5/4^2\}$, $A = 2I_d$, and $h(\xi_1, \xi_2) := n\beta_{n-1}((\xi_1^2 + \xi_2^2 - 1/4^2)4.n)$, where β_{n-1} is the function β-spline of degree $n-1$ whose support is the real interval $[0, n]$. It can be observed that:

(a) $\operatorname{supp} h = Q$, and $|h(\xi_1, \xi_2)|^2 \leq n^2$, (due to the property of β-spline functions: $\sum_{k \in \mathbb{Z}} \beta_s(x - k) = 1$ for all $x \in \mathbb{R}$, for all $s \in \mathbb{N}_0$).

(b) $Q \subset B_{1/2}(0)$, then $\delta(Q) < 1$.

(c) $\mathscr{Q} =: \{2^j Q\}_{j \in \mathbb{Z}}$ covers $\mathbb{R}^2 \backslash \{0\}$ with covering index $\rho_{\mathscr{Q}} = 2$:
- $2^{-j}Q = \{(\xi_1, \xi_2) \in \mathbb{R}^2 : \ 1/4^{j+2} \leq \xi_1^2 + \xi_2^2 \leq 5/4^{j+2}\}$
- intervals $[\frac{1}{4^j}, \frac{5}{4^j}]$ cover the set $\mathbb{R}^+ \backslash \{0\}$ with covering index equal to 2.

From the stated above it is derived that $\rho_{\mathscr{Q}} = 2$.

(d) h is a function of $\mathbb{R}^2 \longrightarrow \mathbb{R}$ determined by the composition of two functions, $\widetilde{\beta}_{n-1}$ and H:
- $\widetilde{\beta}_{n-1} : \mathbb{R} \longrightarrow \mathbb{R}$ defined by $\widetilde{\beta}_{n-1}(t) := n\beta_{n-1}(t)$, and
- $H : \mathbb{R}^2 \longrightarrow \mathbb{R}$ defined by $H(\xi_1, \xi_2) := (\xi_1^2 + \xi_2^2 - 1/16)4n$

i.e. $h = \widetilde{\beta}_{n-1}\ o\ H$, with $\widetilde{\beta}_{n-1} \in \mathscr{C}^{n-2}$ and $H \in \mathscr{C}^{\infty}$, then $h \in \mathscr{C}^{n-2}$.

(e) If for all $j \in \mathbb{Z}$ we define $h_j(\xi_1, \xi_2) := h(2^j(\xi_1, \xi_2))$, then $\mathscr{H} = \{h_j\}_{j\in\mathbb{Z}}$ is a Riesz partition of the unity:
- $\operatorname{supp} h_j = 2^{-j}Q$
- Because to c) given $(\xi_1, \xi_2) \in \mathbb{R}^2$ there exists at least one or at most two consecutive integer indexes, j and $j+1$ such that $(\xi_1, \xi_2) \in 2^{-j}Q \cup 2^{-j-1}Q$. Then:

$$\sum_{t\in\mathbb{Z}} |h_t(\xi_1, \xi_2)|^2 \le |h_j(\xi_1, \xi_2)|^2 + |h_{j+1}(\xi_1, \xi_2))|^2 \le 2.n^2 \tag{4.1}$$

The other inequality is obtained form observing that for each $j \in \mathbb{Z}$ exists at least one point $(\widetilde{\xi}_1, \widetilde{\xi}_2) \in K := 2^{-j}Q \cap 2^{-j-1}Q$ such that:

$$h_j(\widetilde{\xi}_1, \widetilde{\xi}_2) = h_{j+1}(\widetilde{\xi}_1, \widetilde{\xi}_2) \tag{4.2}$$

We define $\widetilde{K} := \{(\widetilde{\xi}_1, \widetilde{\xi}_2) \in K :\ h_j(\widetilde{\xi}_1, \widetilde{\xi}_2) = h_{j+1}(\widetilde{\xi}_1, \widetilde{\xi}_2)\}$.
If $(\widetilde{\xi}_1, \widetilde{\xi}_2) \in \widetilde{K}$ then

$$\beta_{n-1}((4^j(\widetilde{\xi_1}^2 + \widetilde{\xi_2}^2) - 1/16)4n) = \beta_{n-1}((4^{j+1}(\widetilde{\xi_1}^2 + \widetilde{\xi_2}^2) - 1/16)4n) \tag{4.3}$$

Considering the symmetry of β-spline functions with respect to the middle point of its support ($n/2$ in case of β_{n-1}) function, according to equation (4.3) it can be seen that there is $(\widetilde{\xi}_1, \widetilde{\xi}_2)$ which verifies:

$$4^j(\widetilde{\xi_1}^2 + \widetilde{\xi_2}^2) = 1/2 \Rightarrow (\widetilde{\xi}_1^2 + \widetilde{\xi}_2^2) \notin \{1/4^{j+1},\ 5/4^{j+1},\ 1/4^j,\ 5/4^j\} \tag{4.4}$$

Then $m := h_j(\widetilde{\xi}_1, \widetilde{\xi}_2) > 0$[¶].
With the help of (4.2) and the properties of β-spline functions:

$$|h_j(\xi_1, \xi_2)|^2 + |h_{j+1}(\xi_1, \xi_2)|^2 \ge |h_j(\xi_1, \xi_2)|^2 \ge |h_j(\widetilde{\xi}_1, \widetilde{\xi}_2))|^2 = m \tag{4.5}$$

if $\xi_1^2 + \xi_2^2 \ge \widetilde{\xi_1}^2 + \widetilde{\xi_2}^2$, and

$$|h_j(\xi_1, \xi_2)|^2 + |h_{j+1}(\xi_1, \xi_2)|^2 \ge |h_{j+1}(\xi_1, \xi_2))|^2 \ge |h_{j+1}(\widetilde{\xi}_1, \widetilde{\xi}_2))|^2 = m \tag{4.6}$$

if $\xi_1^2 + \xi_2^2 \le \widetilde{\xi_1}^2 + \widetilde{\xi_2}^2$.
Form (4.1), (4.5) and (4.6) it can be derived that:

$$m \le \sum_{j\in\mathbb{Z}} |h_j(\xi_1, \xi_2)|^2 \le 2n^2 \quad \text{for all } (\xi_1, \xi_2) \in \mathbb{R}^2 .$$

[¶]Since $4^j(\xi_1^2 + \xi_2^2) = \frac{1}{2}$ for any j such that $h_j(\xi_1, \xi_2) = h_{j+1}(\xi_1, \xi_2)$. Thus m is unique.

Thus $\mathscr{H} = \{h_j\}_{j\in\mathbb{Z}}$ is PRU with bounds m and $2n^2$. It verifies the conditions of Theorem 2.1 and of the Corollary 2.3. Then:

$$\{4^{\frac{j}{2}}\eta(2^j x - k)\}_{j\in\mathbb{Z},k\in\mathbb{Z}^2} \tag{4.7}$$

is a Parseval frame for $L^2(\mathbb{R}^2)$, with polynomial decay, being

$$\widehat{\eta}(\xi) = \frac{h(\xi)}{\sqrt{\sum_{j\in\mathbb{Z}}((h(2^j\xi))^2}} \quad \text{a.e } \xi \in \mathbb{R}^2 .$$

The next example, following the methodology of Example 4.1, shows the construction of a Gabor's Parseval frame for $L^2(\mathbb{R})$.

Example 4.2. Let $Q := [-5/16, -1/16] \cup [1/16, 5/16] \subset \mathbb{R}$, $B = 1/8$, $C = 1$, and $g(\xi) := n\beta_{n-1}((|\xi| - 1/4^2)4.n)$. If:

(i) $\mathscr{Q} =: \{Q - \frac{1}{8}j\}_{j\in\mathbb{Z}}$, then $\mathscr{Q}$ covers $\mathbb{R}$ with covering index $\rho_{\mathscr{Q}} = 2$.

(ii) $\mathscr{G} = \{g_j := T_{\frac{1}{8}j}\ g\}_{j\in\mathbb{Z}}$, then $\mathscr{G}$ is a RPU for $L^2(\mathbb{R}^d)$ which verifies the requirements of Corollary 3.3.

Thus conditions of Theorem 3.1 and of the Corollary 3.3 are satisfies. Then:

$$\{M_{j/8}T_k\eta = e^{2\pi i<j/8,\ \cdot>}\eta(\ \cdot - k)\}_{j\in\mathbb{Z},k\in\mathbb{Z}} \tag{4.8}$$

is a Parseval frame for $L^2(\mathbb{R})$ with polynomial decay, being

$$\widehat{\eta}(\xi) := \frac{g(\xi)}{\sqrt{\sum_{j\in\mathbb{Z}}|g(\xi - j/8)|^2}} \quad \text{a.e. } \xi . \tag{4.9}$$

5. Appendix

Proof of the Lemma **1.9.** Let V be an open subset such that $0 \in V \subset B_r(0)$, with $0 < r < \frac{1}{2\|A\|}$. See that $\delta(AV) < 1$:

$$\begin{aligned}\delta(AV) &= \sup_{v_1,v_2\in V} \|Av_1 - Av_2\| = \sup_{v_1,v_2\in V} \|A(v_1 - v_2)\| \\ &\le \|A\|\delta(V) < \|A\|\ 2r < 1.\end{aligned}$$

Considering $Q = AV \backslash V$, it is possible to see:

(i) $Q \subset AV$, then $\delta(Q) < 1$

(ii) According to Lemma 1.8 $\{A^jQ\}_{j\in\mathbb{Z}}$ covers $\mathbb{R}^d\backslash\{0\}$ with a finite covering index. □

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