Journal of Informatics and Mathematical Sciences
Volume 4 (2012), Number 3, pp. 319–324

http://www.rgnpublications.com

On Centraloid Operators

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Abstract. In this paper we study centraloid operators, some of their properties and their relation to their adjoint, their Hilbert adjoint, isometries and normal operators. Also we show that the set of all centraloid operators need not be a Banach algebra.

1. **Introduction**

In this paper we define and study a new kind of operators called centraloid operator, where we study some of its properties and its relations with its adjoin, its Hilbert adjoin, isometries and normal operators.

Let X be a normed space over the complex field $\mathbb{C}$. $BL(X)$ will denote the complex normed algebra of bounded linear operators on X. $T^{\times}$ will denote the adjoint of T, $T^{\times} \in BL(X')$ and $\|T\| = \|T^{\times}\|$, where X' is the dual space of X [2, pp. 232]. A bounded linear operator T on X is called an isometry if $\|Tx\| = \|x\|$ for all $x \in X$. The center of $A = BL(X)$ is denoted by $Z(A)$ and is defined by $Z(A) = \{T \in A : TS = ST \text{ for all } S \in A\}$.

Let H be a Hilbert space over the complex field $\mathbb{C}$. The Hilbert adjoint operator of T is denoted by T^*, where $T^* \in BL(H)$ and $\|T\| = \|T^*\|$ [2, pp. 196]. T is called normal in the case $TT^* = T^*T$, and T is called unitary in the case T is bijective and $T^* = T^{-1}$ [2, pp. 201]. Also, T is called unitarily equivalent to $S \in BL(H)$ if there is a unitary operator U on H such that $S = UTU^{-1}$ [2, pp. 207]. Let H and K be two Hilbert spaces and let

$$H \oplus K = \{h \oplus k : h \in H, k \in K\}$$

and

$$\langle h_1 \oplus k_1, h_2 \oplus k_2 \rangle = \langle h_1, h_2 \rangle + \langle k_1, k_2 \rangle.$$

Then $H \oplus K$ is a Hilbert space, is called the direct sum of H and K [3, pp. 24]. If $T_1 \in BL(H)$ and $T_2 \in BL(K)$ we use $T_1 \oplus T_2$ to denote the operator on $H \oplus K$ defined by: $(T_1 \oplus T_2)(x \oplus y) = T_1x \oplus T_2y$ [4, pp. 8]. It is easy see that the norm of $T_1 \oplus T_2$ will be given by

$$\|T_1 \oplus T_2\| = \max\{\|T_1\|, \|T_2\|\}.$$

In [1, Theorem 5], As'ad and Sarsour proved that "If B is a Banach algebra with unity over the complex field $\mathbb{C}$ and if $a \in B$ is such that $\|x(\lambda - a)\| \leq \|(\lambda - a)x\|$ for all $x \in B$ and all $\lambda \in \mathbb{C}$ that satisfies $|\lambda| > \|a\|$, then $a \in Z(B)$."

Throughout this paper, all linear spaces and algebras are assumed to be defined over $\mathbb{C}$, the field of complex numbers.

2. Centraloid Operators on a Normed Space

In this section we define the centraloid operators and study some of its properties. Also we give examples to explain the relation with other known kinds of operators.

Definition 2.1. A bounded linear operator T on a normed space X is called centraloid if it satisfies the condition $\|ST\| \leq \|TS\|$ for all $S \in BL(X)$.

It is clear from the definition that the zero and the identity operators are centraloid operators, moreover any operator in the center of $BL(X)$ is centraloid. However, in Example 2.3 below we show that there is an operator T which is centraloid but is not in the center of $BL(X)$. First we start with the following theorem.

Theorem 2.2. Let $T \in BL(X)$, where X is a normed space.

(i) If $\|TS\| = \|T\|\|S\|$ for all $S \in BL(X)$, then T is centraloid.
(ii) If T is an isometry, then T is centraloid.

Proof. (i) Obvious

(ii) Let T be an isometry on X. For any $S \in BL(X)$ and any $x \in X$, $\|TSx\| = \|Sx\|$, so that $\|TS\| = \|S\|$. However, $\|T\| = 1$. Therefore, by using (i) we get the result. $\square$

Example 2.3. There is a centraloid operator T on ℓ^2 that is not in the center of $BL(\ell^2)$.

Construction. Consider the bounded linear operator $T : \ell^2 \to \ell^2$ that is defined by $T(\xi_1, \xi_2, \xi_3, \ldots) = (0, \xi_1, \xi_2, \xi_3, \ldots)$. By [2, pp. 206], T is an isometry, and by Theorem 2.2(ii) T is centraloid. Now, consider the bounded linear operator $S : \ell^2 \to \ell^2$ is defined by $S(\xi_1, \xi_2, \xi_3, \ldots) = (\xi_1, 0, 0, 0, \ldots)$. It is clear that $ST \neq TS$. Therefore, T is not in the center of $BL(\ell^2)$.

The following proposition shows that under some conditions centraloid operators are in the center. Moreover it explains under what conditions a bounded linear operator T may be centraloid.

Proposition 2.4. If T is a bounded linear operator on a Banach space X and $(T-\lambda I)$ is centraloid for all $\lambda \in C$ satisfies $|\lambda| > \|T\|$, then $(T-\lambda I) \in Z(BL(X))$ for all $\lambda \in \mathbb{C}$.

Proof. Since $(T-\lambda I)$ is centraloid for all $\lambda \in \mathbb{C}$ that satisfies $|\lambda| > \|T\|$, then $\|S(T-\lambda I)\| \leq \|(T-\lambda I)S\|$ for all $S \in BL(X)$ and all $\lambda \in \mathbb{C}$ satisfies $|\lambda| > \|T\|$. However $BL(X)$ is a Banach algebra, then by [1, Theorem 5] $T \in Z(BL(X))$ and so $(T-\lambda I) \in Z(BL(X))$ for all $\lambda \in \mathbb{C}$. □

Theorem 2.5. Let T, W and T_n be centraloid operators on a normed space X for all natural numbers n, and suppose that $\lim T_n = S$. Then the following are centraloid operators

(i) αT for any $\alpha \in \mathbb{C}$.
(ii) TW.
(iii) T^n for any natural number n.
(iv) S, provided that X is complete.

Proof. (i) Obvious.
(ii) For all $S \in BL(X)$, $\|STW\| \leq \|WST\| \leq \|TWS\|$. Hence TW is centraloid.
(iii) By induction.
(iv) Use the continuity of the product, the continuity of the norm and $\|ST_n\| \leq \|T_nS\|$ for all$S \in BL(X)$ and all natural numbers n. □

Corollary 2.6. There is an uncountable number of centraloid operators on a normed space X that are not isometries.

Proof. To see this, consider αI, where I is the identity operator (or any isometry) and α is any element in $\mathbb{C}$ with $|\alpha| \neq 1$.
Hence the converse of (ii) in Theorem 2.2 need not be true in general. □

Proposition 2.7. Let X be a normed space and A be a dense subset of $BL(X)$. If T is a bounded linear operator such that $\|ST\| \leq \|TS\|$ for all $S \in A$, then T is centraloid.

Proof. Left to the reader. □

It would be noted here that Theorem 2.5 implies that the set of all centraloid operators on a Banach space X is an uncountable closed subset of $BL(X)$. Moreover, it is closed under multiplication (composition of operators) and under scalar multiplication. But we show in Example 2.8 below that this set need not be closed under addition. Hence it need not be a Banach algebra.

Example 2.8. There are centraloid operators T and I on ℓ^2, such that $T+I$ is not centraloid, where I is the identity.

Construction. Consider the bounded linear operator $T : \ell^2 \to \ell^2$ that is defined by $Tx = (-\xi_1, \xi_2, -\xi_3, \xi_4, \ldots)$, and let I be the identity operator, where $x = (\xi_1, \xi_2, \xi_3, \xi_4, \ldots) \in \ell^2$. One can easily show that T and I are isometries and so by Theorem 2.2 (ii), they are centraloid. We show that $W = T + I$ is not centraloid. Note that, $Wx = (0, 2\xi_2, 0, 2\xi_4, 0, 2\xi_6, 0, \ldots)$. Consider the bounded linear operator $S : \ell^2 \to \ell^2$ that is defined by $Sx = (\xi_2, 0, \xi_4, 0, \xi_6, 0, \ldots)$. Simple calculations show that $WSx = W((\xi_2, 0, \xi_4, 0, \xi_6, 0, \ldots) = (0, 0, 0, 0, \ldots)$, but $SW(1, 1, 0, 0, 0, 0, \ldots) = S(0, 2, 0, 0, 0, \ldots) = (2, 0, 0, 0, \ldots)$. Hence $\|WS\| = 0$, but $\|SW\| > 0$. Therefore, there is a bounded linear operator S such that $\|SW\| > \|WS\|$, and so W is not a centraloid operator.

Theorem 2.9. If T is an invertible bounded linear operator on a normed space X with $\|T\|\|T^{-1}\| = 1$, then both T and T^{-1} are centraloid.

Proof. For all $S \in BL(X)$, $\|ST\| = \|T^{-1}TST\| \le \|T^{-1}\|\|TS\|\|T\| = \|TS\|$. Hence T is centraloid. Similarly for T^{-1}. □

Proposition 2.10. If a bounded linear operator T on a normed space X and its adjoint $T^\times$ are centraloid then $\|ST\| = \|TS\| = \|S^\times T^\times\| = \|T^\times S^\times\|$ for all $S \in BL(X)$.

Proof. Suppose that T and T^x are centraloid. Then for all $S \in BL(X)$ we have, $\|ST\| \le \|TS\|$ and $\|S^\times T^\times\| \le \|T^\times S^\times\|$ because $S^\times \in BL(X')$. However, $\|S^\times T^\times\| = \|(TS)^\times\| = \|TS\|$ and $\|T^\times S^\times\| = \|(ST)^\times\| = \|ST\|$. Therefore, $\|ST\| = \|TS\| = \|S^\times T^\times\| = \|T^\times S^\times\|$. □

3. **Centraloid Operators on a Hilbert Space**

In this section we study centraloid operators on a Hilbert space, where we study the relation between centraloid operators with its Hilbert adjoint, normal and self adjoint operators. Finally we give a necessary and a sufficient condition for a self adjoint operator to be centraloid. First we give an example to show that the adjoint of a centraloid operator need not be centraloid. Moreover this example shows that the centraloid operator need not be normal.

Example 3.1. There is a nonnormal centraloid operator T on ℓ^2, such that the Hilbert adjoint operator T^* of T is not centraloid.

Construction. Consider the bounded linear operator $T : \ell^2 \to \ell^2$ that is defined by $Tx = (0, \xi_1, \xi_2, \xi_3, \xi_4, \ldots)$, where $x = (\xi_1, \xi_2, \xi_3, \xi_4, \ldots) \in \ell^2$. By direct calculation one can see that T is an isometry ($\|Tx\| = \|x\|$ for all $x \in \ell^2$) and so by Theorem 2.2(ii), it is centraloid. It is clear that the adjoint of T is $T^* : \ell^2 \to \ell^2$ with $T^*x = (\xi_2, \xi_3, \xi_4, \ldots)$. Similar to the proof of Example 2.8, the bounded linear operator $S : \ell^2 \to \ell^2$ defined by $Sx = (\xi_1, 0, 0, 0, \ldots)$, satisfies $\|ST^*\| > \|T^*S\| = 0$.

Hence T^* is not centraloid. Finaly one can show that $TT^* \neq T^*T$. Hence T is not normal.

Note. In Example 3.1 above, define $W : \ell^2 \to \ell^2$ by $Wx = \alpha Tx$, where $\alpha \in \mathbb{C}$ and $|\alpha| \neq 1$, to get a centraloid operator which is neither normal nor isometry.

The following example shows that the self-adjoint and the normal operators need not be centraloid.

Example 3.2. There is a normal operator T on ℓ^2 which is not centraloid.

Construction. Consider the bounded linear operator $T : \ell^2 \to \ell^2$ that is defined by $Tx = (0, \xi_2, 0, \xi_4, 0, \xi_6, 0, \ldots)$, where $x = (\xi_1, \xi_2, \xi_3, \xi_4, \ldots) \in \ell^2$. One can easly show that T is self adjoint and so normal. Similarly as in Example 2.8, T is not centraloid. For this consider the bounded linear operator $S : \ell^2 \to \ell^2$ defined by $Sx = (\xi_2, 0, \xi_4, 0, \xi_6, 0, \ldots)$. Simple calculations show that $\|TS\| = 0$ and $\|ST\| > 0$.

Theorem 3.3. If T is a bounded linear operator on a Hilbert space H, then T and T^* are centraloid if and only if $\|ST\| = \|TS\|$ for all $S \in BL(X)$.

Proof. The proof of the "only if" part is similar to the proof of Proposition 2.10. Conversely, suppose that $\|ST\| = \|TS\|$ for all $S \in BL(X)$. Then $\|T^*S\| = \|(T^*S)^*\| = \|S^*T\| = \|TS^*\| = \|(ST^*)^*\| = \|ST^*\|$. Finally, by using the definition of a centraloid operator we get the result. □

Corollary 3.4. If T is a self adjoint operator on a Hilbert space H, then T is centraloid if and only if $\|ST\| = \|TS\|$ for all $S \in BL(X)$.

Proposition 3.5. Let T and S be bounded linear operators on a Hilbert space H. If S is unitarily equivalent to T, then T is centraloid if and only if S is centraloid.

Proof. To get the result, use the definition of unitarily equivalence, the facts that the inverse of a unitary operator is unitary and that any unitary operator is an isometry and Theorems 2.2(ii) and 2.5(ii). □

Theorem 3.6. Let $T_1 \in BL(H_1)$ and $T_2 \in BL(H_2)$, where H_1 and H_2 are Hilbert spaces. If T_1 and T_2 are centraloid, then $T_1 \oplus T_2$ is centraloid.

Proof. Let $S \in BL(H_1 \oplus H_2)$. Then there are $S_1 \in BL(H_1)$ and $S_2 \in BL(H_2)$ such that $S = (S_1 \oplus S_2)$. Then, $\|S(T_1 \oplus T_2)\| = \|S_1T_1 \oplus S_2T_2)\| = \max\{\|S_1T_1\|, \|S_2T_2\|\} \leq \max\{\|T_1S_1\|, \|T_2S_2\|\} = \|T_1S_1 \oplus T_2S_2\| = \|(T_1 \oplus T_2)S\|$. Therefore, $T_1 \oplus T_2$ is centraloid. □

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Received April 26, 2011
Accepted May 29, 2012