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Invited Research Article

New Analysis of the Time-Fractional and Space-Time Fractional-Order Nagumo Equation

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Abstract. In this paper, we present an algorithm by using the *Adomian Decomposition Method* (ADM) in order to solve the time-fractional Nagumo equation and the space-time fractional-order Nagumo equation. In the space-time fractional case, we expand the $\tanh(\cdot)$ initial condition in the basis functions $e^{-n\zeta}$. The fractional-order derivative could then be easily calculated. An important point in our investigation is that many earlier authors avoided this initial condition as there was no direct method to calculate its fractional derivative. We have studied the convergence analysis and applied it to the time-fractional Nagumo equation and the space-time fractional-order Nagumo equation. We compare the ADM solution with the exact solution and find a very good agreement. We also graphically illustrate the behavior of the ADM solutions.

Keywords. Adomian Decomposition Method (ADM); Time-fractional Nagumo equation; Space-time fractional-order Nagumo equation; Convergence analysis

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1. Introduction

Various operators of fractional calculus have been used in recent years to find solutions to the equations modeling of real-world phenomena in engineering and physical sciences. There exist many phenomena in several fields which are modeled by fractional differential equations.

These include fluid mechanics [45], chemistry [17, 26, 39, 40], biology [30], viscoelasticity [35], engineering, finance, and physics [19,28,36]. This made scientists and engineers give more attention to the modeling problems in which the fractional derivative does arise in physical contexts. In this connection, we refer the interested reader to the earlier works [3,42]. As a result of the difficulty in finding an analytical solution for the fractional differential equations by using analytical techniques, the approximate methods were used (see, for example, [16,24,29,37,41,43,46,49–51]).

The familiar Nagumo equation is given by

$$\psi_\eta = \psi_{\zeta\zeta} + \psi(1-\psi)(\psi-\mu) \qquad (0 \leqq \mu < 1;\ \psi = \psi(\zeta,\eta)) \tag{1.1}$$

involving the space variable ζ and the time variable η. The exact solution of (1.1) is known as a solitary wave in the following form:

$$\psi(\zeta,\eta) = a + b\tanh(\kappa(\zeta - \nu\eta)). \tag{1.2}$$

In fact, there is a class of eight solutions of the form (1.1) that can be obtained by the software Mathematica or otherwise. Here, in this paper, we consider the case when

$$a = \frac{1+\mu}{2},\quad \nu = \frac{1+\mu}{\sqrt{2}},\quad b = \frac{1-\mu}{2} \quad \text{and} \quad \kappa = \frac{\mu-1}{2\sqrt{2}}. \tag{1.3}$$

We mention that the solution (1.2) satisfies the ollowing condition:

$$\psi(-\infty,\eta) = 1 \quad \text{and} \quad \psi(\infty,\eta) = \mu.$$

The Nagumo equation (1.1) has attracted the attention of many researchers (see, for example, [2, 12, 18, 22, 23]). This equation has been applied as a model for the transmission of nerve impulses (see, for details, [15,33]). Furthermore, the equation (1.1) is an important nonlinear reaction-diffusion equation and it has been used in biology, in the area of population genetics, and in circuit theory (see [44]).

Recently, Adomian (see [4,5]) considered a new technique called the *Adomian Decomposition Method* (ADM) for computing the solutions of linear and nonlinear equations. Various authors have studied the convergence of Adomain's method (see [1, 10, 11, 31]). It has recently been proven that the ADM provides a very effective technique and can be applied successfully to many problems such as systems of ordinary and partial differential equations as well as integral equations (see, for example, [6–8,20,21,27,34,47,48]).

This work is organized as follows. Section 2 is devoted to the essential ideas surrounding some operators of fractional calculus, which we propose to use in this paper. Section 3 concentrates upon a description of the standard ADM. In Section 4, we apply the ADM for evaluating the time-fractional Nagumo equation. In Section 5, we apply the ADM for evaluating the space-time fractional-order Nagumo equation. Conclusions are presented in the last section (Section 7).

2. Operators of Fractional Calculus

In this section, we give some basic definitions and properties of the operators of fractional calculus (see, for details, [9,25,32,35]).

Definition 1. If $\psi(\eta) \in L_1(a,b)$, $L_1(a,b)$ being the set of all integrable functions in the interval (a,b), and $\alpha > 0$, then the Riemann-Liouville fractional integral of order α, denoted by J^{α}_{a+}, is defined by

$$J^{\alpha}_{a+}\psi(\eta) = \frac{1}{\Gamma(\alpha)} \int_a^{\eta} (\eta - \varsigma)^{\alpha-1}\, \psi(\varsigma) d\varsigma. \tag{2.1}$$

Correspondingly, the Riemann-Liouville fractional derivative operator D^{α}_{η} is defined by

$$D^{\alpha}_{\eta}\psi(\eta) = \frac{1}{\Gamma(n-\alpha)} \int_a^{\eta} (\eta - \varsigma)^{n-\alpha-1}\, \psi^{(n)}(\varsigma) d\varsigma \tag{2.2}$$
$$(n-1 < \alpha \leqq n;\; n \in \mathbb{N} = \{1,2,3,\cdots\}).$$

Definition 2. For $\alpha > 0$, the Liouville-Caputo fractional derivative of order α, denoted by ${}^{\mathrm{LC}}D^{\alpha}_{a+}$, is defined by

$${}^{\mathrm{LC}}D^{\alpha}_{a+}\psi(\eta) = \frac{1}{\Gamma(n-\alpha)} \int_a^{\eta} (\eta - \varsigma)^{n-\alpha-1}\, D^{n}_{\varsigma}\psi(\varsigma) d\varsigma \tag{2.3}$$
$$(n-1 < \alpha \leqq n;\; n \in \mathbb{N}),$$

where

$$D^{n}_{\eta} := \frac{d^n}{d\eta^n} \qquad (n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}). \tag{2.4}$$

If α is a positive integer, then the Liouville-Caputo fractional derivative becomes the ordinary derivative:

$${}^{\mathrm{LC}}D^{\alpha}_{a+} = D^{\alpha}_{\eta} \qquad (\alpha \in \mathbb{N}). \tag{2.5}$$

Finally, the Liouville-Caputo fractional derivative on the whole space $\mathbb{R} = (-\infty, \infty)$ is defined below.

Definition 3. For $\alpha > 0$, the Liouville-Caputo fractional derivative of order α on the whole space $\mathbb{R}$, denoted by ${}^{\mathrm{LC}}D^{\alpha}_{a+}$, is defined by

$${}^{\mathrm{LC}}D^{\alpha}_{-\infty+}\psi(\zeta) = \frac{1}{\Gamma(n-\alpha)} \int_{-\infty}^{\zeta} (\zeta - \varsigma)^{n-\alpha-1}\, D^{n}_{\varsigma}\psi(\varsigma) d\varsigma \tag{2.6}$$
$$(n-1 < \alpha \leqq n;\; n \in \mathbb{N}).$$

3. Basic Ideas of the Adomian Decomposition Method (ADM)

In this section, we present the basic ideas of the *Adomian Decomposition Method* (ADM) (see, for example, [13]) by considering the following nonlinear partial differential equation:

$$L\big(\psi(\zeta,\eta)\big) + R\big(\psi(\zeta,\eta)\big) + N\big(\psi(\zeta,\eta)\big) = 0 \tag{3.1}$$

together with the condition given by

$$\psi(\zeta,0) = \phi(\zeta), \tag{3.2}$$

where L is the highest-order derivative which is assumed to be invertible, R is the remaining linear operator and N represents a nonlinear operator. Now, by applying the inverse operator

L^{-1} to both sides of (3.1), we get

$$\psi(\zeta,\eta) = \phi(\zeta) - L^{-1}\{R(\psi(\zeta,\eta)) + N(\psi(\zeta,\eta))\}. \tag{3.3}$$

Let

$$\psi(\zeta,\eta) = \sum_{m=0}^{\infty} \psi_m(\zeta,\eta) \tag{3.4}$$

and

$$N(\psi) = \sum_{m=0}^{\infty} \chi_m, \tag{3.5}$$

where χ_m are the Adomian polynomials which depend upon ψ. In view of the equations (3.4) to (3.5), the equation (3.3) takes the following form:

$$\sum_{m=0}^{\infty} \psi_m(\zeta,\eta) = \phi(\zeta) - L^{-1}\{R(\psi(\zeta,\eta))\} + \sum_{m=0}^{\infty} \chi_m(\psi(\zeta,\eta)). \tag{3.6}$$

We set

$$\psi_0(\zeta,\eta) = \phi(\zeta). \tag{3.7}$$

Then

$$\psi_{m+1}(\zeta,\eta) = -L^{-1}\{R(\psi(\zeta,\eta))\} + \sum_{m=0}^{\infty} \chi_m(\psi(\zeta,\eta)) \qquad (m \in \mathbb{N}_0), \tag{3.8}$$

where

$$\chi_m(\psi(\zeta,\eta)) = \left[\frac{1}{m!}\frac{d^m}{d\lambda^m} N\left(\sum_{m=0}^{\infty} \psi_m(\zeta,\eta)\lambda^m\right)\right]_{\lambda=0}. \tag{3.9}$$

Hence, the equations (3.7), (3.8) and (3.9) lead to the following recurrence relations:

$$\psi_0(\zeta,0) = \phi(\zeta), \quad \psi_{m+1}(\zeta,\eta) = -L^{-1}[R(\psi(\zeta,\eta)) + A_m(\psi(\zeta,\eta))]. \tag{3.10}$$

The solution $\psi(\zeta,\eta)$ can be approximated by the truncated series given by

$$\varphi_k(\zeta,\eta) = \sum_{m=0}^{k-1} \psi_m(\zeta,\eta) \tag{3.11}$$

and

$$\lim_{k\to\infty} \varphi_k(\zeta,\eta) = \psi(\zeta,\eta). \tag{3.12}$$

4. The Time-Fractional Nagumo Equation

In this section, we apply the ADM to find the approximation solutions for the time-fractional Nagumo equation. To obtain the time-fractional Nagumo equation, we replace ψ_η in the Naguno equation (1.1) by ψ_η^α, where $n-1 < \alpha \leqq n$ $(n \in \mathbb{N})$. We thus obtain the time-fractional Nagumo equation given by

$$\psi_\eta^\alpha = \psi_{\zeta\zeta} + \psi(1-\psi)(\psi-\mu) \qquad (0 < \alpha \leqq 1;\ 0 \leqq \mu < 1). \tag{4.1}$$

If we operate upon both sides of (4.1) by J_η^α, we obtain

$$\psi(\zeta,\eta) = \psi(\zeta,0) + J_\eta^\alpha\left[\psi_{\zeta\zeta} - \mu\psi + (1+\mu)\psi^2 - \psi^3\right]. \tag{4.2}$$

Now the ADM solutions and the nonlinear functions $N\big(\psi(\zeta,\eta)\big)$ can be presented as infinite series given by

$$\psi(\zeta,\eta)=\psi_0(\zeta,\eta)+\sum_{n=1}^{\infty}\psi_n(\zeta,\eta) \tag{4.3}$$

and

$$N\big(\psi(\zeta,\eta)\big)=(1+\mu)[\psi(\zeta,\eta)]^2-[\psi(\zeta,\eta)]^3=\sum_{n=0}^{\infty}\chi_n, \tag{4.4}$$

where

$$\chi_n=\frac{1}{n!}\left[\frac{d^n}{d\lambda^n}N\left(\sum_{k=0}^{n}\lambda^k\psi_k(\zeta,\eta)\right)\right]_{\lambda=0}. \tag{4.5}$$

Here χ_n are called the Adomian polynomials and the components $\psi_n(\zeta,\eta)$ of the solutions $\psi(\zeta,\eta)$ will be determined by the following recurrence relations:

$$\psi_0(\zeta,\eta)=\psi(\zeta,0) \tag{4.6}$$

and

$$\psi_{n+1}(\zeta,\eta)=J_\eta^\alpha\left[\psi_n(\zeta,\eta)_{\zeta\zeta}-\mu\psi_n(\zeta,\eta)+\chi_n\right]. \tag{4.7}$$

In view of (3.9), and by using the software MATHEMATICA, we evaluate the Adomian polynomials χ_n as follows:

$$\begin{aligned}\chi_0&=(1+\mu)[\psi_0(\zeta,\eta)]^2-[\psi_0(\zeta,\eta)]^3\\ \chi_1&=2(1+\mu)\psi_0(\zeta,\eta)\psi_1(\zeta,\eta)-3[\psi_0(\zeta,\eta)]^2\psi_1(\zeta,\eta),\\ &\vdots\end{aligned} \tag{4.8}$$

Thus, in the case of the first iteration, we have

$$\psi_1(\zeta,\eta)=\frac{1}{\Gamma(\alpha)}\int_0^\eta(\eta-\varsigma)^{\alpha-1}\left[\psi_0(\zeta,\varsigma)_{\zeta\zeta}-\mu\psi_0(\zeta,\varsigma)+\chi_0\right]d\varsigma. \tag{4.9}$$

The initial condition is then taken by setting $\eta=0$ in (1.2), so that

$$\psi(\zeta,0)=\frac{1}{2}\left[1+\mu-(\mu-1)\tanh\left(\frac{(\mu-1)\zeta}{2\sqrt{2}}\right)\right]. \tag{4.10}$$

In view of(4.6) and (4.9) to (4.10), we obtain the first three approximations as follows:

$$\psi_0(\zeta,\eta)=\frac{1}{2}\left[1+\mu-(\mu-1)\tanh\left(\frac{(\mu-1)\zeta}{2\sqrt{2}}\right)\right], \tag{4.11}$$

$$\psi_1(\zeta,\eta)=\frac{(\mu-1)^2(\mu+1)\eta^\alpha}{8\Gamma(\alpha+1)}\operatorname{sech}^2\left(\frac{(\mu-1)\zeta}{2\sqrt{2}}\right) \tag{4.12}$$

and

$$\begin{aligned}\psi_2(\zeta,\eta)=\frac{(\mu-1)^2(\mu+1)\eta^{2\alpha}}{32\pi}\operatorname{sech}^2\left(\frac{(\mu-1)\zeta}{2\sqrt{2}}\right)\Bigg[&\frac{\sqrt{\pi}4^{-\alpha}(\mu-1)^2\cos(\pi\alpha)\Gamma\left(\frac{1}{2}-\alpha\right)}{\Gamma(\alpha+1)}\\ &\cdot\left[\cosh\left(\frac{(\mu-1)\zeta}{\sqrt{2}}\right)-2\right]\operatorname{sech}^2\left(\frac{(\mu-1)\zeta}{2\sqrt{2}}\right)\\ &-\left(4\mu\sin(2\pi\alpha)\Gamma(-2\alpha)-\frac{\sqrt{\pi}4^{-\alpha}\cos(\pi\alpha)\Gamma\left(\frac{1}{2}-\alpha\right)}{\Gamma(\alpha+1)}\right)\\ &\cdot\left[1+\mu-(\mu-1)\tanh\left(\frac{(\mu-1)\zeta}{2\sqrt{2}}\right)\right]\left\{1+\mu+3(\mu-1)\tanh\left(\frac{(\mu-1)\zeta}{2\sqrt{2}}\right)\right\}\Bigg].\end{aligned} \tag{4.13}$$

The following further approximations:

$$\psi_3(\zeta,\eta),\ \psi_4(\zeta,\eta),\ \psi_5(\zeta,\eta),\ \cdots$$

can also be determined similarly and used accordingly. For the sake of brevity, we choose not to record these approximations here. The general form of the above approximations is given by (4.3), that is, by

$$\psi(\zeta,\eta) = \psi_0(\zeta,\eta) + \psi_1(\zeta,\eta) + \psi_2(\zeta,\eta) + \cdots . \tag{4.14}$$

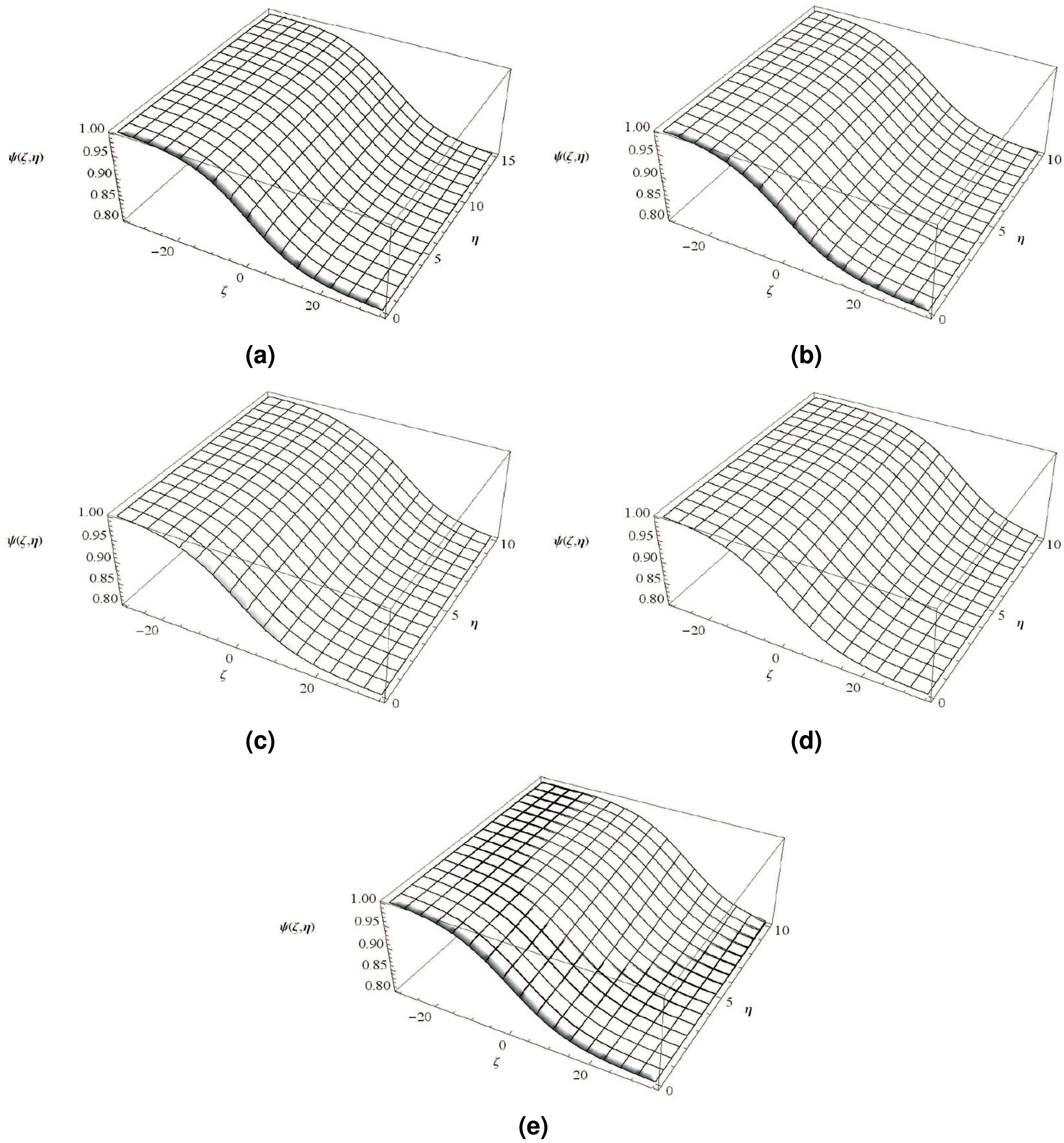

Figure 1. The surface of the 4 terms of the ADM solution for $\alpha = 0.4, 0.6, 0.8$ in (a) to (c), respectively, and the surface of the exact solution (1.2) in (d) and the plotting of (a) to (d) in (e) at $\mu = 0.8$.

In Figure 1, we have graphically illustrated the first four terms of the ADM solution for $\alpha = 0.4, 0.6, 0.8$ in (a) to (c), respectively and the surface of the exact solution (1.2) in (d) at $\mu = 0.8$. It is clear from Figure 1 that, in the limit when $\alpha \to 1$, the behavior of the ADM solution approaches to the exact solution. In Figure 2, we have graphically illustrated the absolute error between the exact solution (1.2) and the ADM solutions for three terms, four terms, five terms and six terms in (a) to (d), respectively. It can be seen from Figure 2 that the absolute error decreases as the number of terms of the ADM solution increases. Obviously, in order to minimize the error involved, more terms need to be considered for the ADM solution.

In Figure 3, we have presented the graphs of the surfaces of the two-term ADM solution (5.18) for different special numerical values of the parameters involved therein.

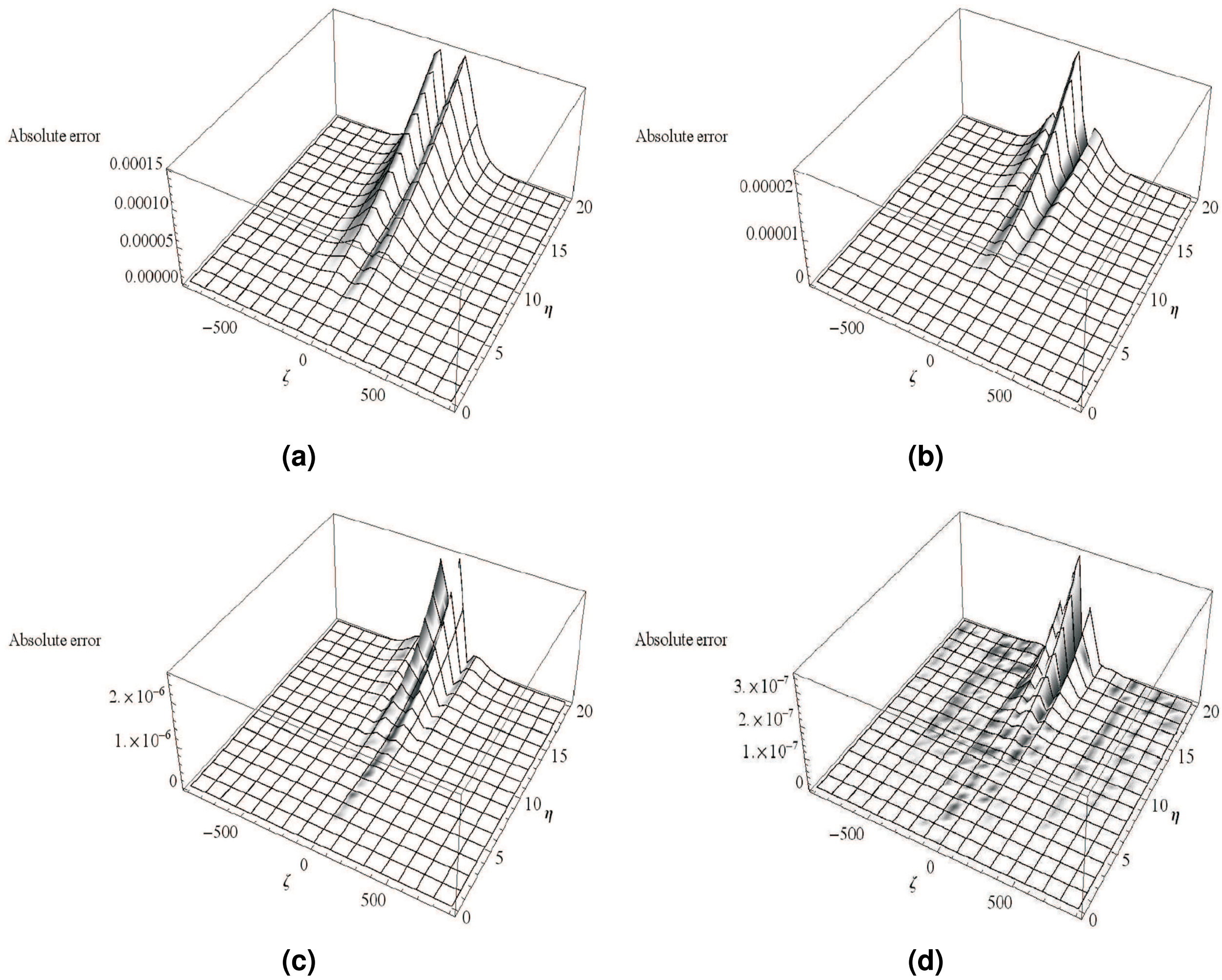

Figure 2. The absolute error of the ADM solutions for three terms, four terms, five terms and six terms in (a) to (d), respectively, at $\mu = 0.98$.

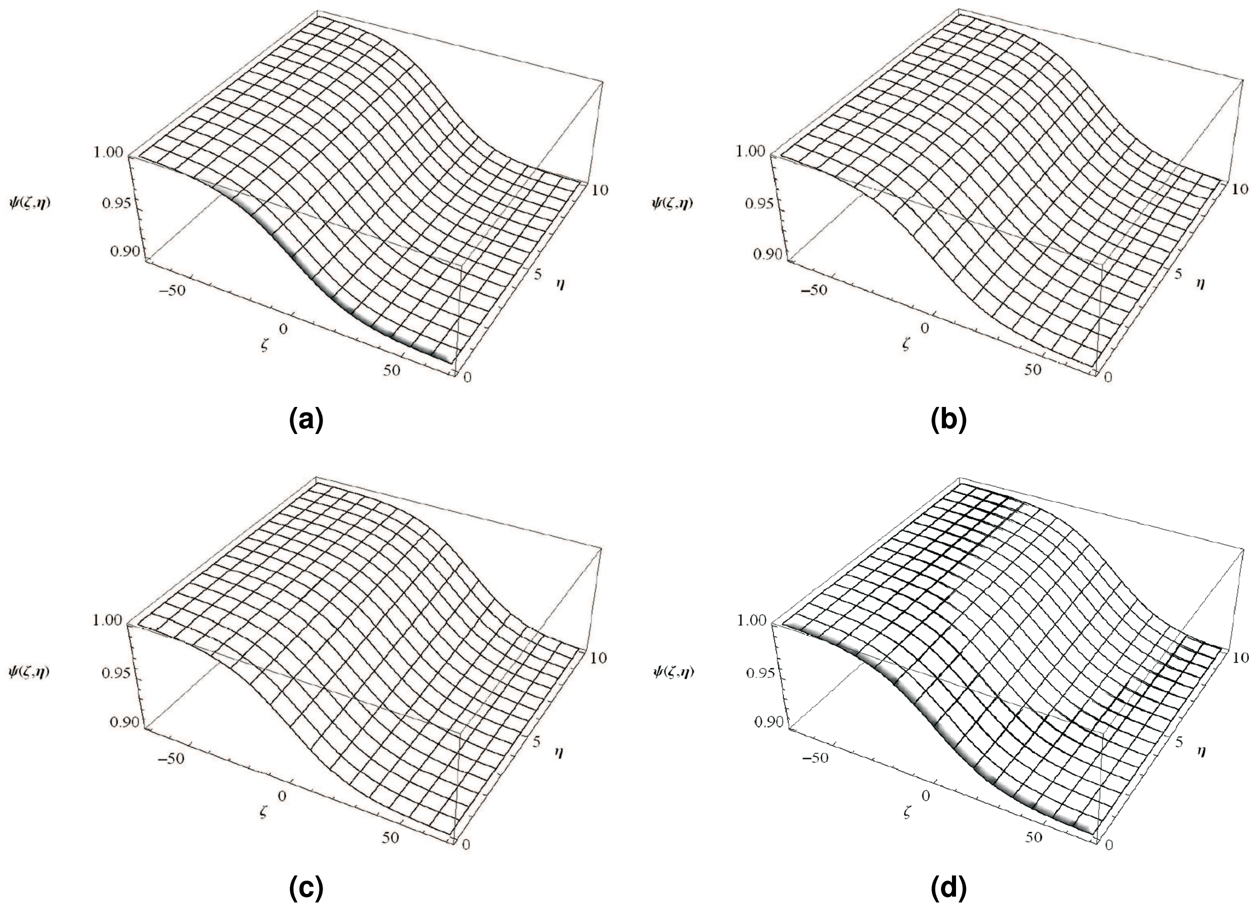

Figure 3. The surface of the two-terms ADM solution (5.18) for $\alpha = 0.4, 0.6$ and $\beta = 0.6, 0.6$ in (a) and (b), respectively, and the surface of the exact solution (1.2) in (c) and the plotting of (a) to (c) in (d) at $\mu = 0.9$.

5. The Space-Time Fractional-Order Nagumo Equation

In this section, we study the ADM solutions for the space-time fractional-order Nagumo equation. We begin by expanding the exact solution given (see, for details, [38])

$$\begin{aligned}\psi(\zeta,\eta) &= \frac{1}{2}\left[1+\mu-(-1+\mu)\tanh\left(\rho(\zeta,\eta)\right)\right] \\ &= 1-(-1+\mu)\sum_{r=1}^{\infty}(-1)^r\, e^{-r\rho(\zeta,\eta)},\end{aligned} \tag{5.1}$$

where

$$\rho(\zeta,\eta) = \frac{(-1+\mu)\left(\zeta - \frac{\eta(1+\mu)}{\sqrt{2}}\right)}{\sqrt{2}} > 0.$$

For convenience, we set $\psi = 1+\varphi$ and rewrite the Nagumo equation (1.1) as follows:

$$\varphi_\eta = \varphi_{\zeta\zeta} - (1-\mu)\varphi - (2-\mu)\varphi^2 - \varphi^3. \tag{5.2}$$

The solution of (5.2) assumes the following form:

$$\varphi(\zeta,\eta) = -(-1+\mu)\sum_{r=1}^{\infty}(-1)^r\, e^{-r\rho(\zeta,\eta)}. \tag{5.3}$$

We take the initial condition by letting $\eta = 0$ in (5.3), so that

$$\varphi(\zeta,0) = -(-1+\mu)\sum_{r=1}^{\infty}(-1)^r\, e^{-r\rho(\zeta,0)}. \tag{5.4}$$

We now consider the ADM for the space-time fractional-order Nagumo equation. If, in the equation (5.2), we replace φ_η by φ_η^α and $\varphi_{\zeta\zeta}$ by $\varphi_\zeta^{2\beta}$, we obtain

$$\varphi_\eta^\alpha = \varphi_\zeta^{2\beta} - (1-\mu)\varphi - (2-\mu)\varphi^2 - \varphi^3 = 0. \tag{5.5}$$

If we operate upon both sides of (5.5) by J_η^α, we obtain

$$\varphi(\zeta,\eta) = \varphi(\zeta,0) + J_\eta^\alpha\left[\varphi_\zeta^{2\beta} - (1-\mu)\varphi - (2-\mu)\varphi^2 - \varphi^3\right]. \tag{5.6}$$

Now the ADM solutions and the nonlinear functions $M\big(\psi(\zeta,\eta)\big)$ can be presented as infinite series given by

$$\varphi(\zeta,\eta) = \varphi_0(\zeta,\eta) + \sum_{n=1}^{\infty}\varphi_n(\zeta,\eta) \tag{5.7}$$

and

$$M\big(\varphi(\zeta,\eta)\big) = -(2-\mu)[\varphi(\zeta,\eta)]^2 - [\varphi(\zeta,\eta)]^3 = \sum_{n=0}^{\infty}\xi_n, \tag{5.8}$$

where

$$\xi_n = \frac{1}{n!}\left[\frac{d^n}{dp^n}M\left(\sum_{k=0}^{n}p^k\psi_k(\zeta,\eta)\right)\right]_{p=0}, \tag{5.9}$$

where ξ_n are called the Adomian polynomials. The components $\xi_n(\zeta,\eta)$ of the solutions $\varphi(\zeta,\eta)$ will be determined by the following recurrence relations:

$$\varphi_0(\zeta,\eta) = \varphi(\zeta,0) \tag{5.10}$$

and

$$\varphi_{n+1}(\zeta,\eta) = J_\eta^\alpha\left[\varphi_{\zeta,n}^{2\beta} - (1-\mu)\varphi_n(\zeta,\eta) + \xi_n\right]. \tag{5.11}$$

In view of (3.9), and by using the software MATHEMATICA, we evaluate the Adomian polynomials ξ_n as follows:

$$\begin{aligned}
&\xi_0 = (\mu-2)[\varphi_0(\zeta,\eta)]^2 - [\varphi_0(\zeta,\eta)]^3,\\
&\xi_1 = 2(\mu-2)\varphi_0(\zeta,\eta)\varphi_1(\zeta,\eta) - 3[\varphi_0(\zeta,\eta)]^2\varphi_1(\zeta,\eta),\\
&\quad\vdots
\end{aligned} \tag{5.12}$$

In view of the equations (5.4), (5.10) and (5.11), the first three components of the decomposition series are derived as follows:

$$\varphi_0(\zeta,\eta) = -(-1+\mu)\sum_{r=1}^{\infty}(-1)^r\, e^{-\lambda\zeta} \qquad \left(\lambda := \frac{\mu-1}{\sqrt{2}}\right), \tag{5.13}$$

$$\varphi_1(\zeta,\eta) = -\frac{\eta^\alpha}{\Gamma(\alpha+1)}\, f_0(\zeta), \tag{5.14}$$

$$\varphi_2(\zeta,\eta) = -\frac{\eta^{2\alpha}}{\Gamma(2\alpha+1)}\left[f_1(\zeta) - (1-\mu)f_0(\zeta) - 2(2-\mu)\varphi_0 f_0(\zeta) - 3\varphi_0^2 f_0(\zeta)\right], \tag{5.15}$$

where

$$f_i(\zeta) = (-1+\mu)\sum_{r=1}^{\infty}(-1)^r\, e^{-\lambda\zeta}(r\lambda)^{(2+2i)\beta} - (1-\mu)(-1+\mu)\sum_{r=1}^{\infty}(-1)^r\, e^{-\lambda\zeta}(r\lambda)^{2i\beta}$$
$$+(2-\mu)(-1+\mu)^2\sum_{r=1}^{\infty}\sum_{r_1=1}^{\infty}(-1)^{r+r_1}\, e^{-(r+r_1)\lambda\zeta}[(r+r_1)\lambda]^{2i\beta}$$
$$-(-1+\mu)^3\sum_{r=1}^{\infty}\sum_{r_1=1}^{\infty}\sum_{r_2=1}^{\infty}(-1)^{r+r_1+r_2}e^{-(r+r_1+r_2)\lambda\zeta}\,[(r+r_1+r_2)\lambda]^{2i\beta}, \tag{5.16}$$

where λ is given as in the equation (5.13), and so on. In this manner, the rest of the components of the decomposition series can be obtained. The first two terms of the decomposition series are given by

$$\psi(\zeta,\eta) = 1+\varphi_0+\varphi_1$$
$$= 1-(-1+\mu)\sum_{r=1}^{\infty}(-1)^r\, e^{-2r\lambda\zeta} - \frac{\eta^\alpha}{\Gamma(\alpha+1)}\, f_0(\zeta). \tag{5.17}$$

In the same manner, we obtain the following other components of approximation:

$$\varphi_3(\zeta,\eta),\ \varphi_4(\zeta,\eta),\ \varphi_4(\zeta,\eta),\ \cdots.$$

When we set $\beta=1$ in the first approximation (5.14) and in the second approximation (5.15), and after some simplifications, we obtain the same approximations as those given by (4.12) and (4.13) after returning to the original variable.

We can obtain the ADM solutions of (5.5) for $\rho(\zeta,\eta)<0$ by using the same procedure as the above for $\rho(\zeta,\eta)>0$. Moreover, by using the software MATHEMATICA, we can simplify (5.17) and we obtain the following ADM solutions for the first few terms:

$$\psi(\zeta,\eta) = 1+\frac{\mu-1}{e^{\frac{(\mu-1)\zeta}{\sqrt{2}}}+1} - \frac{(\mu-1)\eta^\alpha}{\Gamma(\alpha+1)}\left[-\frac{\mu-1}{e^{\frac{(\mu-1)\zeta}{\sqrt{2}}}+1} + \frac{(\mu-1)^2 e^{\frac{3\zeta}{\sqrt{2}}}}{\left(e^{\frac{\mu\zeta}{\sqrt{2}}}+e^{\frac{\zeta}{\sqrt{2}}}\right)^3} - \frac{(\mu-2)(\mu-1)e^{\sqrt{2}\zeta}}{\left(e^{\frac{\mu\zeta}{\sqrt{2}}}+e^{\frac{\zeta}{\sqrt{2}}}\right)^2}\right.$$
$$\left.+2^{-\beta}(1-\mu)^{2\beta}\,\mathrm{Li}_{-2\beta}\left(-e^{-\frac{\zeta(\mu-1)}{\sqrt{2}}}\right)\right], \tag{5.18}$$

where $\mathrm{Li}_s(z)$ denotes the Polylogarithmic function (or *de Jonquière's function*) $\mathrm{Li}_\nu(z)$:

$$\mathrm{Li}_s(z) := \sum_{n=1}^{\infty}\frac{z^n}{n^s} \tag{5.19}$$
$$(s\in\mathbb{C} \text{ when } |z|<1;\ \Re(s)>1 \text{ when } |z|=1).$$

We use the first approximation (5.18) to show the behavior of the ADM solutions. Figure 3 shows the behavior of the ADM solution (5.18) for different values for α and β. It can be seen from Figure 3 that, in the limit as $\alpha\to 1$ and $\beta\to 1$, the ADM solution (5.18) approaches the exact solution of the Nagumo equation (1.1). Therefore, the ADM is an efficient and accurate method which can be used to find approximate analytical solutions of the space-time fractional-order Nagumo equation (5.5).

6. Convergence Analysis

In this section, we establish a lemma and a set of three theorems that guarantee the existence of the ADM solution and provide the maximum absolute truncation error. We define the Banach space as $(C(I),\|\cdot\|)$, which is the space of all continuous functions $\varphi(\zeta,\eta)$ with the norm given by

$$\|\varphi(\zeta,\eta)\| = \max_{(\zeta,\eta)\in I} \left|\varphi(\zeta,\eta)\right|.$$

Lemma. *Suppose that the function $\varphi(\zeta,\eta)$ and their partial derivatives are continuous. Then the derivatives $D_\eta^\alpha\varphi(\zeta,\eta)$ and $D_\zeta^{2\beta}\varphi(\zeta,\eta)$ are bounded.*

Proof. We first prove that $D_\eta^\alpha\varphi(\zeta,\eta)$ is bounded. From the definition (2.2) of the Riemann-Liouville fractional derivative, we have

$$\begin{aligned}\|D_\eta^\alpha\varphi(\zeta,\eta)\| &= \left\|\frac{1}{\Gamma(m-\alpha)}\int_a^\eta (\eta-\varsigma)^{m-\alpha-1}\,\varphi^{(m)}(\zeta,\varsigma)d\varsigma\right\| \\ &\leqq \frac{|\eta-a|}{|(m-\alpha)\Gamma(m-\alpha)|}\,\|\varphi(\zeta,\eta)\| \\ &= L_1\|\varphi(\zeta,\eta)\|, \end{aligned} \tag{6.1}$$

where

$$L_1 = \frac{|\eta-a|}{|(m-\alpha)\Gamma(m-\alpha)|}.$$

In the same manner, we find that

$$\left\|D_\zeta^{2\beta}\varphi(\zeta,\eta)\right\| \leqq L_2\|\varphi(\zeta,\eta)\|.$$

This completes the proof of the lemma. □

Theorem 1. *Let the function $F(\varphi)$ given by*

$$F(\varphi) = -(2-\mu)\varphi^2 - \varphi^3$$

satisfy the Lipschitz condition with the Lipschitz constant L_3. Then the problem (5.5) has a unique solution $\varphi(\zeta,\eta)$ whenever $0<\delta<1$, where

$$\delta := \frac{[L_2+L_3-(1-\mu)]MT}{\Gamma(\alpha)}. \tag{6.2}$$

Proof. Let ψ and φ be two different solutions of the space-time fractional-order Nagumo equation (5.2). Then, for all $\eta\in[0,T]$; $(T>0)$ and $\varsigma\in[0,\eta]$, these solutions are seen to be bounded on using (6.1). We now set

$$M = \max_{0\leqq\varsigma\leqq T;\,0\leqq\eta\leqq T} \left|(\eta-\varsigma)^{\alpha-1}\right|.$$

Then

$$\begin{aligned}\psi-\varphi &= J_\eta^\alpha\left[D_\zeta^{2\beta}\psi(\zeta,\varsigma)-(1-\mu)\psi(\zeta,\varsigma)+F(\psi)\right] - J_\eta^\alpha\left[D_\zeta^{2\beta}\varphi(\zeta,\varsigma)-(1-\mu)\varphi(\zeta,\varsigma)+F(\varphi)\right] \\ &= \frac{1}{\Gamma(\alpha)}\int_0^\eta (\eta-\varsigma)^{\alpha-1}\left[D_\zeta^{2\beta}\psi(\zeta,\varsigma)-(1-\mu)\psi(\zeta,\varsigma)+F(\psi)\right]d\varsigma \\ &\quad -\frac{1}{\Gamma(\alpha)}\int_0^\eta (\eta-\varsigma)^{\alpha-1}\left[D_\zeta^{2\beta}\varphi(\zeta,\varsigma)-(1-\mu)\varphi(\zeta,\varsigma)+F(\varphi)\right]d\varsigma,\end{aligned}$$

so that

$$\begin{aligned}\max|\psi-\varphi| &= \max\left|\frac{1}{\Gamma(\alpha)}\int_0^\eta(\eta-\varsigma)^{\alpha-1}[D_\zeta^{2\beta}\psi(\zeta,\varsigma)-(1-\mu)\psi(\zeta,\varsigma)+F(\psi)]d\varsigma\right.\\ &\qquad\left.-\frac{1}{\Gamma(\alpha)}\int_0^\eta(\eta-\varsigma)^{\alpha-1}[D_\zeta^{2\beta}\varphi(\zeta,\varsigma)-(1-\mu)\varphi(\zeta,\varsigma)+F(\varphi)]d\tau\right|\\ &\leqq\left(\frac{[L_2+L_3-(1-\mu)]}{\Gamma(\alpha)}\right)\max\left\{\int_0^\eta|\eta-\varsigma|^{\alpha-1}\left|\psi-\varphi\right|d\varsigma,\|\psi-\varphi\|\right\}\\ &\leqq\left(\frac{[L_2+L_3-(1-\mu)]MT}{\Gamma(\alpha)}\right)\|\psi-\varphi\|,\end{aligned}\tag{6.3}$$

which yields

$$(1-\delta)\|\psi-\varphi\|\leqq 0,\tag{6.4}$$

where δ is given by (6.2). Since $1-\delta\neq 0$, we have

$$\|\psi-\varphi\|=0.$$

Therefore, we have $\psi=\varphi$, which completes the proof of Theorem 1. □

Theorem 2. *The series solution* $\varphi(\zeta,\eta)$ *of the problem* (5.5) *is given by*

$$\varphi(\zeta,\eta)=\sum_{j=0}^{\infty}\varphi_j(\zeta,\eta)$$

by using the ADM convergence when

$$0<\delta<1 \quad and \quad |\varphi_1(\zeta,\eta)|<\infty.$$

Proof. We first define the sequence s_n of partial sums. Let s_n and s_m be arbitrary partial sums with $n\geqq m$. We propose to prove that s_n is a Cauchy sequence in the Banach space:

$$\begin{aligned}\|s_n-s_m\| &= \max_{(\zeta,\eta)\in I}|s_n-s_m| = \max_{(\zeta,\eta)\in I}\left|\sum_{j=m+1}^{n}u_j\right|\\ &= \max_{(\zeta,\eta)\in I}\left|\frac{1}{\Gamma(\alpha)}\int_0^\eta(\eta-\tau)^{\alpha-1}\left(\sum_{j=m}^{n-1}B_j-\sum_{j=m}^{n-1}C_j+\sum_{j=m}^{n-1}D_j\right)\right|.\end{aligned}\tag{6.5}$$

We find from [14] that

$$\sum_{j=m}^{n-1}B_j=D_\varsigma^{2\beta}(s_{n-1}-s_{m-1}),\qquad \sum_{j=m}^{n-1}C_j=(1-\mu)(s_{n-1}-s_{m-1})$$

and

$$\sum_{j=m}^{n-1}D_j=F(s_{n-1}-s_{m-1}).$$

Thus, clearly, we have

$$\begin{aligned}\|s_n-s_m\| &= \max_{(\zeta,\eta)\in I}\left|\frac{1}{\Gamma(\alpha)}\int_0^\eta(\eta-\varsigma)^{\alpha-1}\Big(D_\zeta^{2\beta}[s_{n-1}-s_{m-1}]-(1-\mu)[s_{n-1}-s_{m-1}]+F[s_{n-1}-s_{m-1}]\Big)d\tau\right|\\ &\leqq\left(\frac{L_2+L_3-(1-\mu)}{\Gamma(\alpha)}\right)\max_{(\zeta,\eta)\in I}\int_0^\eta|\eta-\varsigma|^{\alpha-1}|s_{n-1}-s_{m-1}|\\ &\leqq\left(\frac{[L_2+L_3-(1-\mu)]MT}{\Gamma(\alpha)}\right)\|s_{n-1}-s_{m-1}\|\leqq\delta\|s_{n-1}-s_{m-1}\|.\end{aligned}\tag{6.6}$$

Let $n = m+1$. Then

$$\begin{aligned}\|s_{m+1}-s_m\| &\leqq \delta\|s_m-s_{m-1}\| \\ &\leqq \delta^2\|s_{m-1}-s_{m-2}\| \\ &\leqq \delta^m\|s_1-s_0\|. \end{aligned} \tag{6.7}$$

Thus, by the triangle inequality, we have

$$\begin{aligned}\|s_n-s_m\| &\leqq \|s_{m+1}-s_m\|+\|s_{m+2}-s_{m+1}\|+\cdots+\|s_n-s_{n-1}\| \\ &\leqq \left[\delta^m+\delta^{m+1}+\cdots+\delta^{n-1}\right]\|s_1-s_0\| \\ &\leqq \delta^m\left(\frac{1-\delta^{n-m}}{1-\delta}\right)\|\varphi_1(\zeta,\eta)\|. \end{aligned} \tag{6.8}$$

Since $0<\delta<1$, so that $1-\delta^{n-m}<1$, we have

$$\|s_n-s_m\| \leqq \left(\frac{\delta^m}{1-\delta}\right)\max_{(\zeta,\eta)\in I}|\varphi_1(\zeta,\eta)|. \tag{6.9}$$

But $|\varphi_1(\zeta,\eta)|<\infty$. Therefore, in the limit when $m\to\infty$, we get

$$\|s_n-s_m\|\to 0.$$

We conclude that s_n is a Cauchy sequence in C[I]. Consequently, the series is convergent and the proof of Theorem 2 is complete. □

Theorem 3. *The maximum absolute truncation error of the series solution* $\varphi(\zeta,\eta)$ *given by*

$$\varphi(\zeta,\eta)=\sum_{j=0}^{\infty}\varphi_j(\zeta,\eta)$$

for the problem (5.5) *is estimated as follows:*

$$\max_{(\zeta,\eta)\in I}\left|\varphi(\zeta,\eta)-\sum_{j=0}^{m}\varphi_j(\zeta,\eta)\right| \leqq \left(\frac{\delta^m}{1-\delta}\right)\max_{(\zeta,\eta)\in I}|\varphi_1(\zeta,\eta)|. \tag{6.10}$$

Proof. According to the formula (6.9), we have

$$\|s_n-s_m\| \leqq \left(\frac{\delta^m}{1-\delta}\right)\max_{(\zeta,\eta)\in I}|\varphi_1(\zeta,\eta)|.$$

In the limit as $n\to\infty$, we see that

$$s_n\to\varphi(\zeta,\eta),$$

so we have

$$\|\varphi(\zeta,\eta)-s_m\| \leqq \left(\frac{\delta^m}{1-\delta}\right)\max_{(\zeta,\eta)\in I}|\varphi_1(\zeta,\eta)|. \tag{6.11}$$

Therefore, the maximum absolute truncation error in the interval I is estimated by

$$\max_{(\zeta,\eta)\in I}\left|\varphi(\zeta,\eta)-\sum_{j=0}^{m}\varphi_j(\zeta,\eta)\right| \leqq \left(\frac{\delta^m}{1-\delta}\right)\max_{(\zeta,\eta)\in I}|\varphi_1(\zeta,\eta)|, \tag{6.12}$$

which evidently completes the proof of Theorem 3 □

7. Conclusion

In this paper, we have successfully and efficiently applied the *Adomian Decomposition Method* (ADM) to derive the approximate solutions of the time-fractional and space-time fractional-order Nagumo equation. In the case of space-time fractional-order Nagumo equation, we expanded the tanh-function initial condition in terms of the basis functions $e^{-n\zeta}$. The fractional derivative could then be easily calculated. As there no direct methods to compute the fractional derivatives, many authors seem to have avoided this type of initial conditions. We have studied the convergence analysis and applied our results to the Nagumo equation. The agreement with the numerical solutions are remarkably favorable. Aside from such favorable agreement, the results demonstrate that the ADM provided a fairly accurate technique for solving the time-fractional and space-time fractional-order Nagumo equation. By increasing the number of iterations, we can reach any desired accuracy. We have the software MATHEMATICA 9 in all of our calculations.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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