Journal of Informatics and Mathematical Sciences
Vol. 9, No. 3, pp. 889–902, 2017
ISSN 0975-5748 (online); 0974-875X (print)
Published by RGN Publications

http://www.rgnpublications.com

Proceedings of the Conference
Current Scenario in Pure and Applied Mathematics
December 22-23, 2016
Kongunadu Arts and Science College (Autonomous)
Coimbatore, Tamil Nadu, India

Research Article

Approximate Controllability of Nonlinear Stochastic Integrodifferential Third Order Dispersion System

R. Nirmalkumar* and R. Murugesu

Department of Mathematics, SRMV College of Arts and Science, Coimbatore 641020, Tamilnadu, India
Corresponding author: nirmalkumarsrmvcas@gmail.com

Abstract. In this work, a class of control systems governed by the stochastic nonlinear integrodifferential third order dispersion equations in Hilbert spaces are considered. The existence of mild solutions of stochastic nonlinear integrodifferential third order dispersion equations are proved using fixed point theory, semigroup properties and stochastic analysis techniques. A new set of sufficient conditions are formulated which guarantees the approximate controllability of the main problem. An example is provided to illustrate the application of the main result.

Keywords. Approximate Controllability, Semigroup theory, Stochastic Korteweg-deVries equation, Schauder's fixed point theorem

MSC. 34K50; 35Q53; 34G20

Received: January 15, 2017 **Accepted:** March 18, 2017

1. Introduction

In recent years, there have been a tremendous development in the study of nonlinear waves and a class of nonlinear wave equations which arise frequently in applications. The wide interest in this field comes from the understanding of special waves called solitons and the associated development of a method of solution to a class of nonlinear wave equations are known as nonlinear *Korteweg-de Vries* (KdV) equation. The Kdv equation describes the motion of long, unidirectional, nonlinear water waves on a channel. In fact the KdV equation is a fundamental model of the weakly nonlinear waves in the weakly dispersive media. It is a good model for describing wave phenomena in plasma dynamics. Washimi and Taniuti [21] established in 1966, that the KdV equation governs the propagation of small-amplitude ionacoustic waves. When the surface of the fluid is submitted to a nonconstant pressure, or when the bottom of the layer is not flat, a forcing term has to be added to the equation. This term is given by the gradient of the exterior pressure or of the function whose graph defines the bottom. Considering the forcing term in random, which is a very natural approach if it is assumed that the exterior pressure is generated by a turbulent velocity field for instance. This random force also is assumed of white noise type.

Wadati [18] in 1983 studied the diffusion of soliton of the KdV equation under Gaussian noise and answered the interesting question: 'How does external noise affect the motion of solitons?'. He also studied the behaviors of solitons under the Gaussian white noise of the stochastic KdV equations with and without damping [19]. A Soliton phenomenon is an attractive field of present day research not only in nonlinear physics and mathematics but also in fiber optics and communication engineering. A nonlinear partial differential equation which describes wave propagation in random media was also studied by Wadati [20]. The stochastic KdV equation arises when modelling the propagation of weakly nonlinear waves in a noisy plasma [4]. Physically speaking, the analytic solutions of the stochastic KdV models, especially the solitonic solutions, might help us to understand the physical stochastic mechanisms such as fluid dynamics and plasma physics. Many authors have been studied the stochastic KdV equations [3, 7, 21]. From a mathematical point of view, this equation is recognized as a simple canonical equation for such phenomena because it combines some of the simplest types of dispersion with nonlinearity. When using a convenient set of coordinates and after rescaling the unknown, it can be written as

$$\frac{\partial w}{\partial t}(x,t)+\sigma w(x,t)\frac{\partial w}{\partial x}(x,t)+\frac{\partial^3 w(x,t)}{\partial x^3}=\xi, \tag{1.1}$$

where σ is a real number, ξ is white noise type, w is the amplitude or velocity, x is often proportional to distance in the direction of propagation and t is proportional to elapsed time. Because $\sigma = 0$, (1.1) becomes the stochastic third order dispersion equation.

Mathematical control theory forms a part of application oriented mathematics that deals with the basic principles underlying the analysis and design of control systems. In deterministic case, Russell [14] studied the exact controllability and stabilizability of the KdV equation and Zhang [23] studied the exact boundary controllability of the KdV equations. Many authors have studied on the controllability problems of third-order dispersion equation [1, 13, 17] and references therein. Chalishajar et al. [2] studied the sufficient conditions for the controllability of nonlinear integro-differential third order dispersion equation. But the concept of exact controllability is very limited for many partial differential equations, the approximate controllability is more appropriate for these control systems instead of exact controllability. Muthukumar et al. [9] studied the approximate controllability for semi-linear retarded stochastic systems in Hilbert spaces. Sakthivel et al. [15] proved the approximate controllability of the nonlinear third-order dispersion equation under the assumption that the corresponding linear system is approximately controllable, and the results are obtained by using fixed point approach and semigroup theory. Here, we move from deterministic nonlinear third-order dispersion equation to stochastic nonlinear third-order dispersion equation for the study of controllability. Motivated by the studies [1, 2, 9, 10, 13, 15, 17], the approximate controllability of stochastic nonlinear integrodifferential third-order dispersion equations remains an untreated topic in the literature and hence we are studying the approximate controllability of stochastic nonlinear integrodifferential third order dispersion equations by using the Hypothesis (H_3) in [15].

In this paper some preliminaries, notations are discussed, and we prove the approximate controllability results using Schauder fixed point theorem and an example is discussed at the end.

2. Preliminaries

In this paper, we study the approximate controllability of stochastic nonlinear integrodifferential third order dispersion equations in Hilbert spaces described by

$$\begin{aligned}\frac{\partial w(x,t)}{\partial t}+\frac{\partial^3 w(x,t)}{\partial x^3} &= Bu(x,t)+F\left(t,w(x,t),\int_0^t g(t,s,w(x,s))ds\right)\\ &\quad +G\left(t,w(x,t),\int_0^t h(t,s,w(x,s))ds\right)\partial\beta(t), \quad t>0.\\ w(x,0)&=0 \end{aligned} \tag{2.1}$$

on the domain $t\in[0,b]=J$, $0\le x\le 2\pi$, with the periodic boundary conditions

$$\frac{\partial^k w(0,t)}{\partial x^k}=\frac{\partial^k w(2\pi,t)}{\partial x^k}, k=0,1,2 \tag{2.2}$$

where the state variable $w(\cdot,t)$ takes values in a Hilbert space $H=L_2(0,2\pi)$ with the inner product $\langle\cdot,\cdot\rangle$ and the norm $\|\cdot\|$, and the control function $u(\cdot)$ is given in $L_2(J,U)$, a Banach space of admissible control functions, with $U=L_2(0,2\pi)$ as a Banach space. The notation $L_2(0,2\pi)$ is the space of real valued measurable $\{\mathscr{F}_t\}$-adapted process $f=\{f(x)\}_{0\le x\le 2\pi}$ such that $E\int_0^{2\pi}|f(x)|^2dx<\infty$ (the subscript 2 means that f is square integrable function). From the practical point of view, we restrict the distributed control $u(x,t)$ so that the quantity $[w]=\int_0^{2\pi}w(x,t)dx$ is conserved and in order to conserve this quantity, we define the bounded linear operator B is defined as $(Bu)(x,t)=\left\{g(x)u(x,t)-\int_0^{2\pi}g(s)u(s,t)ds\right\}$ where $g(x)$ is a piecewise continuous non-negative function on $[0,2\pi]$ such that $[g]=\int_0^{2\pi}g(s)ds=1$, for more details [13]. In [13], Russel et al., discussed the bounded linear operator B and exponential decay rates with distributed controls of restricted form and for the equation with boundary dissipation $F : J \times H \times H \to H$, $g : J \times J \times H \to H$, $G : J \times H \times H \to \mathscr{L}(H)$, $h : J \times J \times H \to H$ are nonlinear functions.

The mathematical construction of β can be described as follows(see [3, 6–8, 10, 16, 22] and references therein). Let $(\Omega, \mathscr{F}, P)$ be a complete probability space furnished with a complete family of right continuous increasing sub σ algebras $\{\mathscr{F}_t, t \in J\}$ satisfying $\mathscr{F}_t \subset \mathscr{F}$ and a filtration $\{\mathscr{F}_t, t \in [0, b]\}$ generated by a one dimensional Wiener process $\{\beta(s) : 0 \le s \le t\}$ defined on the probability space $(\Omega, \mathscr{F}, P)$. The Wiener process β on $L_2(0, 2\pi)$ by setting $\beta = \sum_{k=0}^{\infty} \beta_k e_k$, where $\{e_k\}_{k \in N}$ is an orthonormal basis of $L_2(0, 2\pi)$ and $\{\beta_k\}_{k \in N}$ be a sequence of real valued one dimensional standard Browian motions mutually independent over $(\Omega, \mathscr{F}, P)$. Let $\psi \in \mathscr{L}(H)$ and define $\|\psi\|^2 = Tr(\psi^* \psi) = \sum_{k=1}^{\infty} \|\psi e_k\|^2$, where $\mathscr{L}$ is the space of all Hilbert schmidt operators. $\mathscr{C}(J, L_2(\Omega : H))$ denotes the space of all continuous functions from J into $L_2(\Omega : H)$ satisfying the condition $sup_{t \in J} E\|x(t)\|^2 < \infty$, where E is the expectation with respect to the probability measure P.

Let A denote the operator $Aw = -w'''$ on the domain $D(A) \subset L_2(0, 2\pi)$ consisting of functions in $H^3(0, 2\pi)$ satisfying the boundary conditions of (2.2). By Lemma 8.5.2 of Pazy [12], A is infinitesimal generator of C_0-semigroup $T(t)$, $t \ge 0$ isometries on $L_2(0, 2\pi)$. Then for all $w \in D(A)$

$$\begin{aligned}
\langle Aw, w\rangle_H &= \langle -w''', w\rangle \\
&= \int_0^{2\pi} -w''' \cdot w dx \\
&= (-ww'')_0^{2\pi} - \int_0^{2\pi} -w'' \cdot w' dx \\
&= -(-w'w')_0^{2\pi} + \int_0^{2\pi} -w' \cdot w'' dx \\
&= (-ww'')_0^{2\pi} - \int_0^{2\pi} -w \cdot w''' dx \\
&= \int_0^{2\pi} w \cdot w''' dx \\
&= \langle w, w'''\rangle \\
&= -\langle Aw, w\rangle.
\end{aligned}$$

Also, there exists a constant $M > 0$ such that $\|T(t)\| \le M$, $t \in J$.

Throughout this paper, we impose the following hypotheses:

(H_1) The function $F : J \times H \times H \to H$ satisfies the following conditions:

(i) The function $F : J \times H \times H \to H$ satisfies the Lipschitz condition and there exists a constant $M_1 > 0$, such that $\|F(\cdot, w(x_1, \cdot), y_1) - F(\cdot, w(x_2, \cdot), y_2)\|^2 \le M_1(\|w(x_1, \cdot) - w(x_2, \cdot)\|^2 + \|y_1 - y_2\|^2)$ for every $x_i, y_i \in H$, $i = 1, 2$.

(ii) The function $F : J \times H \times H \to H$ is continuous and uniformly bounded and there exists a constant $N_1 > 0$ such that $\|F(t, w(x, s), \int_0^t g(t, s, w(x, s))ds)\|^2 \le N_1$, for all $t \in J$.

(iii) The function $g : J \times J \times H \to H$ is continuous and uniformly bounded and there exists a constant $C_1 > 0$, such that $\|g(t,s,w(x_1,t)) - g(t,s,w(x_2,t))\|^2 \le C_1\|w(x_1,t) - w(x_2,t)\|^2$ for every $x_1, x_2 \in H$

(H$_2$) The function $G : J \times H \times H \to \mathscr{L}(H)$ satisfies the following conditions:

(i) The function $G : J \times H \times H \to \mathscr{L}(H)$ satisfies the Lipschitz condition and there exists a constant $M_2 > 0$ such that $\|G(t,w(x_1,t),z_1) - G(t,w(x_2,t),z_2)\|^2 \le M_2(\|w(x_1,t) - w(x_2,t)\|^2 + \|z_1 - z_2\|^2)$ for every $x_1, x_2, z_1, z_2 \in H$.

(ii) The function $G : J \times H \times H \to \mathscr{L}(H)$ is continuous and uniformly bounded and there exists a constant $N_2 > 0$ such that $\|G(t,w(x,s),\int_0^t h(t,s,w(x,s)))\|^2 \le N_2$, for all $t \in J$.

(iii) The function $h : J \times J \times H \to H$ is continuous and uniformly bounded and there exists a constant $C_2 > 0$, such that $\|h(t,s,w(x_1,t)) - h(t,s,w(x_2,t))\|^2 \le C_2\|w(x_1,t) - w(x_2,t)\|^2$ for every $x_1, x_2 \in H$

By using the variation of constant formula, there exists a unique mild solution of (2.1)

$$w(x,t) = \int_0^t T(t-s)(Bu)(x,s)ds + \int_0^t T(t-s)F\left(s,w(x,s),\int_0^s g(s,\tau,w(x,\tau))ds\right)ds \\ + \int_0^t T(t-s)G\left(s,w(x,s),\int_0^s h(s,\tau,w(x,\tau))ds\right)d\beta(s). \tag{2.3}$$

We define the reachable set of the system (2.1) as defined by $\mathscr{R}_b(\cdot) = w(\cdot,b;u) : u \in L_2(J,U)$.

Definition 2.1. The system (2.1) is approximately controllable on J if $\mathscr{R}_b(\cdot)$ dense in $L_2(\Omega;H)$, that is $\overline{\mathscr{R}_b(\cdot)} = L_2(\Omega;H)$, where $\overline{\mathscr{R}_b(\cdot)}$ is the closure of $\mathscr{R}_b(\cdot)$.

We define the linear operators S_1 from $L_2(J,H)$ to H and S_2 from $L_2(J,H)$ to $L_2(\Omega,H)$ by $S_1 q_1 = \int_0^b T(b-s)q_1(s)ds$ and $S_2 q_2 = \int_0^b T(b-s)q_2(s)d\beta(s)$ for $q_1, q_2 \in L_2(J,H)$ (see [5,9]). The system (2.1) is approximately controllable on J if for any $\epsilon > 0$ and $\xi_b \in L_2(\Omega;H)$, there exists a control $u \in L_2(J,U)$ such that

$$E\|\xi_b - S_1F(\cdot,w(x(\cdot),\cdot),\cdot) - S_2G(\cdot,w(x(\cdot),\cdot),\cdot) - S_1Bu\|^2 < \epsilon.$$

To this purpose, we need the following hypothesis:

(H$_3$) For any $\epsilon > 0$ and $q_1, q_2 \in L_2(J,H)$, there exists a $u \in L_2(J,U)$ such that

$$E\|S_1q_1 - S_2q_2 - S_1Bu\|^2 < \epsilon,$$
$$\|Bu\|^2_{L_2(0,t;H)} \le N\left(\|q_1\|^2_{L_2(0,t;H)} + \|q_2\|^2_{L_2(0,t;H)}\right), \quad 0 \le t \le b,$$

where N is a constant independent of q_1 and q_2.

Remark 2.2. In order to verify the aforementioned hypothesis (H_3), let $U = H$, $0 < \tau < b$, and we define the intercept control operator Φ_τ on $L_2(0,b;H)$ (see [5]) by

$$\Phi_\tau u(t) = \begin{cases} 0, & 0 \le t < \tau, \\ u(t), & \tau \le t \le b, \end{cases}$$

for $u \in L_2(0,b;H)$. For a given $q_1, q_2 \in L_2(0,b;H)$, let us choose a control function u satisfying

$$u(t) = \begin{cases} 0, & 0 \le t < \tau, \\ q_1(t) + q_2(t) + \frac{\tau}{b-\tau} T(t - \frac{\tau}{b-\tau}(t-\tau))\left\{q_1(\frac{\tau}{b-\tau}(t-\tau)) + q_2(\frac{\tau}{b-\tau}(t-\tau))\right\}, & \tau \le t \le b. \end{cases}$$

Then $u \in L_2(0,b;H)$. Let us assume $\beta(t)$ as one dimensional Brownian motion which satisfies

$$E\|S_1 q_1 - S_2 q_2 - S_1 \Phi_\tau u\|^2 < \epsilon.$$

From the following

$$\begin{aligned} E\,\|\Phi_\tau u\|^2_{L_2(0,b;H)} =& E\,\|u\|^2 \\ \le& 4\,\|q_1\|^2_{L_{2(\tau,b;H)}} + 4\,\|q_2\|^2_{L_{2(\tau,b;H)}} \\ &+ 4\left\|\frac{\tau}{b-\tau}\right\|^2 M^2 \left\{\left\|q_1\left(\frac{\tau}{b-\tau}(t-\tau)\right)\right\|^2_{L_{2(\tau,b;H)}} + \left\|q_2\left(\frac{\tau}{b-\tau}(t-\tau)\right)\right\|^2_{L_{2(\tau,b;H)}}\right\} \\ \le& 4\left(1 + \left\|\frac{\tau}{b-\tau}\right\|^2\right) \|q_1\|^2_{L_{2(\tau,b;H)}} + \|q_2\|^2_{L_{2(\tau,b;H)}} \end{aligned}$$

it follows that the controller Φ_τ satisfies Hypothesis (H_3).

Lemma 2.3. *Assume that the Hypotheses* (H_1)–(H_3) *are satisfied, then the system* (2.1) *has a solution on* J.

Proof. Let $Z = \{w(x,t) \in \mathscr{C}(J, L_2(\Omega;H)); w(x,0) = 0\}$ be the space endowed with uniform convergence topology. On the space Z, we consider the set $B_r = \{w(x,t) \in Z; E\|w(x,t)\|^2 \le r\}$, where r is a positive constant.

We define the operator $\Psi : Z \to Z$ by

$$\begin{aligned} (\Psi w)(x,t) = & \int_0^t T(t-s)(Bu)(x,s)ds + \int_0^t T(t-s)F\left(s, w(x,s), \int_0^t g(s,\tau,w(x,s))ds\right)ds \\ & + \int_0^t T(t-s)G\left(s, w(x,s), \int_0^t h(s,\tau,w(x,s))ds\right)d\beta(s). \end{aligned}$$

We shall show that the operator Ψ has a fixed point, which is a solution of (2.1).

Step 1. For all $t > 0$, there exists a positive constant r^* such that Ψ maps B_r into itself.

$$E\,\|(\Psi w)(x,t)\|^2 \le 3M^2 b[bN(N_1+N_2) + bN_1 + N_2],$$

where $\|Bu\|^2 \le N(N_1+N_2)$ (see (H_3)). The aforementioned inequality imply that for large enough $r^* > 0$, the following inequality holds

$$E\,\|(\Psi w)(x,t)\|^2 \le r^*.$$

Therefore Ψ maps B_r into itself.

Step 2. For $t > 0$, the operator Ψ maps B_r into relatively compact subset of B_r. First, we prove that the set $V(t) = \{(\Psi w)(x,t); w \in B_r\}$ is relatively compact in $L_2(\Omega;H)$, for every $t \in J$. The case $t = 0$ is obvious. Let $0 < \epsilon < t \le b$, we define

$$\begin{aligned} (\Psi^\epsilon w)(x,t) =& T(\epsilon)\left[\int_0^{t-\epsilon} T(t-s-\epsilon)(Bu)(x,s)ds\right. \\ &+ \int_0^{t-\epsilon} T(t-s-\epsilon)F\left(s, w(x,s), \int_0^t g(s,\tau,w(x,s))ds\right)ds \end{aligned}$$

$$+\int_0^{t-\epsilon} T(t-s-\epsilon)G\left(s,w(x,s),\int_0^t h(t,s,w(x,s))ds\right)d\beta(s)\Bigg].$$

Because $T(t)$, $t>0$ is compact, the set $V_\epsilon(t)=\{(\Psi^\epsilon w)(x,t);w\in B_r\}$ is relatively compact in $L_2(\Omega;H)$. Moreover, for every $w\in B_r$, we have

$$\begin{aligned}E\left\|(\Psi w)(x,t)-(\Psi^\epsilon w)(x,t)\right\|^2 \le{}& E\Bigg\|\int_{t-\epsilon}^t T(t-s)[(Bu)(x,s)\\ &+F\left(s,w(x,s),\int_0^t g(s,\tau,w(x,s))ds\right)\Bigg]ds\\ &+\int_{t-\epsilon}^t T(t-s)G\left(s,w(x,s),\int_0^t h(t,s,w(x,s)ds)\right)d\beta(s)\Bigg\|^2\\ \le{}& 3M^2\left\{\epsilon\int_{t-\epsilon}^t[N(N_1+N_2)+N_1]ds+\int_{t-\epsilon}^t N_2ds\right\}.\end{aligned}$$

This implies that there are relatively compact sets arbitrary close to the set $V(t)=\{(\Psi w)(x,t);$ $w\in B_r\}$ and hence the set $V(t)$ is relatively compact in $L_2(\Omega;H)$.

Step 3. We prove that the family $\{(\Psi w)(x,t);w\in B_r\}$ is an equicontinuous on J. Let $0<t_1<t_2\le b$.

$$\begin{aligned}E\|(\Psi w)(x,t_1)-(\Psi w)(x,t_2)\|^2 \le{}& E\Bigg\|\int_0^{t_1}[T(t_1-s)-T(t_2-s)]\\ &\times\left[(Bu)(x,s)+F\left(s,w(x,s),\int_0^t g(s,\tau,w(x,s))ds\right)\right]ds\\ &+\int_{t_1}^{t_2}T(t_2-s)[(Bu)(x,s)\\ &+F\left(s,w(x,s),\int_0^t g(s,\tau,w(x,s))ds\right)\Bigg]ds\\ &+\int_0^{t_1}[T(t_1-s)-T(t_2-s)]\\ &\times G\left(s,w(x,s),\int_0^t h(s,\tau,w(x,s)ds))\right)d\beta(s)\\ &+\int_{t_1}^{t_2}T(t_2-s)\times G\left(s,w(x,s),\int_0^t h(s,\tau,w(x,s)ds)\right)d\beta(s)\Bigg\|^2\\ \le{}& 4b\int_0^{t_1}\|[T(t_1-s)-T(t_2-s)]\|^2[N(N_1+N_2)+N_1]ds\\ &+4M^2b\int_{t_1}^{t_2}[N(N_1+N_2)+N_1]ds\\ &+4\int_0^{t_1}\|[T(t_1-s)-T(t_2-s)]\|^2N_2ds+4M^2\int_{t_1}^{t_2}N_2ds.\end{aligned}$$

The right hand side does not depend on any particular choices of $w\in B_r$ and tends to zero as $t_1-t_2\to 0$, because the compactness of $T(t)$ for $t>0$ implies the continuity in the uniform operator topology. Thus, Ψ is equicontinuous on J. Thus $\Psi(B_r)$ is equicontinuous and also bounded. By the Ascoli-Arzela theorem, $\Psi(B_r)$ is relatively compact in $L_2(\Omega;H)$. It is easy to show that for all $t>0$, Ψ is continuous on Z. Hence from Schauder's fixed point theorem Ψ has a fixed point. Thus (2.1) has a solution on J. □

Lemma 2.4. *Let u_1 and u_2 be in $L_2(J,U)$, then under Hypotheses* (H$_1$) *and* (H$_2$)*, we have*

$$E\|w(x,t;u_2)-w(x,t;u_1)\|^2 \le 3M^2 t\|Bu_2-Bu_1\|^2_{L_2(J,H)} \exp\{3M^2 b(bM_F(1+C_1)+M_G(1+C_2))\},\quad 0\le t\le b.$$

Proof.

$$\begin{aligned}E\|w(x,t;u_2)-w(x,t;u_1)\|^2 &\le 3E\left\|\int_0^t T(t-s)[Bu_2(x,s)-Bu_1(x,s)]ds\right\|^2\\ &\quad +3E\left\|\int_0^t T(t-s)[F(s,w(x,s:u_2,),\int_0^t g(s,\tau,w(x,s:u_2))ds\right.\\ &\quad \left.-F(s,w(x,s:u_1),\int_0^t g(s,\tau,w(x,s:u_1))ds\right\|^2\\ &\quad +3E\left\|\int_0^t T(t-s)[G(s,w(x,s;u_2),\int_0^t h(s,\tau,w(x,s:u_2))ds)\right.\\ &\quad \left.-G(s,w(x,s;u_1),\int_0^t h(s,\tau,w(x,s:u_1))ds)]d\beta(s)\right\|^2.\\ &\le 3M^2\int_0^t E\|Bu_2(x,s)-Bu_1(x,s)\|^2ds\\ &\quad +3M^2(bM_1(1+C_1+M_2(1+C_2))\\ &\quad \int_0^t E\|w(x,s:u_2)-w(x,s:u_1)\|^2ds,\end{aligned}$$

by using Grownwall's inequality

$$\le 3M^2t\|Bu_2-Bu_1\|^2_{L_2(J,H)}\exp\{3M^2b(bM_1(1+C_1)+M_2(1+C_2))\}$$

hence the lemma is proved. □

3. Approximate Controllability

Theorem 3.1. *Under hypotheses* (H$_1$)–(H$_3$)*, the system* (2.1) *is approximately controllable on J.*

Proof. The system (2.1) is approximately controllable on J if for any $\epsilon>0$ and $\xi_b\in L_2(\Omega;H)$, there exists a control $u\in L_2(J,U)$ such that

$$E\|\xi_b-S_1F(\cdot,w(x(\cdot),\cdot),\cdot)-S_2G(\cdot,w(x(\cdot),\cdot),\cdot)-S_1Bu\|^2<\epsilon.$$

Because $D(A)$ is dense in $L_2(\Omega;H)$, it is sufficient to prove $D(A)\subset\overline{\mathscr{R}_b(\cdot)}$.

That is, for any given $\epsilon>0$ and $\xi_b\in D(A)$, there exists a $u\in L_2(J,U)$ such that

$$E\|\xi_b-w(x,b;u)\|^2<\epsilon.$$

Because $\xi_b\in D(A)$, there exists a $q_1\in L_2(J,H)$ such that $S_1q_1=\xi_b$, for instance, take $q_1(s)=(\xi_b-sA\xi_b)$. Let $u_1\in L_2(J,U)$ be arbitrary fixed. Because by assumption (H$_3$), there exists a $u_2\in L_2(J,U)$ such that

$$E\|S_1(q_1-F(\cdot,w(x,\cdot;u_1),\cdot))-S_2G(\cdot,w(x,\cdot;u_1),\cdot)-S_1Bu_2\|^2<\frac{\epsilon}{2^4}$$

thus

$$E\,\|\xi_b - S_1F(\cdot,w(x,\cdot;u_1),\cdot)) - S_2G(\cdot,w(x,\cdot;u_1),\cdot) - S_1Bu_2\|^2 < \frac{\epsilon}{2^4}, \tag{3.1}$$

we can also choose $v_2 \in L_2(J,U)$ such that

$$E\,\|S_1(F(\cdot,w(x,\cdot;u_2),\cdot) - F(\cdot,w(x,\cdot;u_2),\cdot)) + S_2(G(\cdot,w(x,\cdot;u_2),\cdot - G(\cdot,w(x,\cdot;u_1),\cdot)) - S_1Bv_2\|^2 < \frac{\epsilon}{2^5}, \tag{3.2}$$

and

$$\|Bv_2\|^2_{L_2(0,t;H)} \leq N\Big\{\|F(\cdot,w(x,\cdot;u_2),\cdot) - F(\cdot,w(x,\cdot;u_1),\cdot)\|^2_{L_2(0,t;H)} + \|G(\cdot,w(x,\cdot;u_2),\cdot) - G(\cdot,w(x,\cdot;u_1),\cdot)\|^2_{L_2(0,t;H)}\Big\}$$

for $0 \leq t \leq b$. Therefore, in view of Hypotheses (H$_1$)-(H$_3$) and Lemma 2.4

$$\begin{aligned}
\|Bv_2\|^2_{L_2(0,t;H)} &\leq N\Big\{\int_0^t E\,\Big\|F(\tau,w(x,\tau;u_2),\int_0^t g(s,\tau,w(x,\tau:u_2))ds) \\
&\quad -F(\tau,w(x,\tau;u_1),\int_0^t g(s,\tau,w(x,\tau:u_1))ds)\Big\|^2 d\tau \\
&\quad + \int_0^t E\,\Big\|G(\tau,w(x,\tau;u_2),\int_0^t h(s,\tau,w(x,\tau;u_2))ds) \\
&\quad -G(\tau,w(x,\tau;u_1),\int_0^t h(s,\tau,w(x,\tau;u_1))ds)\Big\|^2\Big\} \\
&\leq N\Big\{M_1(1+C_1)\int_0^t E\,\|(w(x,\tau;u_2) - w(x,\tau;u_1))\|^2 d\tau \\
&\quad + M_2(1+C_2)\int_0^t E\,\|w(x,\tau;u_2) - w(x,\tau;u_1)\|^2 d\tau\Big\} \\
&\leq N\{M_1(1+C_1) + M_2(1+C_2)\} \times \int_0^t E\,\|(w(x,\tau;u_2) - w(x,\tau;u_1))\|^2 d\tau \\
&\leq N\{M_1(1+C_1) + M_2(1+C_2)\} \\
&\quad \times \int_0^t 3M^2\tau\,\|Bu_2 - Bu_1\|^2_{L_2(J,H)} \exp\{3M^2b(bM_1(1+C_1) + M_2(1+C_2))\}\,d\tau \\
&\leq N\{M_1(1+C_1) + M_2(1+C_2)\} \\
&\quad \times \int_0^t 3M^2 \exp\{3M^2b(bM_1(1+C_1) + M_2(1+C_2))\}\,\frac{t^2}{2}\,\|Bu_2 - Bu_1\|^2_{L_2(J,H)}.
\end{aligned}$$

Putting $u_3 = u_2 - v_2$, we determine v_3 such that

$$E\|S_1(F(\cdot,w(x,\cdot;u_3),\cdot) - F(\cdot,w(x,\cdot;u_2),\cdot)) + S_2(G(\cdot,w(x,\cdot;u_3),\cdot) - G(\cdot,w(x,\cdot;u_2),\cdot)) - S_1Bv_3\|^2 < \frac{\epsilon}{2^5},$$

$$\|Bv_3\|^2_{L_2(0,t;H)} \leq N\Big\{\|F(\cdot,w(x,\cdot;u_3),\cdot) - F(\cdot,w(x,\cdot;u_2),\cdot)\|^2_{L_2(0,t;H)} + \|G(\cdot,w(x,\cdot;u_3),\cdot) - G(\cdot,w(x,\cdot;u_2),\cdot)\|^2_{L_2(0,t;H)}\Big\},$$

for $0 \leq t \leq b$. Hence by Hypotheses (H$_1$)-(H$_2$), we have

$$\|Bv_3\|^2_{L_2(0,t;H)} \leq N\Big\{\int_0^t E\,\Big\|F(\tau,w(x,\tau;u_3),\int_0^t g(s,\tau,w(x,\tau;u_3))ds)$$

$$
\begin{aligned}
& -F(\tau,w(x,\tau;u_2)),\int_0^t g(s,\tau,w(x,\tau;u_2))ds)\Big\|^2 d\tau \\
& + \int_0^t E\Big\|G(\tau,w(x,\tau;u_3)),\int_0^t h(s,\tau,w(x,\tau;u_3)ds)) \\
& -G(\tau,w(x,\tau;u_2)),\int_0^t h(s,\tau,w(x,\tau;u_2)ds))\Big\|^2\Big\} \\
\le\ & N\Big\{M_1(1+C_1)\int_0^t E\,\|(w(x,\tau;u_3)-w(x,\tau;u_2))\|^2 d\tau \\
& + M_2(1+C_2)\int_0^t E\,\|w(x,\tau;u_3)-w(x,\tau;u_2)\|^2 d\tau\Big\} \\
\le\ & N\{M_1(1+C_1)+M_2(1+C_2)\}\int_0^t E\,\|(w(x,\tau;u_3)-w(x,\tau;u_2))\|^2 d\tau \\
\le\ & N\{M_1(1+C_1)+M_2(1+C_2)\} \\
& \times 3M^2\exp\{3M^2b(bM_1(1+C_1)+M_2(1+C_2))\}\int_0^t \tau\,\|Bu_3-Bu_2\|^2_{L_2(J,H)}d\tau \\
\le\ & N\{M_1(1+C_1)+M_2(1+C_2)\} \\
& \times 3M^2\exp\{3M^2b(bM_1(1+C_1)+M_2(1+C_2))\}\int_0^t \tau\,\|Bv_2\|^2_{L_2(J,H)}d\tau \\
\le\ & (N\{M_1(1+C_1)+M_2(1+C_2)\} \\
& \times 3M^2\exp\{3M^2b(bM_1(1+C_1)+M_2(1+C_2))\})^2\int_0^t \frac{\tau^3}{2}\,\|Bu_1-Bu_2\|^2_{L_2(J,H)}d\tau \\
\le\ & (N\{M_1(1+C_1)+M_2(1+C_2)\} \\
& \times 3M^2\exp\{3M^2b(bM_1(1+C_1)+M_2(1+C_2))\})^2\frac{t^4}{2.4}\,\|Bu_1-Bu_2\|^2_{L_2(J,H)}.
\end{aligned}
$$

By proceeding this process, we can obtain a sequence $\{u_n\}_{n\ge1}$ such that $u_{n+1}=u_n-v_n$ and from

$$
\begin{aligned}
\|Bu_n-Bu_{n+1}\|^2_{L_2(0,t;H)} \le\ & \|Bv_n\|^2_{L_2(0,t;H)} \\
\le\ & \{N(M_1(1+C_1)+M_2(1+C_2))3M^2 \\
& \times\{\exp 3M^2b(bM_1(1+C_1)+M_2(1+C_2))\}\}^{n-1} \\
& \times\frac{t^{2n-2}}{2.4\ldots(2n-2)}\,\|Bu_2-Bu_1\|^2_{L_2(0,t;H)} \\
\le\ & \left\{\frac{\begin{pmatrix} N(M_1(1+C_1)+M_2(1+C_2))3M^2 \\ \cdot\{\exp 3M^2b(bM_1(1+C_1)+M_2(1+C_2))\}\end{pmatrix}}{2}\right\}^{n-1} \\
& \times\frac{1}{(n-1)!}\,\|Bu_2-Bu_1\|^2_{L_2(0,t;H)}
\end{aligned}
$$

it follows that

$$
\begin{aligned}
\sum_{n=1}^{\infty}\|Bu_{n+1}-Bu_n\|^2_{L_2(J,H)} \le\ & \sum_{n=0}^{\infty}\left\{\frac{\begin{pmatrix} N(M_1(1+C_1)+M_2(1+C_2))3M^2 \\ \cdot\exp\{3M^2b(bM_1(1+C_1)+M_2(1+C_2))\}t\end{pmatrix}}{2}\right\}^{n} \\
& \times\frac{1}{n!}\,\|Bu_2-Bu_1\|^2_{L_2(0,t;H)} \le\infty.
\end{aligned}
$$

Therefore, there exists $u^* \in L_2(J,H)$ such that $\lim_{n\to\infty} Bu_n = u^*$ in $L_2(J,H)$. From (3.1) and (3.2), it follows that

$$E\left\|\xi_b - S_1F(\cdot,w(x,\cdot;u_2),\int_0^t g(s,\tau,w(x,\tau;u_2))ds)\right.$$
$$\left.-S_2G(\cdot,w(x,\cdot;u_2),\int_0^t h(s,\tau,w(x,\tau;u_2))ds) - S_1Bu_3\right\|^2$$
$$= E\left\|\xi_b - S_1F(\cdot,w(x,\cdot;u_1),\int_0^t g(s,\tau,w(x,\tau;u_1))ds) - S_2G(\cdot,w(x,\cdot;u_1),\int_0^t g(s,\tau,w(x,\tau;u_1))ds)\right.$$
$$-S_1Bu_2 + S_1Bv_2 - S_1[F(\cdot,w(x,\cdot;u_2),\int_0^t g(s,\tau,w(x,\tau;u_2)ds)$$
$$-F(\cdot,w(x,\cdot;u_1),\int_0^t g(s,\tau,w(x,\tau;u_1))ds)] - S_2[G(\cdot,w(x,\cdot;u_2),\int_0^t h(s,\tau,w(x,\tau;u_2))ds)$$
$$\left.-G(\cdot,w(x,\cdot;u_1),\int_0^t h(s,\tau,w(x,\tau;u_1))ds))\right\|^2$$
$$< \left(\frac{1}{2^3}+\frac{1}{2^4}\right)\epsilon.$$

Choosing $v_n \in L_2(J,U)$ by hypothesis (H$_3$) such that

$$E\left\|S_1[F(\cdot,w(x,\cdot;u_n),\int_0^t g(s,\tau,w(x,\tau;u_n))ds)] - F(\cdot,w(x,\cdot;u_{n-1}),\int_0^t g(s,\tau,w(x,\tau;u_{n-1}))ds)\right.$$
$$+S_2[G(\cdot,w(x,\cdot;u_n),\int_0^t h(s,\tau,w(x,\tau;u_n))ds)$$
$$\left.-G(\cdot,w(x,\cdot;u_{n-1}),\int_0^t h(s,\tau,w(x,\tau;u_{n-1}))ds)] - S_1Bv_n\right\|^2 < \frac{\epsilon}{2^{n+2}}.$$

putting $u_{n+1} = u_n - v_n$, we have

$$E\left\|\xi_b - S_1F(.,w(x,.;u_n),\int_0^t g(s,\tau,w(x,\tau;u_n))ds)\right.$$
$$\left.-S_2G(.,w(x,.;u_n),\int_0^t h(s,\tau,w(x,\tau;u_n))ds) - S_1Bu_{n+1}\right\|^2$$
$$< \left(\frac{1}{2^3}+\frac{1}{2^4}+\ldots+\frac{1}{2^{n+2}}\right)\epsilon, \quad n = 1,2,\ldots.$$

Therefore, for $\epsilon > 0$, there exists an integer N such that

$$E\,\|S_1Bu_{N+1} - S_1Bu_N\|^2 < \frac{\epsilon}{2^2}$$

and

$$E\left\|\xi_b - S_1F(\cdot,w(x,\cdot;u_N),\int_0^t g(s,\tau,w(x,\tau;u_N))ds)\right.$$
$$\left.-S_2G(\cdot,w(x,\cdot;u_N,\int_0^t h(s,\tau,w(x,\tau;u_N))ds) - S_1Bu_N\right\|^2$$
$$= E\left\|\xi_b - S_1F(\cdot,w(x,\cdot;u_N),\int_0^t g(s,\tau,w(x,\tau;u_N))ds)\right.$$
$$\left.-S_2G(\cdot,w(x,\cdot;u_N),\int_0^t h(s,\tau,w(x,\tau;u_N))ds) - S_1Bu_{N+1} + S_1Bu_{N+1} - S_1Bu_N\right\|^2$$

$$\leq 2\Big\{E\Big\|\xi_b - S_1F(\cdot,w(x,\cdot;u_N)),\int_0^t g(s,\tau,w(x,\tau;u_N))ds)$$
$$-S_2G(\cdot,w(x,\cdot;u_N)),\int_0^t h(s,\tau,w(x,\tau;u_N))ds) - S_1Bu_{N+1}\Big\|^2$$
$$+E\|S_1Bu_{N+1} - S_1Bu_N\|^2\Big\}$$
$$< 2\left(\frac{1}{2^3}+\frac{1}{2^4}+\ldots+\frac{1}{2^{n+2}}\right)\epsilon + 2\left(\frac{\epsilon}{2^2}\right)$$
$$\leq \epsilon.$$

Thus, the system (2.1) is approximately controllable on J as N tends to infinity. Hence the theorem is proved.

4. Example

In this section, we provide a simple example to illustrate the application of our main result. Consider the following system of partial stochastic nonlinear differential system in Hilbert spaces of the form

$$\begin{aligned}\partial z(t,y) = &\left\{\frac{-\partial^3 Z}{\partial y^3}(t,y) + \mu(t,y) + \frac{1}{2}e^{-t}\sin z(t,y)\int_0^t g(t,s,z(t,y)dy\right\}\\ &+\frac{1}{2}\cos t z(t,y)\int_0^t h(t,s,z(t,y))dy)\partial\beta(s),\end{aligned} \tag{4.1}$$
$$z(0,y) = 0,\ \ 0 \leq y \leq 2\pi,\ t \in [0,1],$$
$$\frac{\partial^k z(t,0)}{\partial y^k} = \frac{\partial^k z(t,2\pi)}{\partial y^k},\qquad k = 0,1,2.$$

Let $H = L_2(0,2\pi)$, $\beta(t)$ stands for a one dimensional wiener process in H defined on a stochastic space $(\Omega,\mathscr{F},P)$. Define an operator A on $L_2(0,2\pi)$ with domain $D(A)$ defined by

$$D(A) = \left\{z \in H^3(0,2\pi);\ \frac{\partial^k z}{\partial y^k}(0) = \frac{\partial^k}{\partial y^k}z(2\pi),\ k = 0,1,2.\right\}$$

Such that $Az = -\frac{\partial^3 z}{\partial y^3}$. It is well known that A is the infinitesimal generator of a C_0-semigroup $(T(t))_{t\geq 0}$ on H and define the bounded linear operator $Bu(t,y) = \mu(t,y)$, $0 \leq y \leq 2\pi$. Next, we consider the problem of approximately for the system (4.1) and to treat this system in the abstract form (1.1) by using the results [8, 11]. Moreover the functions

$$F\left(t,z(t,y),\int_0^t g(t,s,z(t,y))dy\right) = \frac{1}{2}e^{-t}\sin z(t,y)\int_0^t g(t,s,z(t,y))dy,$$
$$G\left(t,z(t,y),\int_0^t h(t,s,z(t,y))dy\right) = \frac{1}{2}\cos t z(t,y)\int_0^t h(t,s,z(t,y))dy,$$

satisfy hypotheses (H_1)-(H_2). Assume that hypothesis (H_3) is satisfied [24]. Further, all the conditions stated in Theorem 3.1 are satisfied. Hence by Theorem 3.1 the system (4.1) is approximately controllable.

5. Conclusion

This article addresses the problem of approximate controllability of stochastic nonlinear integrodifferential third order dispersion equations. A fixed point approach with semigroup theory is employed for achieving the existence for the mild solutions of stochastic nonlinear integrodifferential third order dispersion equations. A new set of sufficient conditions are also formulated for the approximate controllability of control system.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

References

[1] D. N. Chalishajar, R. K. George and A. K. Nandakumaran, Exact controllability of the third order nonlinear dispersion system, *Journal of Mathematical Analysis and Applications* **332** (2007), 1028 – 1044.

[2] D. N. Chalishajar, Controllability of nonlinear integro-differential third order dispersion system, *Journal of Mathematical Analysis and Applications* **348** (2008), 480 – 486.

[3] A. Debussche and J. Printems, Numerical simulation of the stochastic Korteweg-de Vries equation, *Physica D* **134** (1999), 200 – 226.

[4] R. Herman, The stochastic, Damped Korteweg-de Vries equation (1990), *Journal of Physics A: Mathematical and General* **23** (1990), 1063 – 1084.

[5] J.M. Jeong and H.H. Roh, Approximate controllability for semilinear retarded systems, *Journal of Mathematical Analysis and Applications* **321** (2006), 961 – 975.

[6] J. Klamka, Stochastic controllability of linear systems with delay in control, *Bulletin of the Polish Academy of Sciences* **55** (2007), 23 – 29.

[7] G. Lin, L. Grinberg and G.E. Karniadakis, Numerical studies of the stochastic Korteweg-de Vries equation, *Journal of Computational Physics* **213** (2006), 676 – 703.

[8] X. Mao, *Stochastic Differential Equations and Applications*, Chichester, Horwood (1997).

[9] P. Muthukumar and P. Balasubramaniam, Approximate controllability for semi-linear retarded stochastic systems in Hilbert spaces, *IMA Journal of Mathematical Control and Information* **26** (2009), 131 – 140.

[10] P. Muthukumar and C. Rajivganthi, Approximate Controllability of stochastic nonlinear third order dispersion equation, *International Journal of Robust and nonlinear Control* (**2014**), doi:10.1002/rnc.2908.

[11] B. Oksendal, *Stochastic Differential Equations: An Introduction with Applications*, Springer-Verlag, New York (1995).

[12] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Springer-Verlag, New York (1983).

[13] D.L. Russell and B.Y. Zhang, Controllability and stabilizability of the third-order linear dispersion equation on a periodic domain, *SIAM Journal on Control and Optimization* **31** (1993), 659 – 676.

[14] D.L. Russell, Exact controllability and stabilizability of the Korteweg-de Vries equation, *Transactions of the American Mathematical Society* **348** (1996), 3643 – 3672.

[15] R. Sakthivel, N.I. Mahmudov and Y. Ren, Approximate controllability of the nonlinear third-order dispersion equation, *Applied Mathematics and Computation* **217** (2011), 8507 – 8511.

[16] L. Shen and J. Sun, Approximate controllability of abstract stochastic impulsive systems with multiple time-varying delays, *International Journal of Robust and Nonlinear Control*, (**2012**) doi:10.1002/rnc.2789.

[17] N.K. Tomar and N. Sukavanam, Exact controllability of semilinear third order dispersion equation, *The Journal of Nonlinear Sciences and Applications* **4** (2011), 308 – 314.

[18] M. Wadati, Stochastic Korteweg-de Vries equation, *Journal of the Physical Society of Japan* **52** (1983), 2642 – 2648.

[19] M. Wadati and Y. Akutsu, Stochastic Korteweg-de Vries equatiowith and without damping, *Journal of the Physical Society of Japan* **53** (1984), 3342 – 3350.

[20] M. Wadati, Deformation of solitons in random media, *Journal of the Physical Society of Japan* **59** (1990), 4201 – 4203.

[21] H. Washimi and T. Taniuti, Propagation of ion-acoustic solitary waves of small amplitude, *Physical Review Letters* **17** (1996), 996 – 998.

[22] Y.C. Xie, Exact solutions for stochastic KdV equations, *Physics Letters A* **310** (2003), 161 – 167.

[23] B.Y. Zhang, Exact boundary controllability of the KdV equations, *SIAM Journal on Control and Optimization* **37** (1999), 523 – 565.

[24] H.X. Zhou, Approximate controllability for a class of semilinear abstract equations, *SIAM Journal on Control and Optimization* **21** (1983), 551 – 565.