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Research Article

Spectral Properties of *k*-Quasi *Parahyponormal Operators

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Abstract. In this paper, we prove some basic properties of k-quasi-*parahyponormal operators and spectrum of class of k-quasi-*parahyponormal operators is continuous. Also, we proved the non zero points of its approximate point spectrum and joint approximate point spectrum are identical.

Keywords. Parahyponormal operator; Approximate point spectrum and Joint approximate point spectrum

MSC. 47A10; 47B20

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1. Introduction

Let H be a separable complex Hilbert space and $B(H)$ denote the C^*-algebra of all bounded linear operators acting on H. Recall that, $T \in B(H)$ is called p-hyponormal for $p > 0$, if $(T^*T)^p \geq (TT^*)^p$ [1], when $p = 1$ T is called hyponormal operator, when $p = \frac{1}{2}$ T is called semihyponormal operators. An operator T is called log hyponormal operator if T is invertible and satisfies $\log(T^*T) \geq \log(TT^*)$. An operator T is paranormal operator if $\|Tx\|^2 \leq \|T^2x\|\|x\|$, for all $x \in H$ [7,8]. It was originally introduced as an intermediate class between hyponormal operators and normaloid one, i.e., $\|T\| = r(T)$, where $r(T)$ denotes the spectral radius of T. Extensive studies on paranormal have been done by many authors, Halmos has discussed many problems on paranormal and hyponormal operators. As a generalisation of the class of hyponormal operators the class of p-hyponormal operators has been defined and studied by Aluthge. T is called normaloid if $\|T^n\| = \|T\|^n$, for all $n \in N$ (equivalently $\|T\| = r(T)$, the spectral radius of T). Mahmoud M. Kutkut introduced parahyponormal operator. An operator T is parahyponormal operator if $\|Tx\|^2 \leq \|TT^*x\|\|x\|$ for all $x \in H$ [14]. Spectral properties of p-hyponormal operators, quasi hyponormal operators and paranormal operators have been studied by many authors and they have also proved many interesting properties similar to those of hyponormal operators [6,11,17]. The relations between paranormal and p-hyponormal and log hyponormal operators, Furuta *et al*. introduced a very interesting class of bounded linear Hilbert space operators: class A and they showed that class A is a subclass of paranormal and contains p-hyponormal and log-hyponormal operators. One of the recent trends in operator theory is studying natural extension of an operators. We introduce some of the operators as follows.

For every positive integer k an operator T is said to be *parahyponormal operator, if $\|T^*x\|^2 \leq \|T^*Tx\|\|x\|$ for all $x \in H$. For every positive integer k an operator T is said to be k-quasi *parahyponormal operator, if $T^{*k}((T^*T)^2 - 2\lambda TT^* + \lambda^2)T^k \geq 0$ for $\lambda > 0$ and when $k = 1$, it is quasi-*parahyponormal operator. Generally the following implications hold: parahyponormal $\subset$ *parahyponormal $\subset$ quasi *parahyponormal $\subset$ k-quasi-*parahyponormal

In this paper, we prove some basic properties of k-quasi-*parahyponormal operators and spectrum of class of k-quasi-*parahyponormal operators is continuous. Also, we proved the non zero points of its approximate point spectrum and joint approximate point spectrum are identical.

2. Basic Properties of k-Quasi-*Parahyponormal Operators

We derived some basic properties of k-quasi-*parahyponormal operators as follows.

Theorem 2.1. *Let $T \in B(H)$ be k-quasi-*parahyponormal operator for any positive integer $k > 0$ and let $T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}$ on $H = \overline{ran(T^k)} \oplus \ker T^{*k}$ be 2×2 matrix expression. Assume that $ran(T^k)$ is not dense if and only if $(T_1^*T_1)^2 - 2\lambda(T_1T_1^* + T_2T_2^*) + \lambda^2) \geq 0$ on $\overline{ran(T^k)}$ and $T_3^k = 0$. Furthermore, $\sigma(T) = \sigma(T_1) \cup \{0\}$.*

Proof. Let P be the projection of H onto $\overline{ran(T^k)}$. Then $T_1 = TP = PTP$.
Since T is k-quasi-*parahyponormal operator, we have $P((T^*T)^2 - 2\lambda TT^* + \lambda^2)P \geq 0$. Then

$$P(T^*T)^2P - 2\lambda P(TT^*)P + P\lambda^2P \geq 0$$

$$(PT^*TP)^2 - 2\lambda P(TT^*)P + \lambda^2 \geq 0$$

$$(T_1^*T_1)^2 - 2\lambda(T_1T_1^* + T_2T_2^*) + \lambda^2 \geq 0$$

For any $x = (x_1, x_2) \in H$

$$\begin{aligned}\langle T_3^k x_2, x_2\rangle &= \langle T^k(I-P)x, (I-P)x\rangle \\ &= \langle (I-P)x, T^{*k}(I-P)x\rangle \\ &= 0\end{aligned}$$

This implies $T_3^k = 0$.

Since $\sigma(T) \cup M = \sigma(T_1) \cup \sigma(T_3)$ where M is the union of the holes in $\sigma(T)$, which happens to be a subset of $\sigma(T_1) \cup \sigma(T_3)$ by [1, Corollary 7]. $\sigma(T_3) = 0$ and $\sigma(T_1) \cup \sigma(T_3)$ has no interior points we have $\sigma(T) = \sigma(T_1) \cup \{0\}$.

Suppose that $T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}$ on $H = \overline{ran(T^k)} \oplus \ker T^{*k}$

where $((T_1^*T_1)^2 - 2\lambda(T_1T_1^* + T_2T_2^*) + \lambda^2) \geq 0$, for every $\lambda > 0$ and $T_3^k = 0$

$$T^k = \begin{pmatrix} T_1^k & \sum_{j=0}^{k-1} T_1^j T_2 T_3^{k-1-j} \\ 0 & 0 \end{pmatrix}, \qquad (T^*T)^2 = \begin{pmatrix} (T_1^*T_1)^2 + T_1^*T_2 + T_2^*T_1 & B \\ B^* & D \end{pmatrix}$$

where $D = T_2^*T_1T_1^*T_2 + (T_2^*T_2)^2 + (T_3^*T_3)^2 + T_3^*T_3T_2^*T_2 + T_2^*T_2T_3^*T_3$
and $B = T_1^*T_1T_1^*T_2 + T_1^*T_2T_2^*T_2 + T_1^*T_2T_3^*T_3$

$$\begin{aligned}T^kT^{*k} &= \begin{pmatrix} (T_1^kT_1^{*k}) + \sum_{j=0}^{k-1} T_1^jT_2T_3^{k-1-j}(\sum_{j=0}^{k-1} T_1^jT_2T_3^{k-1-j})^* & 0 \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}\end{aligned}$$

where $A = A^* = (T_1^kT_1^{*k}) + \sum_{j=0}^{k-1} T_1^jT_2T_3^{k-1-j}(\sum_{j=0}^{k-1} T_1^jT_2T_3^{k-1-j})^* \geq 0$ for every $\lambda > 0$.

Therefore,

$$T^kT^{*k}((T^*T)^2 - 2\lambda TT^* + \lambda^2)T^kT^{*k} = \begin{pmatrix} (A((T_1^*T_1)^2 - 2\lambda(T_1T_1^* + T_2T_2^*) + \lambda^2)A) & 0 \\ 0 & 0 \end{pmatrix} \geq 0$$

It follows that $T^{*k}((T^*T)^2 - 2\lambda TT^* + \lambda^2)T^k \geq 0$ for $\lambda > 0$ on $H = \overline{ran(T^k)} \oplus \ker T^{*k}$. Thus T is k-quasi-*parahyponormal operator. □

Corollary 2.2. *Let T be k-quasi-*parahyponormal operator and $ran(T)$ is not dense and $T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}$ on $H = \overline{ran(T^k)} \oplus \ker T^{*k}$. Then T_1 is a parahyponormal operator, $T_3^k = 0$.Furthermore, $\sigma(T) = \sigma(T_1) \cup \{0\}$.*

Corollary 2.3. *If T be k-quasi-*parahyponormal operator and $ran(T)$ is not dense and $T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}$ on H. Then T_1 is a parahyponormal operator on $\overline{ran(T)}$.*

Corollary 2.4. *Let T be k-quasi-*parahyponormal operator and $0 \neq \mu \in \sigma_p(T)$. If T is of the form $T = \begin{pmatrix} \mu & B \\ 0 & C \end{pmatrix}$ on $H = N(T-\mu) \oplus N(T-\mu)^{\perp}$ then $B = 0$.*

Proof. Let P be the projection onto $N(T-\mu)$ and $x \in N(T-\mu)$. Since T is k-quasi-*parahyponormal operator and $x = \frac{1}{\mu^k}T^k x \in R(T^k)$, we have $P((T^*T)^2 - 2\lambda TT^* + \lambda^2)P \geq 0$. Then

$$P(T^*T)^2P - 2\lambda P(TT^*)P + P\lambda^2 P \geq 0$$
$$(PT^*TP)^2 - 2\lambda P(TT^*)P + \lambda^2 \geq 0$$
$$(T_1^*T_1)^2 - 2\lambda(T_1T_1^* + T_2T_2^*) + \lambda^2 \geq 0$$

which gives that

$$\mu^4 - 2\lambda\mu^2 + \lambda^2 \geq 2\lambda BB^* \quad \text{for all } \lambda > 0$$

Hence $B = 0$. □

Corollary 2.5. *Let $T \in B(H)$ be a k-quasi-*parahyponormal operator for a positive integer k. If $M \subset H$is an invariant subspace of T, then the restriction $T|_M$ is also k-quasi-*parahyponormal operator.*

Proof. Let P be the orthogonal projection of H onto M, and let $T_1 = T|_M$. Then $T^kP = PT^kP$ and $T_1 = PTP|_M$. Since T is a k-quasi-*parahyponormal operator and by Theorem 2.1, we have T_1 is k-quasi-*parahyponormal operator. □

3. The Spectral Continuity of k-Quasi-*Parahyponormal Operators

For every $T \in B(H)$, $\sigma(T)$ is compact subset of C. The function σ viewed as a function from $B(H)$ into the set of all compact subset of C, equipped with the Housdorff metric, is well known to be upper semi continuous, but fails to be continuous in general. Conway and Morrel [3] have carried out a detailed study of spectral continuity in $B(H)$. Recently, the continuity of spectrum was considered when restricted to some subsets of the entire manifold of Toeplitz operators in [13]. It has been proved that is continuous in the set of normal operators and hyponormal operators in [9]. And this result has been extended to quasi hyponormal operators by Djordjevic in [8], to p-hyponormal operators, (p,k)-quasi hyponormal operators, *-paranormal and paranormal operators by many authors. In this section we extend this result to k-quasi-*parahyponormal operators.

A complex number λ is said to be in the point spectrum $\sigma_p(T)$ of T if there is a non zero $x \in H$ such that $(T-\lambda)x = 0$. If in addition $(T^* - \overline{\lambda})x = 0$ then λ is said to be in the joint point spectrum $\sigma_{jp}(T)$ of T. If T is hyponormal then $\sigma_{jp}(T) = \sigma_p(T)$.

The approximate point spectrum of an operator T is defined as follows $\sigma_{ap}(T) = \{\lambda \in C : \exists$ a sequence of unit vectors x_n such that $\|x_n - \lambda x_n\| \to 0$ as $n \to 0\}$.

Lemma 3.1. *Let T be a k-quasi-*parahyponormal operator. Then the following assertions hold.*

(i) *If T is quasi nilpotent, then $T^{k+1} = 0$.*

(ii) *For every nonzero $\lambda \in \sigma_p(T)$, the matrix representation of T with respect to the decomposition $H = N(T-\lambda)+N(T-\lambda)^{\perp}$ is $T = \begin{pmatrix} \lambda & 0 \\ 0 & B \end{pmatrix}$ for some operator satisfying $\lambda \notin \sigma_p(B)$ and $\sigma(T) = \lambda \cup \sigma(B)$.*

Proof. (i) Suppose T is k-quasi-*parahyponormal operator. If the range of T^k is dense, then T is parahyponormal operator, which leads to that T is normaloid. Hence $T = 0$.
If the range of T^k is not dense, then $T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}$ on $H = \overline{ran(T^k)} \oplus \ker T^{*k}$ where T_1 is a parahyponormal operator, $T_3^k = 0$ and $\sigma(T) = \sigma(T_1) \cup \{0\}$ (by Theorem 2.1).
Since $\sigma(T_1) = 0$, we have $T_1 = 0$.
Thus

$$\begin{aligned} T^{k+1} &= \begin{pmatrix} 0 & T_2 \\ 0 & T_3 \end{pmatrix} \\ &= \begin{pmatrix} 0 & T_2T_3^k \\ 0 & T_3^{k+1} \end{pmatrix} \\ &= 0 \end{aligned}$$

(ii) If $\lambda \neq 0$ and $\lambda \in \sigma_p(T)$, we have that $N(T-\lambda)$ reduces T by Corollary 2.4. So, we have that $\begin{pmatrix} \lambda & 0 \\ 0 & B \end{pmatrix}$ for some operator B satisfying $\lambda \notin \sigma_p(B)$ and $\sigma(T) = \lambda \cup \sigma(B)$. □

Lemma 3.2 ([1])**.** *Let H be a complex Hilbert space. Then there exists a Hilbert space K such that $H \subset K$and a map $\phi : B(H) \to B(K)$ such that*

(i) *ϕ is a faithful *representation of the algebra $B(H)$ on K.*

(ii) *$\phi(A) \geq 0$ for any $A \geq 0$ in $B(H)$*

(iii) *$\sigma_a(T) = \sigma_a(\phi(T)) = \sigma_p(\phi(T))$ for any $T \in B(H)$.*

Lemma 3.3 ([1])**.** *Let $\varphi : B(H) \to B(K)$ be Berberian's faithful $*$ representation. Then $\sigma_{ja}(T) = \sigma_{jp}(\varphi(T))$.*

Lemma 3.4. *The spectrum σ is continuous on the set of k-quasi-*parahyponormal operators.*

Proof. Suppose T is k-quasi-*parahyponormal operator. Let $\varphi : B(H) \to B(K)$ be Berberian's faithful $*$ representation of Lemma 3.2.
Now, we will show that $\varphi(T)$ is also k-quasi-*parahyponormal operator.
Since

$$T^{*k}((T^*T)^2 - 2\lambda TT^* + \lambda^2)T^k \geq 0 \qquad \text{for every } \lambda > 0$$

$$(\varphi(T))^{*k}((\varphi(T)^*\varphi(T))^2 - 2\lambda\varphi(T)\varphi(T)^* + \lambda^2)(\varphi(T))^k \geq 0 \qquad \text{for every } \lambda > 0$$

$$\varphi(T^{*k}((T^*T)^2 - 2\lambda TT^* + \lambda^2)T^k) \geq 0 \qquad \text{for every } \lambda > 0$$

(by Lemma 3.2).

Therefore, $\varphi(T)$ is also k-quasi-*parahyponormal operator by Lemma 3.1, we have T belongs to the set $C(i)$ [5].

Therefore, we have that the spectrum σ is continuous on the set of k-quasi-*parahyponormal operators (by [5, Corrollary 7]).

Lemma 3.5. *Let T be a k-quasi-*parahyponormal operator and $\lambda \neq 0$ then $Tx = \lambda x$ implies $T^* x = \overline{\lambda} x$.*

Proof. We may assume that $x \neq 0$. Let M_0 be a span of $\{0\}$ then is an invariant subspace of T and $T = \begin{pmatrix} \mu & T_2 \\ 0 & T_3 \end{pmatrix}$ on $H = M_0 \oplus M_0^{\perp}$.

Let P be the projection of H onto M_0. It sufficient to show that $T_2 = 0$ in the above equation. Since T is a k-quasi-parahyponormal operator, we have

$$P((T^*T)^2 - 2\lambda TT^* + \lambda^2)P \geq 0\,.$$

By expanding this and by simple calculations we have $\sum T_2 T_3^* = 0$.

Since T is k-quasi-parahyponormal operator,

$$T^{*k}((T^*T)^2 - 2\lambda TT^* + \lambda^2)T^k \geq 0\,.$$

Recall that $\begin{pmatrix} X & Y \\ Y^* & Z \end{pmatrix} \geq 0$ if and only if $X, Z \geq 0$ and $Y = X^{\frac{1}{2}} W Y^{\frac{1}{2}}$ for some contractions W. Therefore, $T_2 T_3^k = 0$.

Since $\lambda \neq 0$ and $T_2 = 0$, we have $Tx = \lambda x$ and $T^* x = \overline{\lambda} x$.

Hence $(T - \lambda)x = 0$ and $(T^* - \overline{\lambda})x = 0$. □

Theorem 3.6. *Let T be a k-quasi-*parahyponormal operator then $\sigma_{jp} | \{0\} = \sigma_p | \{0\}$ and if $(T - \lambda)x = 0$, $(T - \mu)y = 0$ and $\lambda \neq \mu$, then $\langle x, y \rangle = 0$.*

Proof. Suppose T is k-quasi-*parahyponormal operator. Then

$$T^{*k}((T^*T)^2 - 2\lambda TT^* + \lambda^2)T^k \geq 0$$

$$(T - \lambda)x = 0 \text{ and } (T^* - \overline{\lambda})x = 0 \quad \text{for } x \neq 0 \in H \qquad \text{(by Lemma 3.5)}$$

By the definition of joint point spectrum and point spectrum and by the above equation, we have $\sigma_{jp} | \{0\} = \sigma_p | \{0\}$.

Without the loss of generality, we may assume that $\mu \neq 0$. Then, we have $(T - \mu)^* y = 0$ (by Lemma 3.5).

Then, we have $\mu \langle x, y \rangle = \langle x, T^* y \rangle = \langle Tx, y \rangle = \langle x, y \rangle$.

Since $\lambda \neq \mu$, $\langle x, y \rangle = 0$. □

Theorem 3.7. *Let T be a k-quasi-parahyponormal operator for a positive integer. Then*

$$\sigma_{ja}(T) | \{0\} = \sigma_a(T) | \{0\}\,.$$

Proof. By continuity of the spectrum and by Lemma 3.4, Lemma3.5 the result is true. That is the non zero points of its approximate point spectrum and joint approximate point spectrum are identical. □

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Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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