Journal of Informatics and Mathematical Sciences
Volume 4 (2012), Number 3, pp. 293–303

http://www.rgnpublications.com

Projective Change between Randers Metric and Special (α, β)-metric

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Abstract. In the present paper, we find the conditions to characterize projective change between two (α,β)-metrics, such as special (α,β)-metric, $L = \alpha - \frac{\beta^2}{\alpha} + \beta$ and Randers metric $\bar{L} = \bar{\alpha} + \bar{\beta}$ on a manifold with dim $n \geq 3$, where α and $\bar{\alpha}$ are two Riemannian metrics, β and $\bar{\beta}$ are two non-zero 1-forms. Further, we study the special curvature properties of two classes of (α,β)-metrics.

1. Introduction

The projective change between two Finsler spaces have been studied by many authors ([3], [12], [10], [13], [15] [20]). An interesting result concerned with the theory of projective change was given by Rapscak's paper [18]. He proved the necessary and sufficient condition for projective change. In 1994, S. Bacso and M. Matsumoto [3] studied the projective change between Finsler spaces with (α,β)-metric. In 2008, H.S. Park and Y. Lee [13] studied on projective changes between a Finsler space with (α,β)-metric and the associated Riemannian metric. The authors Z. Shen and Civi Yildirim [20] studied on a class of projectively flat metrics with constant flag curvature in 2008. In 2009, Ningwei Cui and Yi-Bing Shen [12] studied projective change between two classes of (α,β)-metrics. The author N. Cui (2006) studied S-curvature of some (α,β)-metrics [4]. Some results on a class of (α,β)-metrics with constant flag curvature have been studied recently by Z. Lin (2009) [7].

The first part of the present paper is devoted to the study of projective change between two classes of Finsler spaces with (α,β)-metric (Theorem 3.1). The second part is devoted to investigate the special curvature properties of these Finsler metrics under projective change (Theorem 4.2).

2010 *Mathematics Subject Classification.* 53B40, 53C60.
Key words and phrases. Projective change; Randers metric; Douglas metric; Projective invariant; Locally Minkowski space.

2. **Preliminaries**

The terminology and notations are referred to ([8], [19], [1]). Let $F^n = (M, L)$ be a Finsler space on a differential manifold M endowed with a fundamental function $L(x, y)$. We use the following notations:

(a) $g_{ij} = \frac{1}{2}\dot{\partial}_i\dot{\partial}_j L^2, \quad \dot{\partial}_i = \frac{\partial}{\partial y^i}$,

(b) $C_{ijk} = \frac{1}{2}\dot{\partial}_k g_{ij}$,

(c) $h_{ij} = g_{ij} - l_i l_j$,

(d) $\gamma^i_{jk} = \frac{1}{2}g^{ir}(\partial_j g_{rk} + \partial_k g_{rj} - \partial_r g_{jk})$,

(e) $G^i = \frac{1}{2}\gamma^i_{jk} y^j y^k, \quad G^i_j = \dot{\partial}_j G^i, \quad G^i_{jk} = \dot{\partial}_k G^i_j, \quad G^i_{jkl} = \dot{\partial}_l G^i_{jk}$.

The concept of (α, β)-metric $L(\alpha, \beta)$ was introduced in 1972 by M. Matsumoto and studied by many authors like ([11], [16], [6], [9], [22], [17]). The Finsler space $F^n = (M, L))$ is said to have an (α, β)-metric if L is a positively homogeneous function of degree one in two variables $\alpha^2 = a_{ij}(x)y^i y^j$ and $\beta = b_i(x)y^i$. A change $L \to \bar{L}$ of a Finsler metric on a same underlying manifold M is called projective change if any geodesic in (M, L) remains to be a geodesic in $(M, \bar{L})$ and viceversa. We say that a Finsler metric is projectively related to another metric if they have the same geodesics as point sets. In Riemannian geometry, two Riemannian metrics α and $\bar{\alpha}$ are projectively related if and only if their spray coefficients have the relation [12]

$$G^i_\alpha = G^i_{\bar{\alpha}} + \lambda_{x^k} y^k y^i, \tag{2.1}$$

where $\lambda = \lambda(x)$ is a scalar function on the based manifold, and (x^i, y^j) denotes the local coordinates in the tangent bundle TM.

Two Finsler metrics F and $\bar{F}$ are projectively related if and only if their spray coefficients have the relation [12]

$$G^i = \bar{G}^i + P(y)y^i, \tag{2.2}$$

where $P(y)$ is a scalar function on $TM \setminus \{0\}$ and homogeneous of degree one in y. A Finsler metric is called a projectively flat metric if it is projectively related to a locally Minkowskian metric.

For a given Finsler metric $L = L(x, y)$, the geodesics of L satisfy the following ODEs:

$$\frac{d^2x^i}{dt^2} + 2G^i\left(x, \frac{dx}{dt}\right) = 0,$$

where $G^i = G^i(x, y)$ are called the geodesic coefficients, which are given by

$$G^i = \frac{1}{4}g^{il}\{[L^2]_{x^m y^l} y^m - [L^2]_{x^l}\}.$$

Let $\phi=\phi(s)$, $|s|<b_0$, be a positive C^∞ function satisfying the following

$$\phi(s)-s\phi'(s)+(b^2-s^2)\phi''(s)>0,\quad (|s|\le b<b_0). \tag{2.3}$$

If $\alpha=\sqrt{a_{ij}y^iy^j}$ is a Riemannian metric and $\beta=b_iy^i$ is 1-form satisfying $\|\beta_x\|_\alpha<b_0$ for all $x\in M$, then $L=\phi(s)$, $s=\beta/\alpha$, is called an (regular) (α,β)-metric. In this case, the fundamental form of the metric tensor induced by L is positive definite.

Let $\nabla\beta=b_{i|j}dx^i\otimes dx^j$ be covariant derivative of β with respect to α.

Denote

$$r_{ij}=\frac{1}{2}(b_{i|j}+b_{j|i}),\qquad s_{ij}=\frac{1}{2}(b_{i|j}-b_{j|i}).$$

β is closed if and only if $s_{ij}=0$ [21]. Let $s_j=b^is_{ij}$, $s^i_j=a^{il}s_{lj}$, $s_0=s_iy^i$, $s^i_0=s^i_jy^j$ and $r_{00}=r_{ij}y^iy^j$.

The relation between the geodesic coefficients G^i of L and geodesic coefficients G^i_α of α is given by

$$G^i=G^i_\alpha+\alpha Qs^i_0+\{-2Q\alpha s_0+r_{00}\}\{\Psi b^i+\Theta\alpha^{-1}y^i\}, \tag{2.4}$$

where

$$\Theta=\frac{\phi\phi'-s(\phi\phi''+\phi'\phi')}{2\phi((\phi-s\phi')+(b^2-s^2)\phi'')},$$
$$Q=\frac{\phi'}{\phi-s\phi'},$$
$$\Psi=\frac{1}{2}\frac{\phi''}{(\phi-s\phi')+(b^2-s^2)\phi''}.$$

Definition 2.1 ([12]). Let

$$D^i_{jkl}=\frac{\partial^3}{\partial y^j\partial y^k\partial y^l}\left(G^i-\frac{1}{n+1}\frac{\partial G^m}{\partial y^m}y^i\right), \tag{2.5}$$

where G^i are the spray coefficients of L. The tensor $D=D^i_{jkl}\partial_i\otimes dx^j\otimes dx^k\otimes dx^l$ is called the Douglas tensor. A Finsler metric is called Douglas metric if the Douglas tensor vanishes.

We know that the Douglas tensor is a projective invariant [14]. Note that the spray coefficients of a Riemannian metric are quadratic forms and one can see that the Douglas tensor vanishes form (2.5). This shows that Douglas tensor is a non-Riemannian quantity.

In the following, we use quantities with a bar to denote the corresponding quantities of the metric $\bar{L}$. First, we compute the Douglas tensor of a general (α,β)-metric.

Let

$$\widehat{G}^i=G^i_\alpha+\alpha Qs^i_0+\Psi\{-2Q\alpha s_0+r_{00}\}b^i.$$

Then (2.4) becomes

$$G^i = \widehat{G}^i + \Theta\{-2Q\alpha s_0 + r_{00}\}\alpha^{-1}y^i.$$

Clearly, G^i and $\widehat{G}^i$ are projective equivalent according to (2.2), they have the same Douglas tensor.
Let

$$T^i = \alpha Q s^i_0 + \Psi\{-2Q\alpha s_0 + r_{00}\}b^i. \tag{2.6}$$

Then $\widehat{G}^i = G^i_\alpha + T^i$, thus

$$\begin{aligned} D^i_{jkl} &= \widehat{D}^i_{jkl} \\ &= \frac{\partial^3}{\partial y^j \partial y^k \partial y^l}\left(G^i_\alpha - \frac{1}{n+1}\frac{\partial G^m_\alpha}{\partial y^m}y^i + T^i - \frac{1}{n+1}\frac{\partial T^m}{\partial y^m}y^i\right) \\ &= \frac{\partial^3}{\partial y^j \partial y^k \partial y^l}\left(T^i - \frac{1}{n+1}\frac{\partial T^m}{\partial y^m}y^i\right). \end{aligned} \tag{2.7}$$

To simplify (2.7), we use the following identities

$$\alpha_{y^k} = \alpha^{-1}y_k, \quad s_{y^k} = \alpha^{-2}(b_k\alpha - s y_k),$$

where $y_i = a_{il}y^l$, $\alpha_{y^k} = \frac{\partial \alpha}{\partial y^k}$. Then

$$\begin{aligned} [\alpha Q s^m_0]_{y^m} &= \alpha^{-1}y_m Q s^m_0 + \alpha^{-2}Q'[b_m\alpha^2 - \beta y_m]s^m_0 \\ &= Q's_0 \end{aligned}$$

and

$$\begin{aligned} [\Psi(-2Q\alpha s_0 + r_{00})b^m]_{y^m} = \Psi'\alpha^{-1}(b^2 - s^2)[r_{00} - 2Q\alpha s_0] \\ + 2\Psi[r_0 - Q'(b^2 - s^2)s_0 - Qss_0], \end{aligned}$$

where $r_j = b^i r_{ij}$ and $r_0 = r_i y^i$. Thus from (2.6), we obtain

$$\begin{aligned} T^m_{y^m} = Q's_0 + \Psi'\alpha^{-1}(b^2 - s^2)[r_{00} - 2Q\alpha s_0] \\ + 2\Psi[r_0 - Q'(b^2 - s^2)s_0 - Qss_0]. \end{aligned} \tag{2.8}$$

Now, we assume that the (α, β)-metrics L and $\bar{L}$ have the same Douglas tensor, that is, $D^i_{jkl} = \bar{D}^i_{jkl}$. Thus from (2.5) and (2.7), we get

$$\frac{\partial^3}{\partial y^j \partial y^k \partial y^l}\left(T^i - \bar{T}^i - \frac{1}{n+1}(T^m_{y^m} - \bar{T}^m_{y^m})y^i\right) = 0.$$

Then there exists a class of scalar functions $H^i_{jk} = H^i_{jk}(x)$, such that

$$H^i_{00} = T^i - \bar{T}^i - \frac{1}{n+1}(T^m_{y^m} - \bar{T}^m_{y^m})y^i, \tag{2.9}$$

where $H^i_{00} = H^i_{jk}y^j y^k$, T^i and $T^m_{y^m}$ are given by the relations (2.6) and (2.8) respectively.

3. Projective Change between Randers Metric and Special (α, β)-metric

In this section, we find the projective relation between two (α, β)-metrics that is special (α, β)-metric $L = \alpha - \frac{\beta^2}{\alpha} + \beta$ and Randers metric $\bar{L} = \bar{\alpha} + \bar{\beta}$ on a same underlying manifold M of dimension $n \geq 3$. For (α, β)-metric $L = \alpha - \frac{\beta^2}{\alpha} + \beta$, one can prove by (2.3) that L is a regular Finsler metric if and only if 1-form β satisfies the condition $\|\beta_x\|_\alpha < 1$ for any $x \in M$. The geodesic coefficients are given by (2.4) with

$$\theta = \frac{1 + 3s^2 - 4s^3}{2(1 + s - s^2)(1 - 2b^2 + 3s^2)},$$
$$Q = \frac{1 - 2s}{1 + s^2},$$
$$\Psi = \frac{-1}{1 - 2b^2 + 3s^2}. \tag{3.1}$$

Substituting (3.1) in to (2.4), we get

$$G^i = G^i_\alpha + \frac{1}{\alpha^2 - 2b^2\alpha^2 + 3\beta^2}\left[\frac{-2(\alpha - 2\beta)\alpha^2 s_0}{\alpha^2 + \beta^2} + r_{00}\right] \times \left[-\alpha^2 b^i + \frac{(\alpha^3 + 3\alpha\beta^2 - 4\beta^3)y^i}{2(\alpha^2 + \alpha\beta - \beta^2)}\right] + \frac{\alpha^2(\alpha - 2\beta)s^i_0}{\alpha^2 + \beta^2}. \tag{3.2}$$

For Randers metric $\bar{L} = \bar{\alpha} + \bar{\beta}$, one can also prove by (2.3) that $\bar{L}$ is a regular Finsler metric if and only if $\|\beta_x\|_\alpha < 1$ for any $x \in M$. The geodesic coefficients are given by (2.4) with

$$\bar{\theta} = \frac{1}{2(1 + s)}, \quad \bar{Q} = 1, \quad \bar{\Psi} = 0. \tag{3.3}$$

First, we prove the following lemma:

Lemma 3.1. *Let $L = \alpha - \frac{\beta^2}{\alpha} + \beta$ and $\bar{L} = \bar{\alpha} + \bar{\beta}$ be two (α, β)-metrics on a manifold M with dimension $n \geq 3$. Then they have the same Douglas tensor if and only if both the metrics L and $\bar{L}$ are Douglas metrics.*

Proof. First, we prove the sufficient condition. Let L and $\bar{L}$ be Douglas metrics and corresponding Douglas tensors be D^i_{jkl} and $\bar{D}^i_{jkl}$. Then by the definition of Douglas metric, we have $D^i_{jkl} = 0$ and $\bar{D}^i_{jkl} = 0$, that is, both L and $\bar{L}$ have same Douglas tensor. Next, we prove the necessary condition. If L and $\bar{L}$ have the same Douglas tensor, then (2.9) holds. Substituting (3.1) and (3.3) in to (2.9), we obtain

$$H^i_{00} = \frac{A^i\alpha^9 + B^i\alpha^8 + C^i\alpha^7 + D^i\alpha^6 + E^i\alpha^5 + F^i\alpha^4 + G^i\alpha^3 + H^i\alpha^2 + J^i}{K\alpha^8 + L\alpha^6 + M\alpha^4 + N\alpha^2 + P} - \bar{\alpha}\bar{s}^i_0, \tag{3.4}$$

where

$$A^i = (1 - 2b^2)[(1 - 2b^2)s^i_0 + 2b^i s_0],$$

$$
\begin{aligned}
B^i &= -(1-2b^2)\{b^i r_{00} + 2\beta(1-2b^2)s^i_0 + 4b^i\beta s_0 + 2\lambda y^i[(1-4b^2)s_0 - r_0]\},\\
C^i &= \beta\{\beta(1-2b^2)(7-2b^2)s^i_0 - 4[b^i\beta(b^2-2) - 3\lambda y^i b^2]s_0\},\\
D^i &= \beta\{\beta(1-2b^2)[2\beta(2b^2-7)s^i_0 + 2\lambda y^i[(2b^2+5)s_0 + 2r_0] - b^i r_{00}]\\
&\quad + 2\beta(b^2-2)[4(b^i\beta + b^2\lambda y^i)s_0 + b^i r_{00}] - 6\lambda y^i[b^2 r_{00} - \beta r_0]\},\\
E^i &= 3\beta^3\{\beta[(5-4b^2)s^i_0 + 2b^i s_0] - 4\lambda y^i(1-b^2)s_0\},\\
F^i &= \beta^3\{6\beta^2[(4b^2-5)s^i_0 - 2b^i s_0] + b^i\beta(2b^2-7)r_{00}\\
&\quad - 2\lambda y^i[\beta(14b^2-19)s_0 + 3(2b^2-1)r_{00} - \beta(7-2b^2)r_0]\},\\
G^i &= 3\beta^5(3\beta s^i_0 - 4\lambda y^i s_0),\\
H^i &= -3\beta^5\{\beta[6\beta s^i_0 + b^i r_{00}] + 2\lambda y^i[(b^2-2)r_{00} - \beta(5s_0 + r_0)]\},\\
J^i &= 6\lambda y^i\beta^7 r_{00},\\
\lambda &= \frac{1}{n+1}
\end{aligned}
\tag{3.5}
$$

and

$$
\begin{aligned}
K &= (1-2b^2)^2,\\
L &= 4\beta^2(1-2b^2)(2-b^2),\\
M &= \beta^4[(1-2b^2)^2 + 3(7-8b^2)],\\
N &= -12\beta^6(b^2-2),\\
P &= 9\beta^8.
\end{aligned}
\tag{3.6}
$$

Then (3.4) is equivalent to

$$
\begin{aligned}
&A^i\alpha^9 + B^i\alpha^8 + C^i\alpha^7 + D^i\alpha^6 + E^i\alpha^5 + F^i\alpha^4 + G^i\alpha^3 + H^i\alpha^2 + J^i\\
&\quad = (K\alpha^8 + L\alpha^6 + M\alpha^4 + N\alpha^2 + P)(\bar{\alpha}\bar{s}^i_0 + H^i_{00}).
\end{aligned}
\tag{3.7}
$$

Replacing y^i in (3.7) by $-y^i$ yields

$$
\begin{aligned}
&-A^i\alpha^9 + B^i\alpha^8 - C^i\alpha^7 + D^i\alpha^6 - E^i\alpha^5 + F^i\alpha^4 - G^i\alpha^3 + H^i\alpha^2 + J^i\\
&\quad = (K\alpha^8 + L\alpha^6 + M\alpha^4 + N\alpha^2 + P)(H^i_{00} - \bar{\alpha}\bar{s}^i_0).
\end{aligned}
\tag{3.8}
$$

Subtracting (3.8) from (3.7), we obtain

$$
A^i\alpha^9 + C^i\alpha^7 + E^i\alpha^5 + G^i\alpha^3 = (K\alpha^8 + L\alpha^6 + M\alpha^4 + N\alpha^2 + P)(\bar{\alpha}\bar{s}^i_0). \tag{3.9}
$$

From (3.9), $P\bar{\alpha}\bar{s}^i_0$ has the factor α^2, that is, the term $P\bar{\alpha}\bar{s}^i_0 = 9\beta^8\bar{\alpha}\bar{s}^i_0$ has the factor α^2. Now, we can study two cases for Riemannian metric.

Case (i): If $\bar{\alpha} \neq \mu(x)\alpha$, then $P\bar{s}^i_0 = 9\beta^8\bar{s}^i_0$ has the factor α^2.
Note that β^2 has no factor α^2. Then the only possibility is that $\beta\bar{s}^i_0$ has the factor α^2.

Then for each i there exists a scalar function $\tau^i = \tau(x)$ such that $\beta\bar{s}^i_0 = \tau^i\alpha^2$ which is equivalent to $b_j\bar{s}^i_k + b_k\bar{s}^i_j = 2\tau^i\alpha_{jk}$.

When $n \geq 3$ and we assume that $\tau^i \neq 0$, then

$$\begin{aligned}
2 &\geq \text{rank}(b_j\bar{s}^i_k) + \text{rank}(b_k\bar{s}^i_j) \\
&\geq \text{rank}(b_j\bar{s}^i_k + b_k\bar{s}^i_j) \\
&= \text{rank}(2\tau^i a_{jk}) \geq 3,
\end{aligned} \tag{3.10}$$

which is impossible unless $\tau^i = 0$. Then $\beta\bar{s}^i_0 = 0$. Since $\beta \neq 0$, we have $\bar{s}^i_0 = 0$, which says that $\bar{\beta}$ is closed.

Case (ii): If $\bar{\alpha} = \mu(x)\alpha$, then (3.9) reduces to

$$A^i\alpha^8 + C^i\alpha^6 + E^i\alpha^4 + G^i\alpha^2 = \mu(x)\bar{s}^i_0[K\alpha^8 + L\alpha^6 + M\alpha^4 + N\alpha^2 + P],$$

which is written as

$$\mu(x)P\bar{s}^i_0 = [A^i\alpha^6 + C^i\alpha^4 + E^i\alpha^2 + G^i - \mu(x)\bar{s}^i_0(K\alpha^6 + L\alpha^4 + M\alpha^2 + N)]\alpha^2. \tag{3.11}$$

From (3.11), we can see that $\mu(x)P\bar{s}^i_0$ has the factor α^2. i.e., $\mu(x)P\bar{s}^i_0 = 9\mu(x)\bar{s}^i_0\beta^8$ has the factor α^2. Note that $\mu(x) \neq 0$ for all $x \in M$ and β^2 has no factor α^2. The only possibility is that $\beta\bar{s}^i_0$ has the factor α^2. As the similar reason in case (i), we have $\bar{s}^i_0 = 0$, when $n \geq 3$, which says that $\bar{\beta}$ is closed.

M. Hashiguchi [5] proved that Randers metric $\bar{L} = \bar{\alpha} + \bar{\beta}$ is a Douglas metric if and only if $\bar{\beta}$ is closed. Thus $\bar{L} = \bar{\alpha} + \bar{\beta}$ is a Douglas metric. Since L is projectively related to $\bar{L}$, then both L and $\bar{L}$ are Douglas metrics.

Now, we prove the following main theorem:

Theorem 3.1. *The Finsler metric $L = \alpha - \frac{\beta^2}{\alpha} + \beta$ is projectively related to $\bar{L} = \bar{\alpha} + \bar{\beta}$ if and only if the following conditions are satisfied*

$$\begin{aligned}
&G^i_\alpha = G^i_{\bar{\alpha}} + \theta y^i - \tau\alpha^2 b^i, \\
&b_{i|j} = \tau[(-1 + 2b^2)a_{ij} - 3b_i b_j], \\
&d\bar{\beta} = 0,
\end{aligned} \tag{3.12}$$

where $b^i = a^{ij}b_j$, $b = \|\beta\|_\alpha$, $b_{i|j}$ denote the coefficients of the covariant derivatives of β with respect to α, $\tau = \tau(x)$ is a scalar function and $\theta = \theta_i y^i$ is a 1-form on a manifold M with dimension $n \geq 3$.

Proof. First, we prove the necessary condition. Since Douglas tensor is an invariant under projective changes between two Finsler metrics, if L is projectively related to $\bar{L}$, then they have the same Douglas tensor. According to Lemma (3.1), we obtain that both L and $\bar{L}$ are Douglas metrics.

We know that Randers metric $\bar{L} = \bar{\alpha} + \bar{\beta}$ is a Douglas metric if and only if $\bar{\beta}$ is closed , that is

$$d\bar{\beta} = 0 \tag{3.13}$$

and $L=\alpha-\frac{\beta^2}{\alpha}+\beta$ is a Douglas metric if and only if

$$b_{i|j}=\tau[(-1+2b^2)a_{ij}-3b_ib_j], \tag{3.14}$$

for some scalar function $\tau=\tau(x)$ [2], where $b_{i|j}$ denote the coefficients of the covariant derivatives of $\beta=b_iy^i$ with respect to α. In this case, β is closed. Since β is closed, $s_{ij}=0\Rightarrow b_{i|j}=b_{j|i}$. Thus $s^i_0=0$ and $s_0=0$.
By using (3.14), we have $r_{00}=\tau[(-1+2b^2)\alpha^2-3\beta^2]$. Substituting all these in (3.2), we obtain

$$G^i=G^i_\alpha-\tau\left[\frac{\alpha^3+3\alpha\beta^2-4\beta^3}{2(\alpha^2+\alpha\beta-\beta^2)}\right]y^i+\tau\alpha^2b^i. \tag{3.15}$$

Since L is projective to $\bar{L}=\bar{\alpha}+\bar{\beta}$, this is a Randers change between L and $\bar{\alpha}$. Noticing that $\bar{\beta}$ is closed, then L is projectively related to $\bar{\alpha}$. Thus there is a scalar function $P=P(y)$ on $TM\setminus\{0\}$ such that

$$G^i=G^i_{\bar{\alpha}}+Py^i. \tag{3.16}$$

From (3.15) and (3.16), we have

$$\left[P+\tau\left(\frac{\alpha^3+3\alpha\beta^2-4\beta^3}{2(\alpha^2+\alpha\beta-\beta^2)}\right)\right]y^i=G^i_\alpha-G^i_{\bar{\alpha}}+\tau\alpha^2b^i. \tag{3.17}$$

Note that the RHS of the above equation is a quadratic form. Then there must be a one form $\theta=\theta_iy^i$ on M, such that

$$P+\tau\left(\frac{\alpha^3+3\alpha\beta^2-4\beta^3}{2(\alpha^2+\alpha\beta-\beta^2)}\right)=\theta.$$

Thus (3.17) becomes

$$G^i_\alpha=G^i_{\bar{\alpha}}+\theta y^i-\tau\alpha^2b^i. \tag{3.18}$$

From (3.13) and (3.14) together with (3.18) complete the proof of the necessity.

For the sufficiency, noticing that $\bar{\beta}$ is closed, it suffices to prove that L is projectively related to $\bar{\alpha}$. Substituting (3.14) in to (3.2) yields (3.15).

From (3.15) and (3.18), we have

$$G^i=G^i_{\bar{\alpha}}+\left[\theta-\tau\left(\frac{\alpha^3+3\alpha\beta^2-4\beta^3}{2(\alpha^2+\alpha\beta-\beta^2)}\right)\right]y^i,$$

i.e., L is projectively related to $\bar{\alpha}$.

From the above theorem, immediately we get the following corollaries

Corollary 3.1. *The Finsler metric $L=\alpha-\frac{\beta^2}{\alpha}+\beta$ is projectively related to $\bar{L}=\bar{\alpha}+\bar{\beta}$ if and only if they are Douglas metrics and the spray coefficients of α and $\bar{\alpha}$ have the following relation*

$$G^i_\alpha=G^i_{\bar{\alpha}}+\theta y^i-\tau\alpha^2b^i,$$

where $b^i=a^{ij}b_j$, $\tau=\tau(x)$ is a scalar function and $\theta=\theta_iy^i$ is a one form on a manifold M with dimension $n\geq 3$.

Further, we assume that the Randers metric $\bar{L}=\bar{\alpha}+\bar{\beta}$ is locally Minkowskian, where $\bar{\alpha}$ is an Euclidean metric and $\bar{\beta}=\bar{b_i}y^i$ is a one form with $\bar{b_i}$=constants. Then (3.12) can be written as

$$G^i_\alpha=\theta y^i-\tau\alpha^2 b^i,$$
$$b_{i|j}=\tau[(-1+2b^2)a_{ij}-3b_ib_j]. \tag{3.19}$$

Thus, we state

Corollary 3.2. *The Finsler metric $L=\alpha-\frac{\beta^2}{\alpha}+\beta$ is projectively related to $\bar{L}$ if and only if L is projectively flat, in other words, L is projectively flat if and only if* (3.19) *holds.*

4. **Special Curvature Properties of Two (α,β)-metrics**

We know that, the Berwald curvature tensor of a Finsler metric L is defined by [12]

$$B=B^i_{jkl}dx^j\otimes\partial_i\otimes dx^k\otimes dx^l,$$

where $B^i_{jkl}=[G^i]_{y^jy^ky^l}$ and G^i are the spray coefficients of L. The mean Berwald curvature tensor is defined by

$$E=E_{ij}dx^i\otimes dx^j,$$

where $E_{ij}=\frac{1}{2}B^m_{mij}$. A Finsler metric is said to be of *isotropic mean Berwald curvature* if

$$E_{ij}=\frac{n+1}{2}c(x)L_{y^iy^j},$$

for some scalar function $c(x)$ on M.

In this section, we assume that (α,β)-metric $L=\alpha-\frac{\beta^2}{\alpha}+\beta$ has some special curvature properties. Randers metric $\bar{L}=\bar{\alpha}+\bar{\beta}$ is projectively related to L.

First, we assume that L has isotropic S-curvature, i.e., $S=(n+1)c(x)L$ for some scalar function $c(x)$ on M. The (α,β)-metric, $L=\alpha+\epsilon\beta+k(\frac{\beta^2}{\alpha})$ of isotropic curvature has been characterized in [4], where ϵ and k are non zero constants. We use the following theorem proved by N. Cui [4].

Theorem 4.1. *For the special form of (α,β)-metric, $L=\alpha+\epsilon\beta+k(\frac{\beta^2}{\alpha})$, where ϵ, k are non zero constants, the following are equivalent:*

(a) *L has isotropic S-curvature, i.e., $S=(n+1)c(x)L$ for some scalar function $c(x)$ on M.*
(b) *L has isotropic mean Berwald curvature.*
(c) *β is a Killing one form of constant length with respect to α. This is equivalent to $r_{00}=s_0=0$.*
(d) *L has vanished S-curvature, i.e., $S=0$.*
(e) *L is a weak Berwald metric, i.e., $E=0$.*

The above theorem is valid for $L = \alpha - \frac{\beta^2}{\alpha} + \beta$ when we take $\epsilon = 1$ and $k = -1$. Then we have

Theorem 4.2. *Let $L = \alpha - \frac{\beta^2}{\alpha} + \beta$ has isotropic S-curvature or isotropic mean Berwald curvature. Then the Finsler metric L is projectively related to $\bar{L} = \bar{\alpha} + \bar{\beta}$ if and only if the following conditions hold:*

(a) *α is projectively related to $\bar{\alpha}$,*

(b) *β is parallel with respect to α, i.e., $b_{i|j} = 0$,*

(c) *$\bar{\beta}$ is closed, i.e., $d\bar{\beta} = 0$,*

where $b_{i|j}$ denote the coefficients of the covariant derivatives of β with respect to α.

Proof. The sufficiency is obvious from Theorem 3.1. For the necessity, from Theorem 3.1 we have that if L is projectively related to $\bar{L}$, then

$$b_{i|j} = \tau[(-1 + 2b^2)a_{ij} - 3b_i b_j],$$

for some scalar function $\tau = \tau(x)$. Contracting above equation with y^i and y^j yields

$$r_{00} = \tau[(-1 + 2b^2)\alpha^2 - 3\beta^2]. \tag{4.1}$$

By the Theorem 4.1, if L has isotropic S-curvature or equivalently isotropic mean Berwald curvature, then $r_{00} = 0$. If $\tau \neq 0$, then (4.1) gives

$$(-1 + 2b^2)\alpha^2 - 3\beta^2 = 0, \tag{4.2}$$

which is equivalent to

$$(-1 + 2b^2)a_{ij} - 3b_i b_j = 0. \tag{4.3}$$

Contracting the above equation with a^{ij} yields $-n + (2n - 3)b^2 = 0$, which is impossible.

Thus $\tau = 0$. Substituting in to Theorem 3.1, we complete the proof. □

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Received March 20, 2011
Accepted October 2011