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Research Article

Solution of Black-Scholes Equation on Barrier Option

S. Meena[1] and J. Vernold Vivin[2,*]

[1] *Research & Development Centre, Bharathiar University, Coimbatore 641046, Tamilnadu, India*
[2] *Department of Mathematics, University College of Engineering Nagercoil, (Anna University Constituent College), Nagercoil 629004, Tamilnadu, India*
Corresponding author: vernoldvivin@yahoo.in

Abstract. In this article, a solution of the Black-Scholes partial differential equation corresponding to barrier options is proposed. Semigroup theory techniques and Mellin transform method are used to discuss its solution.

Keywords. European option; Barrier option; Black Scholes equation; Co-semigroups; Mellin transform

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1. Introduction

In finance, an option is a contract which gives the right, to buy or sell an underlying asset subject to certain conditions within a specified period of time. The price that is paid for the asset when the option is exercised is called the "exercise price" or the "strike price". The last day on which the option may be exercised is called the "expiry date" or "maturity date".

Any option with the general characteristic that the underlying security's price must pass a certain level or barrier before it can be exercised is called a barrier option. There are two types of barrier options, Knock-out options and Knock-in options. A Knock-out option becomes worthless if at any time before expiry, the stock price reaches the barrier, while a Knock-in option only provides a pay-off once the stock price crosses the barrier.

Since the theory for the pricing of Knock-out option and Knock-in option are identical, apart from the final value, we will address the former in this paper. For more details on option theory refer to [9].

In [1], Black and Scholes published their seminal work on option pricing in which they described a mathematical frame work for finding the fair price of an european option. The Black Scholes model for pricing options has been applied to many different commodities and payoff structures.

In this work, we consider the Black-Scholes equation corresponding to barrier option to discuss its solution.

2. Barrier Option

A barrier option is essentially a normal option with an extra constraint. It satisfies the Black-Scholes equation [1].

$$\frac{\partial C}{\partial t}(S,t)+\frac{1}{2}\sigma^2 S^2\frac{\partial^2 C}{\partial S^2}(S,t)+rS\frac{\partial C}{\partial S}(S,t)-rC(S,t)=0,\quad 0<S<\infty, \tag{2.1}$$

$$C(S,T)=\max(S-K,0), \tag{2.2}$$

with the additional condition $C(B,t)=0$, $0\le t<T$, where B is the barrier, K is the strike price, S is the price of the underlying asset at time t, T is the expiry time. The value of the option C also depends on the volatility σ, and the interest rate r, where r and σ are constants in this work.

Using several changes of variables equation (2.1) and (2.2) can be reduced to the heat equation [5]

$$\frac{\partial U}{\partial \tau}=\frac{\partial^2 U}{\partial x^2}\quad \text{for}\ \ 0<x<\infty,\ \tau>0 \tag{2.3}$$

with the boundary conditions

$$U(x,0)=U(x)=\max\left(e^{x-\alpha x}-\frac{K}{B}e^{-\alpha x},0\right),\quad x>0, \tag{2.4}$$

$$U(0,\tau)=0.$$

3. Solution of Barrier Option using Semigroup Theory

In this section, we find the solution of (2.3) and (2.4) using the theory of Co-semigroups. We recall some basic facts in the Co-semigroup theory.

A family $T = T(t)_{t\geq 0}$ of bounded linear operators from a Banach space X into itself is called a Co-semigroup on X if

(1) $T(0) = I$, the identity operator on X,

(2) $T(t+s) = T(t)T(s)$ for all $t, s \geq 0$,

(3) $\lim_{t\to 0^+} T(t)x = x$ for all $x \in X$.

The family $S(t)_{t\in R}$ of bounded linear operators from X into itself is called a Co-group, if

(1) $S(0) = I$, the identity operator on X,

(2) $S(t+s) = S(t)S(s)$ for all $t, s \in R$,

(3) $\lim_{t\to 0} S(t)x = x$ for all $x \in X$.

The infinitesimal generator of Co-semigroup $T = (T(t))_{t\geq 0}$ is the operator given by

$$D(G) = \left\{x \in X : \lim_{t\to 0^+} \frac{1}{t}(T(t)x - x) \in X\right\},$$

$$Gx = \lim_{t\to 0^+} \frac{1}{t}(T(t)x - x), \quad x \in D(G).$$

Growth bound of the Co-semigroup $T = (T(t))_{t\geq 0}$ is given by

$$w_0(T) = \inf\{w \in R : \exists\ M > 0 : \|T(t)\| \leq Me^{\omega t}\}.$$

Further, we have

$$w_0(T) = \lim_{t\to\infty} \frac{ln\|T(t)\|}{t} = \inf_{t>0} \frac{ln\|T(t)\|}{t}.$$

For more details on semi-group theory, refer [4].

We define the operators

$$A_0 : D(A_0) \to L^2(R)$$

and

$$B_0 : D(B_0) \to L^2(R)$$

by

$$D(A_0) = \left\{f \in L^2(R) : f \text{ is absolutely continuous with } \frac{\partial f}{\partial x} \in L^2(R)\right\},$$

$$D(B_0) = \left\{f \in L^2(R) : f, \frac{\partial f}{\partial x} \text{ are absolutely continuous with } \frac{\partial f}{\partial x}, \frac{\partial^2 f}{\partial x^2} \in L^2(R)\right\},$$

$$A_0 f = \frac{\partial f}{\partial x}, \quad B_0 f = \frac{\partial^2 f}{\partial x^2}.$$

From [2], it is observed that A_0 is the infinitesimal generator of the Co-group

$$(S_0(t)f)(s) = f(t+s) \text{ where } S_0 : R \to B(L^2(R)), \quad s, t \in R.$$

Further B_0 is the infinitesimal generator of a Co-semigroup T_0 given by

$$(T_0(t)f)(x) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{\frac{-s^2}{4t}} f(x+s)ds$$

for all $t > 0$, $x \in R$, $f \in L^2(R)$.

Now, the semigroup $(S_0(t))_{t\geq 0}$ commute as do the resolvents of A_0 and hence of $B_0 = A_0^2$.

Then the semigroup $(T_0(t))_{t\geq 0}$ generated by B_0, are analytic and commute and has an analytic extension $(T(t))_{t\geq 0}$.

We now obtain the solution of (2.3)

Theorem 3.1. *If $f \in L^2(R)$, then the function given by $U_f : R \times R \to C$ given by*

$U_f(x,\tau) = (T(t))f)(x)$ is a solution of (2.3).

Proof. Clearly $(T(t))_{t\geq 0}$ is a bounded analytic semigroup of angle $\frac{\pi}{2}$. The domain $D(A)$ of the generator A of $(T(t))_{t\geq 0}$ contains $D(A_0^2)$.

In particular it contains

$$D_0 = \left\{ f \in L^2(R)/D^\alpha f \in L^2(R) \ \text{ for every multi index } \alpha \text{ with } |\alpha| \leq 2 \right\}$$

and for every $f \in D_0$ the generator is given by $Af = \frac{\partial^2 f}{\partial x^2}$.

Since, $\frac{\partial U_f}{\partial \tau} = \frac{\partial^2 U_f}{\partial x^2}$, it follows $\frac{\partial U_f}{\partial \tau} = Af$.

Hence $U_f(x,\tau) = (T(t))f)(x)$ is the solution. □

4. Valuation of Barrier option using Mellin Transform Method

The Mellin transform method is one of the most popular method for solving diffusion equations in many areas of science and technology. The Mellin transforms in option theory were introduced by Panini and Srivastav [7]. An application of Mellin transform techniques can be found in [6,8]. Let $M\{f(x);w\}$ denote the Mellin transform of a function $f(x) \in R^+$ given by,

$$\overline{f}(w) := M\{f(x);w\} = \int_0^{\infty} f(x)x^{w-1}dx$$

where complex variable w exists on an appropriate strip of convergence in C. Conversely, the inverse Mellin transform of a function $\overline{f}(x) \in C$ is defined by

$$f(x) := M^{-1}\left\{\overline{f}(w);x\right\} = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \overline{f}(w)x^{-w}dw\,.$$

where $c \in R(w)$, the real part of $c \in C$. Now, the Mellin transform for barrier option is given by

$$\overline{C}(w,t) = \int_0^{\infty} C(S,t)S^{w-1}dS$$

where w is the complex variable with $0 < Re(w) < \infty$. The inversion of the Mellin transform is also given by

$$C(S,t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \overline{C}(w,t)S^{-w}dw.$$

Taking the Mellin transform of equation (2.1), we get

$$M\left(\frac{\partial C}{\partial t}(S,t)\right) + M\left(\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2}(S,t)\right) + M\left(rS\frac{\partial C}{\partial S}(S,t)\right) - M(rC(S,t)) = M(0). \tag{4.1}$$

From the properties of Mellin transform equation (4.1) is transformed as

$$\frac{\partial \overline{C}(w,t)}{\partial t} - wr\overline{C}(w,t) + w(w+1)\frac{\sigma^2}{2}\overline{C}(w,t) - r\overline{C}(w,t) = 0, \tag{4.2}$$

$$-\frac{\partial \overline{C}(w,t)}{\partial t} = \frac{\sigma^2}{2}\left[\left(w^2 + w(1-\frac{2r}{\sigma^2}) - \frac{2r}{\sigma^2}\right)\overline{C}(w,t)\right]. \tag{4.3}$$

Put $z = \frac{2r}{\sigma^2}$ in equation (4.3), we get

$$-\frac{\partial \overline{C}(w,t)}{\partial t} = \frac{\sigma^2}{2}[(w^2 + w - wz - z)\overline{C}(w,t)]. \tag{4.4}$$

Substituting $G(w) = (w^2 + w - wz - z)$ in equation (4.4), we get

$$\frac{\partial \overline{C}(w,t)}{\partial t} = \frac{-\sigma^2}{2}G(w)\overline{C}(w,t).$$

Seperating the variables and integrating we get

$$\overline{C}(w,t) = \overline{C}(w,0)e^{\frac{-1}{2}\sigma^2 G(w)t} \tag{4.5}$$

where $\overline{C}(w,0)$ is a constant.
Also, we have $\overline{C}(w,0) = \overline{J}(w,t)e^{\frac{1}{2}\sigma^2 G(w)T}$, where $\overline{J}(w,t) = \frac{k^{1-w}}{w(w-1)}$ is called the Mellin transform of the boundary condition (2.2)
Hence equation (4.5) becomes $\overline{C}(w,t) = \overline{J}(w,t)e^{\frac{1}{2}\sigma^2 G(w)(T-t)}$.
Using the inverse Mellin transform, the price of a barrier option is

$$C(S,t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \overline{J}(w,t)e^{\frac{1}{2}\sigma^2 G(w)(T-t)}S^{-w}dw$$

where $c \in (0,\infty)$ and $(S,t) \in (0,\infty)\times(0,T)$.

5. Conclusion

This paper establish a connection between the solution of the heat equation (2.3) equivalent with Black-Scholes equation (2.1) and the Co-semigroup, $T(t)$. Also, it provides, the applications of semigroup theory in finance.

Competing Interests

The author declares that he has no competing interests.

Authors' Contributions

The author wrote, read and approved the final manuscript.

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