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Fuzzy Economic Order Quantity for Items with Imperfect Quality and Inspection Errors in an Uncertain Environment on Fuzzy Parameters

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Abstract. This article investigates the inventory problem for item received with imperfect quality and inspection errors in an uncertain environment. Two inventory models are discussed with fuzzy parameters for crisp order quantity, or for fuzzy order quantity. Function principle is proposed as an arithmetic operation of fuzzy trapezoidal number to obtain fuzzy economic order quantity and fuzzy annual profit. Graded mean integration method is used for defuzzification of the annual profit. Extension of Lagrangian method is used to find optimal order quantity. Numerical examples are provided to illustrate the results of proposed models.

1. Introduction

Based on the setup cost and the inventory carrying cost we obtain an economic order quantity or EPQ. First assume that the quality of all the items produced were perfect. Secondly assume that the screening process is error free which find out the defective items which are an idealistic approach. The defective occurs due to the bad material quality, improper in process control, inefficient machines and transit damage. Hence it requires optimal order quantity which gives error during the screening of a defective lot.

A number of researchers have not used the perfect quality assumption. Portens (1986) studied the effect of defective items on the basic of EOQ model. Rosenblatt and Lee (1986) assumed that the time between the in control and the out of control state of a process follows an exponential distribution and that the defective items are reworked instantaneously. Lee and Rosenblatt (1987) studied a joint lot sizing and inspection policy for an EOQ model with a fixed percentage of defective products. Rekik *et al.* (2007) extended the work of inderfurth (2004) for two cases:

(a) an additive errors case where the variability of errors is independent of the order quantity, and

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(b) a multiplicative errors case where the variability of errors is proportional to the order quantity.

Recently, Salameh and Jaber (2000), has been receiving attention. They studied a joint lot sizing and inspection policy for an EOQ model when a random proportion of the units in a lot are defective. They assumed a 100% screening process with no human error. The S and J model suggested that the imperfect items are not reworked but just withdrawn from the received lot. It is also assumed that there is no human error in the screening process. Raouf *et al.* (1983) studied human error in inspection planning.

They extended the Raouf *et al.* (1983) inspection plan for the case of a number of misclassifications. Duffuaa and Khan (2005) carried out a sensitivity analysis to study the effect of different types of misclassification on the optimal inspection plan.

The S and J model by assuming that the screening process is not error-free. An imperfect process is utilized to describe the defective proportion of the received lot. i.e., the inspector may commit error while screening. The probability of misclassification errors is assumed to be known. The inspection process would consist of three cost (a) cost of inspection, (b) cost of Type I error, (c) cost of Type II error. The defective items classified by the inspector and the buyer would be salvaged as a single batch that is sold at a lower price.

In this article investigate the inventory model for items with imperfect quality and inspection errors. Two inventory models are discussed. The first model is fuzzy inventory model for crisp order quantity. The second model, fuzzy inventory model for fuzzy order quantity. To find the estimate of total net profit in the fuzzy sense and then derive the corresponding optimal lot size.

2. **Notations**

The following notations are used

D : Number of units demanded per year.
y : Order size.
C : Unit variable cost.
K : Fixed ordering cost.
A : A parameter used for simplifying the holding cost.
S : Unit selling price of a non defective item.
A : Unit selling price of a defective item.
x : Screening rate.
d : Unit screening cost
h : Unit holding cost
T : Cycle length

m_1	:	Probability of Type I error (classifying a nondefective item as defective)
m_2	:	Probability of Type II error (classifying a defective item as a nondefective)
p	:	Probability that an item is defective.
t_1	:	Inspection time in a cycle.
t_2	:	the remaining time in a cycle, after the defective items are screened out.
$f(P)$	:	probability density function of P.
$f(m_1)$	:	probability density function of m_1.
$f(m_2)$	:	probability density function of m_2.
B_1	:	number of items that are classified as defective in one cycle.
B_2	:	number of defective items that are returned from the market in one cycle.
C_a	:	cost of accepting a defective item.
C_r	:	cost of rejecting a nondefective item.
$*$	:	the super script representing optimal value.

3. **Mathematical Model**

Figure 1 shows how the inventory behaves with a buyer or a retailer. It should be noted here that this behavior was suggested by S and J (2000). The screening and consumption of the inventory continues until time t_1, after which all the defectives (B_1) are withdrawn from inventory as a single batch and are sold to the secondary market. The consumption process continues at the demand rate until the end of cycle time T. Due to inspection error, some of the items used to fulfill the demand would be defective. These defective items are later returned to the inventory and are shown in Figure 1 as B_2. To avoid shortages, it is assumed that the number of non-defective items is at least equal to adjusted demand, that is the sum of the actual demand and items that are replaced for the ones returned from the market over T. i.e.

$$y(1-p)(1-m_1) > DT$$

So for the limiting case, the cycle length can be written as $T = \frac{y(1-p)(1-m)}{D}$. It should be noted that the above expression is unaffected by the Type II error and reduces to the cycle length in S and J if the Type I error becomes zero.

The total cost per cycle C, consists of procurement cost, screening cost, holding cost.

$$\begin{aligned} C &= K + Cy + dy + C_r(1-p)m_1 y + C_p p y m_2 \\ &\quad + \frac{h}{2}\left(\frac{2}{x} - \frac{D}{x^2} + \frac{A^2}{D}\right) y^2 + y p m_2 T \end{aligned}$$

where $A = 1 - D/x - (m_1 + p) + p(m_1 + m_2)$.

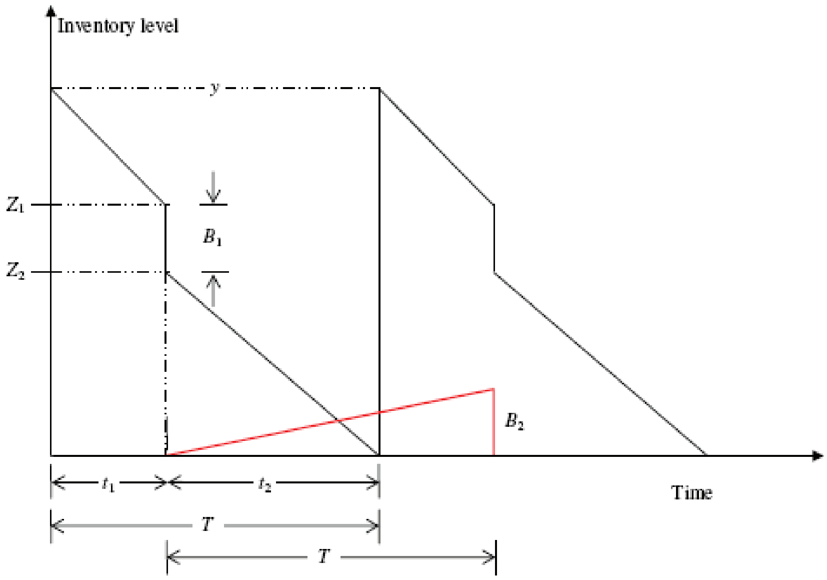

Figure 1. Behavior of the inventory level over time.

Figure 1 depicts the behavior of different types of inventory in the order cycle. The (red) triangle at the bottom represents the defective lot that is returned by the market and is accumulated into the salvaged lot.

The total profit per cycle can now be written as the different between the total revenue and total cost per cycle is

$$\begin{aligned} TP(y) = {} & Sy(1-p)(1-m_1) + Sypm_2 + Vy(1-p)m_1 + Vyp \\ & - \bigg[K + Cy + dy + C_r(1-p)ym_1 + C_a pym_2 \\ & + \frac{h}{2}\left(\frac{2}{x} - \frac{D}{x^2} + \frac{A^2}{D}\right)y^2 + ypm_2T\bigg]. \end{aligned}$$

Since p, m_1 and m_2 are random variables with probability density functions $f(p)$, $f(m_1)$ and $f(m_2)$, the annual profit can be written as

$$\begin{aligned} TPU(y) &= \frac{TP(y)}{T} \\ &= SD + \frac{SDpm_2}{(1-p)(1-m_1)} + \frac{VDm_1}{1-m_1} + \frac{VDp}{(1-p)(1-m_1)} \\ &\quad + \frac{1}{(1-p)(1-m_1)}\bigg[-\frac{KD}{y} - CD - dD - C_r(1-p)m_1D \\ &\quad - C_a pm_2D - \frac{h}{2}\left(\frac{2}{x} - \frac{D}{x^2} + \frac{A^2}{D}\right)yD\bigg] - \frac{h}{2}ypm_2. \end{aligned}$$

Also, the optimal order size that represents the maximum annual profit, is determined by setting the first derivative equal to zero and solving for y to get

$$y^* = \sqrt{\frac{2KD}{hpm_2(1-p)(1-m_1) + hD\left(\frac{2}{x} - \frac{D}{x^2} + \frac{A^2}{D}\right)}}.$$

4. **Methodology**

4.1. *Graded Mean Integration Representation Method*

S.H. Chen and C.H. Hsien (1999) introduced Graded Mean Integration Representation method based on the integral value of graded mean h-level of generalized fuzzy number for defuzzifying generalized fuzzy number.

Suppose A is a generalized fuzzy number as shown in Figure 2. It is described as any fuzzy subset of the real line R, whose membership function, $\mu_{\widetilde{A}}$ satisfies the following conditions:

(i) $\mu_{\widetilde{A}}(x)$ is a continuous mapping from R to the closed interval $[0,1]$.
(ii) $\mu_{\widetilde{A}}(x) = 0$, $-\infty < x \le a_1$,
(iii) $\mu_{\widetilde{A}}(x) = L(x)$ is strictly increasing on $[a_1, a_2]$,
(iv) $\mu_{\widetilde{A}}(x) = w_A$, $a_2 \le x \le a_3$,
(v) $\mu_{\widetilde{A}}(x) = R(x)$ is strictly decreasing on $[a_3, a_4]$,
(vi) $\mu_{\widetilde{A}}(x) = 0$, $a_4 \le x < \infty$,

where $0 < w_A \le 1$, and a_1, a_2, a_3 and a_4 are real numbers.

Also this type of generalized fuzzy number be denoted as $\widetilde{A} = (a_1, a_2, a_3, a_4; w_A)LR$. When $w_A = 1$, it can be simplified as $\widetilde{A} = (a_1, a_2, a_3, a_4)LR$.

Second, by Graded Mean Integration Representation Method L^{-1} and R^{-1} are the inverse functions of L and R, respectively, and the graded mean h-level

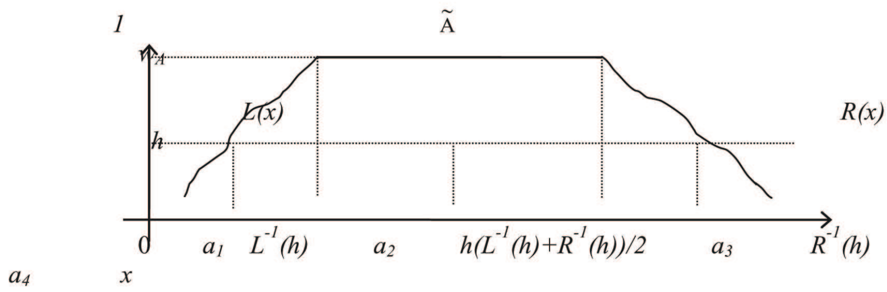

Figure 2. The graded mean h-level value of generalized fuzzy number.

$\widetilde{\mathbf{A}} = (\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4; \mathbf{w}_\mathbf{A})\boldsymbol{LR}$

Value of generalized fuzzy number $\widetilde{A} = (a_1, a_2, a_3, a_4; w_A)L$ is h is $h(L^{-1}(h) + R^{-1}(h))/2$. Then Graded Mean Integration Representation of A and P $(\widetilde{A})$ with grate W_A, where

$$P(\widetilde{A}) = \int_0^{w_A} h\left(\frac{L^{-1}(h) + R^{-1}(h)}{2}\right) dh \bigg/ \int_0^{w_A} h\, dh$$

with $0 < h \leq w_A$ and $0 < w_A \leq 1$.

Throughout this paper, we have use and only popular trapezoidal fuzzy number as the type of all fuzzy parameters in our proposed fuzzy production inventory models. Let $\widetilde{B}$ be a trapezoidal fuzzy number, and be denoted as $\widetilde{B} = (b_1, b_2, b_3, b_4)$. Then we can get the Graded Mean Integration Representation of $\widetilde{B}$ by formula (1) as

$$P(\widetilde{B}) = \int_0^1 h\left(\frac{b_1 + b_4 + (b_2 - b_1 - b_4 + b_3)h}{2}\right) dh \bigg/ \int_0^1 h\, dh$$
$$= \frac{b_1 + 2b_2 + 2b_3 + b_4}{6}.$$

4.2. *The Fuzzy Arithmetical Operations under Function Principle*

The fuzzy arithmetical operations under function principle.

In S.H. Chen (1985), Function principle is proposed to be as the fuzzy arithmetical operations by trapezoidal fuzzy numbers. We describe some fuzzy arithmetical operations under Function principle as follows:

Suppose $\widetilde{A} = (a_1, a_2, a_3, a_4)$ and $\widetilde{B} = (b_1, b_2, b_3, b_4)$ are two fuzzy trapezoidal fuzzy numbers. Then

(a) The addition of $\widetilde{A}$ and $\widetilde{B}$ is

$$\widetilde{A} \oplus \widetilde{B} = (c_1, c_2, c_3, c_4)$$

where $T = \{a_1 + b_1, a_2 + b_2, a_3 + b_3, a_4 + b_4\}$
where a_1, a_2, a_3, a_4, b_1, b_2, b_3 and b_4 are any real numbers.

(b) The multiplication of $\widetilde{A}$ and $\widetilde{B}$ is

$$\widetilde{A} \oplus \widetilde{B} = (c_1, c_2, c_3, c_4)$$

where $T = \{a_1b_1, a_2b_2, a_3b_3, a_4b_4\}$, $T_1 = \{a_2b_2, a_2b_3, a_3b_2, a_3b_3\}$,
$C_1 = \min T$, $C_2 = \min T_1$, $C_3 = \max T$, $C_4 = \max T_1$.
If a_1, a_2, a_3, a_4, b_1, b_2, b_3 and b_4 are all non zero positive real numbers, then

$$\widetilde{A} \otimes \widetilde{B} = \{a_1b_1, a_2b_2, a_3b_3, a_4b_4\}$$

(c) $-\widetilde{B} = (-b_4, -b_3, -b_2, -b_1)$, then the subtraction of $\widetilde{A}$ and $\widetilde{B}$ is

$$\widetilde{A} \ominus \widetilde{B} = \{a_1 - b_4, a_2 - b_3, a_3 - b_2, a_4 - b_1\}$$

where a_1, a_2, a_3, a_4, b_1, b_2, b_3 and b_4 are any real numbers.

(d) $\frac{1}{\widetilde{B}} = \widetilde{B}^{-1} = \left(\frac{1}{b_4}, \frac{1}{b_3}, \frac{1}{b_2}, \frac{1}{b_1}\right)$, where b_1, b_2, b_3 and b_4 are any real numbers. If a_1, a_2, a_3, a_4, b_1, b_2, b_3 and b_4 are all non zero positive real numbers, then the division of $\widetilde{A}$ and $\widetilde{B}$ is

$$\widetilde{A} \oslash \widetilde{B} = \left(\frac{a_1}{b_4}, \frac{a_2}{b_3}, \frac{a_3}{b_2}, \frac{a_4}{b_1}\right)$$

(e) Let $x \in R$. Then

(i) $\alpha \geq 0$, $\alpha \otimes \widetilde{A} = (\alpha a_1, \alpha a_2, \alpha a_3, \alpha a_4)$

(ii) $\alpha < 0$, $\alpha \otimes \widetilde{A} = (\alpha a_4, \alpha a_3, \alpha a_2, \alpha a_1)$

5. **Fuzzy Inventory Model**

5.1. *The Fuzzy Inventory Model for Crisp Order Size*

Let $\widetilde{D} = (D_1, D_2, D_3, D_4)$, $\widetilde{S} = (S_1, S_2, S_3, S_4)$, $\widetilde{V} = (V_1, V_2, V_3, V_4)$, $\widetilde{K} = (K_1, K_2, K_3, K_4)$, $\widetilde{A} = (A_1, A_2, A_3, A_4)$, $\widetilde{h} = (h_1, h_2, h_3, h_4)$, $\widetilde{d} = (d_1, d_2, d_3, d_4)$, $C_r = (C_{r_1}, C_{r_2}, C_{r_3}, C_{r_4})$, $C_a = (C_{a_1}, C_{a_2}, C_{a_3}, C_{a_4})$ be trapezoidal numbers.

Net profit per unit time in fuzzy sense is given by

$$\begin{aligned}
\widetilde{TPU}(y) = \Bigg(& S_1 D_1 + \frac{S_1 D_{_1p} m_2}{(1-p)(1-m_1)} + \frac{V_1 D_1 m_1}{1-m_1} + \frac{V_1 D_{_1p}}{(1-p)(1-m_1)} \\
& + \frac{1}{(1-p)(1-m_1)} \Bigg[-\frac{D_4 K_4}{y} - C_4 D_4 - d_4 D_4 - C_{r_4}(1-p) m_1 D_4 \\
& - C_{a_4} p m_2 D_4 - \frac{h_4}{2}\frac{2}{x} y + \frac{h_1}{2}\frac{D_1^2}{x^2} y - \frac{h_4}{2} A_4^2 y \Bigg] - \frac{h_4}{2} y p m_2, \\
& S_2 D_2 + \frac{S_2 D_2 p m_2}{(1-p)(1-m_1)} + \frac{V_2 D_2 m_1}{1-m_1} + \frac{V_2 D_2 p}{(1-p)(1-m_1)} \\
& + \frac{1}{(1-p)(1-m_1)} \Bigg[-\frac{D_3 K_3}{y} - C_3 D_3 - d_3 D_3 - C_{r_3}(1-p) m_1 D_3 \\
& - C_{a_3} p m_2 D_3 - \frac{h_3}{2}\frac{2}{x} y + \frac{h_2}{2}\frac{D_2^2}{x^2} y - \frac{h_3}{2} A_3^2 y \Bigg] - \frac{h_3}{2} y p m_2, \\
& S_3 D_3 + \frac{S_3 D_3 p m_3}{(1-p)(1-m_1)} + \frac{V_3 D_3 m_1}{1-m_1} + \frac{V_3 D_3 p}{(1-p)(1-m_1)} \\
& + \frac{1}{(1-p)(1-m_1)} \Bigg[-\frac{D_2 K_2}{y} - C_2 D_2 - d_2 D_2 - C_{r_2}(1-p) m_1 D_2 \\
& - C_{a_2} p m_2 D_2 - \frac{h_2}{2}\frac{2}{x} y + \frac{h_3}{2}\frac{D_3^2}{x^2} y - \frac{h_2}{2} A_2^2 y \Bigg] - \frac{h_2}{2} y p m_2, \\
& S_4 D_4 + \frac{S_4 D_4 p m_2}{(1-p)(1-m_1)} + \frac{V_4 D_4 m_1}{1-m_1} + \frac{V_4 D_4 p}{(1-p)(1-m_1)} \\
& + \frac{1}{(1-p)(1-m_1)} \Bigg[-\frac{D_1 K_1}{y} - C_1 D_1 - d_1 D_1 - C_{r_1}(1-p) m_1 D_1
\end{aligned}$$

$$- C_{a_1} p m_2 D_1 - \frac{h_1}{2}\frac{2}{x} y + \frac{h_4}{2}\frac{D_4^2}{x^2} y - \frac{h_1}{2} A_1^2 y \Big] - \frac{h_1}{2} y p m_2 \Big) \tag{5.1}$$

$$\begin{aligned}
P(\widetilde{TPU}(y)) = \frac{1}{6}\Bigg[& \Bigg(S_1 D_1 + \frac{S_1 D_1 p m_2}{(1-p)(1-m_1)} + \frac{V_1 D_1 m_1}{1-m_1} + \frac{V_1 D_1 p}{(1-p)(1-m_1)} \\
& + \frac{1}{(1-p)(1-m_1)} \Bigg[- \frac{D_4 K_4}{y} - C_4 D_4 - d_4 D_4 - C_{r_4}(1-p) m_1 D_4 \\
& - C_{a_4} p m_2 D_4 - \frac{h_4}{2}\frac{2}{x} y + \frac{h_1}{2}\frac{D_1^2}{x^2} y - \frac{h_4}{2} A_4^2 y \Bigg] - \frac{h_4}{2} y p m_2 \Bigg) \\
& + 2\Bigg(S_2 D_2 + \frac{S_2 D_2 p m_2}{(1-p)(1-m_1)} + \frac{V_2 D_2 m_1}{1-m_1} + \frac{V_2 D_2 p}{(1-p)(1-m_1)} \\
& + \frac{1}{(1-p)(1-m_1)} \Bigg[- \frac{D_3 K_3}{y} - C_3 D_3 - d_3 D_3 - C_{r_3}(1-p) m_1 D_3 \\
& - C_{a_3} p m_2 D_3 - \frac{h_3}{2}\frac{2}{x} y + \frac{h_2}{2}\frac{D_2^2}{x^2} y - \frac{h_3}{2} A_3^2 y \Bigg] - \frac{h_3}{2} y p m_2 \Bigg) \\
& + 2\Bigg(S_3 D_3 + \frac{S_3 D_3 p m_3}{(1-p)(1-m_1)} + \frac{V_3 D_3 m_1}{1-m_1} + \frac{V_3 D_3 p}{(1-p)(1-m_1)} \\
& + \frac{1}{(1-p)(1-m_1)} \Bigg[- \frac{D_2 K_2}{y} - C_2 D_2 - d_2 D_2 - C_{r_2}(1-p) m_1 D_2 \\
& - C_{a_2} p m_2 D_2 - \frac{h_2}{2}\frac{2}{x} y + \frac{h_3}{2}\frac{D_3^2}{x^2} y - \frac{h_2}{2} A_2^2 y \Bigg] - \frac{h_2}{2} y p m_2 \Bigg) \\
& + \Bigg(S_4 D_4 + \frac{S_4 D_4 p m_2}{(1-p)(1-m_1)} + \frac{V_4 D_4 m_1}{1-m_1} + \frac{V_4 D_4 p}{(1-p)(1-m_1)} \\
& + \frac{1}{(1-p)(1-m_1)} \Bigg[- \frac{D_1 K_1}{y} - C_1 D_1 - d_1 D_1 - C_{r_1}(1-p) m_1 D_1 \\
& - C_{a_1} p m_2 D_1 - \frac{h_1}{2}\frac{2}{x} y + \frac{h_4}{2}\frac{D_4^2}{x^2} y - \frac{h_1}{2} A_1^2 y \Bigg] - \frac{h_1}{2} y p m_2 \Bigg) \Bigg].
\end{aligned} \tag{5.2}$$

Differentiating (5.2) partially with respect to y and equating to zero for maximum profit we have,

$$y* = \sqrt{\frac{2(K_4 D_4 + 2K_3 D_3 + 2K_2 D_2 + K_1 D_1)}{\left(\begin{array}{l} \frac{2}{x}(h_1 D_4 + 2h_2 D_2 + 2h_3 D_3 + h_4 D_4) \\ -\frac{1}{x^2}(h_1 D_1^2 + 2h_2 D_2^2 + 2h_3 D_3^2 + h_4 D_4^2) \\ \times(h_1 A_1^2 + 2h_2 A_2^2 + 2h_3 A_3^2 + h_4 A_4^2) \\ +(1-p)(1-m_1) p m_2 (h_1 + 2h_2 + 2h_3 + h_4) \end{array}\right)}}. \tag{5.3}$$

5.2. *Fuzzy Inventory Model for Fuzzy Order Size*

In this section, we introduced the fuzzy inventory model, by changing the crisp order size quantity in section 5.1 into fuzzy in order size.

Suppose fuzzy order size quantity $\widetilde{y}$ to be trapezoidal fuzzy number.

$\widetilde{y} = (y_1, y_2, y_3, y_4)$ with $0 < y_1 \le y_2 \le y_3 \le y_4$.

$$\begin{aligned}
\widetilde{TPU}(y) = \Bigg(& S_1 D_1 + \frac{S_1 D_1 p m_2}{(1-p)(1-m_1)} + \frac{V_1 D_1 m_1}{1-m_1} + \frac{V_1 D_1 p}{(1-p)(1-m_1)} \\
& + \frac{1}{(1-p)(1-m_1)} \Bigg[-\frac{D_4 K_4}{y_1} - C_4 D_4 - d_4 D_4 - C_{r_4}(1-p) m_1 D_4 \\
& - C_{a_4} p m_2 D_4 - \frac{h_4}{2}\frac{2}{x} y_4 D_4 + \frac{h_1}{2}\frac{D_1^2}{x^2} y_1 - \frac{h_4}{2} A_4^2 y_4 \Bigg] - \frac{h_4}{2} y_4 p m_2, \\
& S_2 D_2 \frac{S_2 D_2 p m_2}{(1-p)(1-m_1)} + \frac{V_2 D_2 m_1}{1-m_1} + \frac{V_2 D_2 p}{(1-p)(1-m_1)} \\
& + \frac{1}{(1-p)(1-m_1)} \Bigg[-\frac{D_3 K_3}{y_2} - C_3 D_3 - d_3 D_3 - C_{r_3}(1-p) m_1 D_3 \\
& - C_{a_3} p m_2 D_3 - \frac{h_3}{2}\frac{2}{x} y_3 D_3 + \frac{h_2}{2}\frac{D_2^2}{x^2} y_2 - \frac{h_3}{2} A_3^2 y_3 \Bigg] - \frac{h_3}{2} y_3 p m_2, \\
& S_3 D_3 + \frac{S_3 D_3 p m_3}{(1-p)(1-m_1)} + \frac{V_3 D_3 m_1}{1-m_1} + \frac{V_3 D_3 p}{(1-p)(1-m_1)} \\
& + \frac{1}{(1-p)(1-m_1)} \Bigg[-\frac{D_2 K_2}{y_3} - C_2 D_2 - d_2 D_2 - C_{r_2}(1-p) m_1 D_2 \\
& - C_{a_2} p m_2 D_2 - \frac{h_2}{2}\frac{2}{x} y_2 D_2 + \frac{h_3}{2}\frac{D_3^2}{x^2} y_3 - \frac{h_2}{2} A_2^2 y_2 \Bigg] - \frac{h_2}{2} y_2 p m_2, \\
& S_4 D_4 + \frac{S_4 D_4 p m_2}{(1-p)(1-m_1)} + \frac{V_4 D_4 m_1}{1-m_1} + \frac{V_4 D_{4P}}{(1-p)(1-m_1)} \\
& + \frac{1}{(1-p)(1-m_1)} \Bigg[-\frac{D_1 K_1}{y_4} - C_1 D_1 - d_1 D_1 - C_{r_1}(1-p) m_1 D_1 \\
& - C_{a_1} p m_2 D_1 - \frac{h_1}{2}\frac{2}{x} y_1 D_1 + \frac{h_4}{2}\frac{D_4^2}{x^2} y_4 - \frac{h_1}{2} A_1^2 y_1 \Bigg] - \frac{h_1}{2} y_1 P^{m_2} \Bigg)
\end{aligned} \tag{5.4}$$

$$\begin{aligned}
P(\widetilde{TPU}(y)) = \frac{1}{6} \Bigg[\Bigg(& S_1 D_1 + \frac{S_1 D_1 p m_2}{(1-p)(1-m_1)} + \frac{V_1 D_1 m_1}{1-m_1} + \frac{V_1 D_1 p}{(1-p)(1-m_1)} \\
& + \frac{1}{(1-p)(1-m_1)} \Bigg[-\frac{D_4 K_4}{y_1} - C_4 D_4 - d_4 D_4 - C_{r_4}(1-p) m_1 D_4 \\
& - C_{a_4} p m_2 D_4 - \frac{h_4}{2}\frac{2}{x} y_4 D_4 + \frac{h_1}{2}\frac{D_1^2}{x^2} y_1 - \frac{h_4}{2} A_4^2 y_4 \Bigg] - \frac{h_4}{2} y_4 p m_2 \\
& + 2\Bigg(S_2 D_2 + \frac{S_2 D_2 p m_2}{(1-p)(1-m_1)} + \frac{V_2 D_2 m_1}{1-m_1} + \frac{V_2 D_{2P}}{(1-p)(1-m_1)} \\
& + \frac{1}{(1-p)(1-m_1)} \Bigg[-\frac{D_3 K_3}{y_2} - C_3 D_3 - d_3 D_3 - C_{r_3}(1-p) m_1 D_3 \\
& - C_{a_3} p m_2 D_3 - \frac{h_3}{2}\frac{2}{x} y_3 D_3 + \frac{h_2}{2}\frac{D_2^2}{x^2} y_2 - \frac{h_3}{2} A_3^2 y_3 \Bigg] - \frac{h_3}{2} y_3 p m_2 \Bigg)
\end{aligned}$$

$$+2\bigg(S_3D_3+\frac{S_3D_3pm_3}{(1-p)(1-m_1)}+\frac{V_3D_3m_1}{1-m_1}+\frac{V_3D_3p}{(1-p)(1-m_1)}$$
$$+\frac{1}{(1-p)(1-m_1)}\bigg[-\frac{D_2K_2}{y_3}-C_2D_2-d_2D_2-C_{r_2}(1-p)m_1D_2$$
$$-C_{a_2}pm_2D_2-\frac{h_2}{2}\frac{2}{x}y_2D_2+\frac{h_3}{2}\frac{D_3^2}{x^2}y_3-\frac{h_2}{2}A_2^2y_2\bigg]-\frac{h_2}{2}y_2pm_2\bigg)$$
$$+\bigg(S_4D_4+\frac{S_4D_4pm_2}{(1-p)(1-m_1)}+\frac{V_4D_4m_1}{1-m_1}+\frac{V_4D_4p}{(1-p)(1-m_1)}$$
$$+\frac{1}{(1-p)(1-m_1)}\bigg[-\frac{D_1K_1}{y_1}-C_1D_1-d_1D_1-C_{r_1}(1-p)m_1D_1$$
$$-C_{a_1}pm_2D_1-\frac{h_1}{2}\frac{2}{x}y_1D_1+\frac{h_4}{2}\frac{D_4^2}{x^2}y_4-\frac{h_1}{2}A_1^2y_1\bigg]-\frac{h_1}{2}y_1pm_2\bigg)\bigg]. \tag{5.5}$$

with $0 < y_1 \le y_2 \le y_3 \le y_4$.

It will not change the meaning of formula (5.5) if we replace inequality conditions with $0 < y_1 \le y_2 \le y_3 \le y_4$ into the following inequality $y_2 - y_1 \ge 0$, $y_3 - y_2 \ge 0$, $y_4 - y_3 \ge 0$ and $y_1 \ge 0$. Extension of the Lagrangian Method is used to find the solution of y_1, y_2, y_3, and y_4.

Step 1:

Solve the constraint problem

$$P(\widetilde{TPU}(y))=\frac{1}{6}\bigg[\bigg(S_1D_1+\frac{S_1D_1pm_2}{(1-p)(1-m_1)}+\frac{V_1D_1m_1}{1-m_1}+\frac{V_1D_1p}{(1-p)(1-m_1)}$$
$$+\frac{1}{(1-p)(1-m_1)}\bigg[-\frac{D_4K_4}{y_1}-C_4D_4-d_4D_4-C_{r_4}(1-p)m_1D_4$$
$$-C_{a_4}pm_2D_4-\frac{h_4}{2}\frac{2}{x}y_4D_4+\frac{h_1}{2}\frac{D_1^2}{x^2}y_1-\frac{h_4}{2}A_4^2y_4\bigg]-\frac{h_4}{2}y_4pm_2,$$
$$2\bigg(S_2D_2+\frac{S_2D_2pm_2}{(1-p)(1-m_1)}+\frac{V_2D_2m_1}{1-m_1}+\frac{V_2D_2p}{(1-p)(1-m_1)}$$
$$+\frac{1}{(1-p)(1-m_1)}\bigg[-\frac{D_3K_3}{y_2}-C_3D_3-d_3D_3-C_{r_3}(1-p)m_1D_3$$
$$-C_{a_3}pm_2D_3-\frac{h_3}{2}\frac{2}{x}y_3D_3+\frac{h_2}{2}\frac{D_2^2}{x^2}y_2-\frac{h_3}{2}A_3^2y_3\bigg]-\frac{h_3}{2}y_3pm_2\bigg)$$
$$+2\bigg(S_3D_3+\frac{S_3D_3pm_3}{(1-p)(1-m_1)}+\frac{V_3D_3m_1}{1-m_1}+\frac{V_3D_3p}{(1-p)(1-m_1)}$$
$$+\frac{1}{(1-p)(1-m_1)}\bigg[-\frac{D_2K_2}{y_3}-C_2D_2-d_2D_2-C_{r_2}(1-p)m_1D_2$$
$$-C_{a_2}pm_2D_2-\frac{h_2}{2}\frac{2}{x}y_2D_2+\frac{h_3}{2}\frac{D_3^2}{x^2}y_3-\frac{h_2}{2}A_2^2y_2\bigg]-\frac{h_2}{2}y_2pm_2\bigg)$$

$$+\left(S_4D_4+\frac{S_4D_4pm_2}{(1-p)(1-m_1)}+\frac{V_4D_4m_1}{1-m_1}+\frac{V_4D_4p}{(1-p)(1-m_1)}\right.$$
$$+\frac{1}{(1-p)(1-m_1)}\left[-\frac{D_1K_1}{y_1}-C_1D_1-d_1D_1-C_{r_1}(1-p)m_1D_1\right.$$
$$\left.\left.\left.-C_{a_1}pm_2D_1-\frac{h_1}{2}\frac{2}{x}y_1D_1+\frac{h_4}{2}\frac{D_4^2}{x^2}y_4-\frac{h_1}{2}A_1^2y_1\right]-\frac{h_1}{2}y_1pm_2\right)\right].$$

By taking the derivative of $P(\widetilde{TPU}(y))$ with respect to y_1, y_2, y_3, and y_4, and let all the partial derivatives equal to zero. We get

$$y_1=\sqrt{\frac{2K_4D_4}{h_1\left[\frac{2}{x}D_1-\frac{D_1^2}{x^2}+A_1^2\right]+pm_2h_1(1-p)(1-m_1)}},$$
$$y_2=\sqrt{\frac{4K_3D_3}{2h_2\left[\frac{2}{x}D_2-\frac{D_2^2}{x^2}+A_2^2\right]+2pm_2h_2(1-p)(1-m_1)}},$$
$$y_3=\sqrt{\frac{4K_2D_2}{2h_3\left[\frac{2}{x}D_3-\frac{D_3^2}{x^2}+A_3^2\right]+2pm_2h_3(1-p)(1-m_1)}},$$
$$y_4=\sqrt{\frac{2K_1D_1}{h_4\left[\frac{2}{x}D_4-\frac{D_4^2}{x^2}+A_4^2\right]+pm_2h_4(1-p)(1-m_1)}}.$$

Because the above show that $y_1 > y_2 > y_3 > y_4$, it does not satisfy the constraint $0 < y_1 \le y_2 \le y_3 \le y_4$.

Step 2:

Convert the inequality constraint $y_2-y_1 \ge 0$ into equality constraint $y_2-y_1=0$ and optimize $P(\widetilde{TPU}(y))$ subject $y_2-y_1=0$ by the Lagrangian Method. We have Lagrangian function as $L(y_1,y_2,y_3,y_4,\lambda)=P(\widetilde{TPU}(y))-\lambda_1(y_2-y_1)$.

Taking the partial derivatives of $L(y_1,y_2,y_3,y_4,\lambda)$ with respect to y_1, y_2, y_3, y_4 and λ_1. Let all the partial derivatives equal to zero and solve y_1, y_2, y_3, y_4 and λ_1.

$$y_1=y_2=\sqrt{\frac{2(K_4D_4+2K_3D_3)}{\left(\begin{array}{l}\frac{2}{x}[h_1D_1+2h_2D_2]+(h_1A_1^2+2h_2A_2^2)-\frac{1}{x^2}(h_1D_1^2+2h_2D_2^2)\\+(h_1+2h_2)pm_2h_4(1-p)(1-m_1)\end{array}\right)}},$$
$$y_3=\sqrt{\frac{4K_2D_2}{2h_3\left[\frac{2}{x}D_3-\frac{D_3^2}{x^2}+A_3^2\right]+2h_3pm_2(1-p)(1-m_1)}},$$

$$y_4 = \sqrt{\frac{2K_1D_1}{h_4\left[\frac{2}{x}D_4 - \frac{D_4^2}{x^2} + A_2^2\right] + h_4pm_2(1-p)(1-m_1)}}.$$

Because the above results show that $y_1 > y_4$, is does not satisfy the constraint $0 < y_1 \leq y_2 \leq y_3 \leq y_4$. Therefore it is not a local optimum. Similarly, we can get the same result if we select any other one inequality constraint to be equality constraint.

Step 3:

Convert the inequality constraint $y_2 - y_1 \geq 0$ and $y_3 - y_2 \geq 0$ into equality constraint $y_2 - y_1 = 0$ and $y_3 - y_2 = 0$. We optimize $P(\widetilde{TPU}(y))$ subject to $y_2 - y_1 = 0$ and $y_3 - y_2 = 0$ by the Lagrangian Method. Then the Lagrangian function is

$$L(y_1, y_2, y_3, y_4, \lambda_1, \lambda_2) = P(\widetilde{TPU}(y)) - \lambda_1(y_2 - y_1) - \lambda_2(y_3 - y_2).$$

We take the partial derivatives of $L(y_1, y_2, y_3, y_4, \lambda_1, \lambda_2)$ with respect to y_1, y_2, y_3, y_4, λ_1 and λ_2 and let all the partial derivatives equal to zero and solve y_1, y_2, y_3, y_4. Then we get

$$y_1 = y_2 = y_3 = \sqrt{\frac{2(K_4D_4 + 2K_3D_3 + 2K_2D_2)}{\begin{pmatrix} \frac{2}{x}[h_1D_1 + 2h_2D_2 + 2h_3D_3] + (h_1A_1^2 + 2h_2A_2^2 + 2h_3A_3^2) \\ -\frac{1}{x^2}(h_1D_1^2 + 2h_2D_2^2 + 2h_3D_3^2) \\ +(h_1 + 2h_2 + 2h_3)pm_2(1-p)(1-m_1) \end{pmatrix}}},$$

$$y_4 = \sqrt{\frac{2K_1D_1}{h_4\left[\frac{2}{x}D_4 - \frac{D_4^2}{x^2} + A_4^2\right] + h_4pm_2(1-p)(1-m_1)}}.$$

But $y_{p_1} > y_{p_4}$ it does not satisfy the constraint $< y_{p_1} \leq y_{p_2} \leq y_{p_3} \leq y_{p_4}$, therefore it is not a local optimum. Similarly, we can get the same result if we select any other two inequality constraint to be equality constraint.

Step 4:

Convert the inequality constraint $y_2 - y_1 \geq 0$, $y_3 - y_2 \geq 0$ and $y_4 - y_3 \geq 0$ into equality constraint $y_2 - y_1 = 0$, $y_2 - y_2 = 0$ and $y_4 - y_3 = 0$. We optimize $P(\widetilde{TPU}(y))$ subject to $y_2 - y_1 = 0$, $y_3 - y_2 = 0$ and $y_4 - y_3 = 0$ by the Lagrangian Method. Then the Lagrangian function is

$$\begin{aligned} &L(y_1, y_2, y_3, y_4, \lambda_1, \lambda_2, \lambda_3) \\ &\quad = P(\widetilde{TPU}(y)) - \lambda_1(y_2 - y_1) - \lambda_2(y_3 - y_2) - \lambda_3(y_4 - y_3). \end{aligned}$$

We take the partial derivatives of $L(y_1, y_2, y_3, y_4, \lambda_1, \lambda_2, \lambda_3)$ with respect to y_1, y_2, y_3, y_4, λ_1, λ_2 and λ_3 and let all the partial derivatives equal to zero y_1, y_2, y_3

and y_4. Then we get

$$y_1 = y_2 = y_3 = y_4 = \sqrt{\frac{2(K_4D_4 + 2K_3D_3 + 2K_2D_2 + K_1D_1)}{\left(\begin{array}{l} \frac{2}{x}(h_1D_4 + 2h_2D_2 + 2h_3D_3 + h_4D_4) \\ -\frac{1}{x^2}(h_1D_1^2 + 2h_2D_2^2 + 2h_3D_3^2 + h_4D_4^2) \\ \times(h_1A_1^2 + 2h_2A_2^2 + 2h_3A_3^2 + h_4A_4^2) \\ +(1-p)(1-m_1)pm_2(h_1 + 2h_2 + 2h_3 + h_4) \end{array}\right)}}. \tag{5.6}$$

Because the above solution $\widetilde{y} = (y_1, y_2, y_3, y_4)$. Satisfy all the inequality constraints, the procedure terminate with $\widetilde{y}$ as a local optimum solution to the problem.

Since the above local optimum solution is the only one feasible solution of formula (5.6). So it is an optimum solution of the inventory model with fuzzy order size quantity according to extension of the Lagrangian Method.

Let $y_1 = y_2 = y_3 = y_4 = y$. Then the optimal fuzzy order size

$$y^* = \sqrt{\frac{2(K_4D_4 + 2K_3D_3 + 2K_2D_2 + K_1D_1)}{\left(\begin{array}{l} \frac{2}{x}(h_1D_4 + 2h_2D_2 + 2h_3D_3 + h_4D_4) \\ -\frac{1}{x^2}(h_1D_1^2 + 2h_2D_2^2 + 2h_3D_3^2 + h_4D_4^2) \\ \times(h_1A_1^2 + 2h_2A_2^2 + 2h_3A_3^2 + h_4A_4^2) \\ +(1-p)(1-m_1)pm_2(h_1 + 2h_2 + 2h_3 + h_4) \end{array}\right)}}. \tag{5.7}$$

6. **Numerical Analysis**

Consider a production system that replenishes the buyer's orders instantly. This system is not perfect, i.e. it produces some defective items. The inspection process that screens out the defective items is also imperfect. The probability density functions for the fraction of defective items and the inspection errors are mostly taken from the history of a supplier and workers. In the case when these values are not known. The fraction of defectives in a lot can be determined by using the lot size or the time at which a process goes out of control in a cycle. Similarly, the parameters for inspection errors can be determined by the methods suggested by Cary *et al.* (1994) or Jaraiedo (1983). In the following analysis, most of the data is taken from the S and J model.

$$\begin{array}{ll} D = 50{,}000\,\text{units/year}; & C = \$25/\text{unit}; \\ K = \$100/\text{cycle}; & S = \$50/\text{unit}; \\ V = \$20/\text{unit}; & x = 1\,\text{unit/min}; \\ d = \$0.5/\text{unit}; & h = \$5/\text{unit}; \\ C_a = \$500/\text{unit}; & C_r = \$100/\text{unit}; \end{array}$$

$$f(p)=\begin{cases}25 & 0\le p\le 0.05\\ 0 & \text{otherwise;}\end{cases}\qquad f(m_1)=\begin{cases}25 & 0\le m_1\le 0.05\\ 0 & \text{otherwise;}\end{cases}$$

$$f(m_2)=\begin{cases}25 & 0\le m_2\le 0.05\\ 0 & \text{otherwise.}\end{cases}$$

Consider $p=0.02, m_1=0.02, m_2=0.02$.

Assuming that the buyer operates for 8 hours per day for 365 days per year, the annual screening rate would be, $x=1,75,200$ units.

Let

$$\begin{aligned}
&\widetilde{D}=(47500,50000,50000,52500); && \widetilde{K}=(95,100,100,105);\\
&\widetilde{h}=(4.75,5,5,5.25); && \widetilde{A}=(0.6613,0.6755,0.6755,0.6897);\\
&\widetilde{C}=(23.75,25,25,26.25); && \widetilde{S}=(47.5,50,50,52.5);\\
&\widetilde{V}=(19,20,20,21); && \widetilde{d}=(0.475,0.5,0.5,0.525);\\
&\widetilde{C}_a=(475,500,500,525); && \widetilde{C}_r=(95,100,100,105).
\end{aligned}$$

Substituting above value in (5.7) we obtain the optimal values of $\widetilde{y}^*=\mathbf{1454}$ **units**. $\widetilde{\mathbf{TPU}}(\mathbf{y})^\star=\mathbf{1095090/year}$.

7. Conclusion

In the fuzzy environment it may be possible and reasonable to discuss the imperfect quality and inspection errors with trapezoidal fuzzy number for crisp order quantity y, or for fuzzy order quantity $\widetilde{y}$. In addition, we find that the optimal fuzzy order quantity $\widetilde{y}^*=(\widetilde{y}^*,\widetilde{y}^*,\widetilde{y}^*,\widetilde{y}^*)$ is the special type of trapezoidal fuzzy number. It can also be considered as crisp real number and the optimal solution of our proposed models, $\widetilde{y}^*$ and y^* are real numbers. The optimal fuzzy order quantity $\widetilde{y}^*$ or the optimal crisp order quantity y^* will become

$$\sqrt{\frac{2KD}{hpm_2(1-p)(1-m_1)+hD\left(\frac{2}{x}-\frac{D}{x^2}+\frac{A^2}{D}\right)}}.$$

It means that the optimal solution of our proposed models can be specified to meet the classical inventory models. Hence these fuzzy inventory models are executable and useful in the real world.

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