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I-Convergence and Summability in Topological Group

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Abstract. In this article we introduce the I-convergence of sequences in topological groups and give certain characterizations of I-convergent sequences in topological groups and prove some fundamental theorems for topological groups.

1. Introduction

The notion of statistical convergence is a very useful functional tool for studying the convergence problems of numerical sequences/matrices (double sequences) through the concept of density. It was first introduced by Fast [7], independently for the real sequences. Later on it was further investigated from sequence point of view and linked with the summability theory by Fridy [8] and many others. The idea is based on the notion of natural density of subsets of N, the set of positive integers, which is defined as follows: The natural density of a subset of N is denoted by $\delta(E)$ and is defined by $\delta(E) = \lim_{n\to\infty} \frac{1}{n}|\{k \in E : k \leq n\}|$, where the vertical bar denotes the cardinality of the respective set. This notion was used by Cakalli [5] to extend to topological Hausdroff groups.

The notion of I-convergence (I denotes the ideal of subsets of N, the set of positive integers), which is a generalization of statistical convergence, was introduced by Kastyrko, Salat and Wilczynski [9] and further studied by many other authors. Later on it was further investigated from sequence space point of view and linked with summability theory by Salat, Tripathy and Ziman [11, 12], Tripathy and Hazarika [13, 14, 15, 16], Hazarika [17], Hazarika and Savas [18] and many other authors.

The purpose of this article is to give certain characterizations of I-convergent sequences in topological groups and to obtain fundamental theorems in topological groups.

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2. Definitions and preliminaries

Definition 2.1. Let S be a non-empty set. A non-empty family of sets $I \subseteq P(S)$ (power set of S) is called an *ideal* in S if (i) for each $A, B \in I$, we have $A \cup B \in I$; (ii) for each $A \in I$ and $B \subseteq A$, we have $B \in I$.

Definition 2.2. Let S be a non-empty set. A family $F \subseteq P(S)$ (power set of S) is called a *filter* on S if (i) $\phi \notin F$; (ii) for each $A, B \in F$, we have $A \cap B \in F$; (iii) for each $A \in F$ and $B \supseteq A$, we have $B \in F$.

Definition 2.3. An ideal I is called *non-trivial* if $I \neq \phi$ and $S \notin I$. It is clear that $I \subseteq P(S)$ is a non-trivial ideal if and only if the class $F = F(I) = \{S - A : A \in I\}$ is a filter on S.
The filter $F(I)$ is called the filter associated with the ideal I.

Definition 2.4. A non-trivial ideal $I \subseteq P(S)$ is called an *admissible ideal* in S if it contains all singletons, i.e., if it contains $\{\{x\} : x \in S\}$.

Definition 2.5. A sequence (x_k) of points in X is said to be *I-convergent* to an element x_0 of X if for each neighbourhood V of 0 such that the set

$$\{k \in N : x_k - x_0 \notin V\} \in I$$

and it is denoted by $I\text{-}\lim_{k\to\infty} x_k = x_0$.

Definition 2.6. A sequence (x_k) of points in X is said to be *I-Cauchy* in X if for each neighbourhood V of 0, there is an integer $n(V)$ such that the set

$$\{k \in N : x_k - x_{n(V)} \notin V\} \in I$$

Definition 2.7. Let $A \subset X$ and $x_0 \in X$. Then x_0 is in the I-sequential closure of A if there is a sequence (x_k) of points in A such that $I\text{-}\lim_{k\to\infty} x_k = x_0$. We denote I-sequential closure of a set A by $\bar{A}^I$. We say that a set is I-sequentially closed if it contains all of the points in its I-sequential closure.

Throughout the article $s(X)$, $c^I(X)$ and $C^I(X)$ denote the set of all X-valued sequences, the set of all X-valued I-convergent sequences and the set of all X-valued I-Cauchy sequences in X, respectively.

By a method of sequential convergence, we mean an additive function B defined on a subgroup of $s(X)$, denoted by $c_B^I(X)$ into X.

Definition 2.8. A sequence $x = (x_k)$ is said to be *B-convergent* to x_0 if $x \in c_B^I(X)$ and $B(x) = x_0$.

Definition 2.9. A method B is called *regular* if every convergent sequence $x = (x_k)$ is B-convergent with $B(x) = \lim x$.

Definition 2.10. A point x_0 is called a *B-sequential accumulation point* of A (or is in the B-sequential derived set) if there is a sequence $x = (x_k)$ of points in $A - \{x_0\}$ such that $B(x) = x_0$.

Definition 2.11. A subset A of X is called *B-sequentially countably compact* if any infinite subset A has at least one B-sequentially accumulation point in A.

Definition 2.12. A subset A of X is called *B-sequentially compact* if $x = (x_k)$ is a sequence of points of A, there is a subsequence $y = (y_{k_n})$ of x with $B(y) = x_0$.

3. **Main results**

Theorem 3.1. *A sequence (x_k) is I-convergent if and only if for each neighbourhood V of 0 there exists a subsequence $(x_{k'(r)})$ of (x_k) such that $\lim_{r\to\infty} x_{k'(r)} = x_0$ and*

$$\{k \in N : x_k - x_{k'(r)} \notin V\} \in I.$$

Proof. Let $x = (x_k)$ be a sequence in X such that $I\text{-}\lim_{k\to\infty} x_k = x_0$. Let $\{V_n\}$ be a sequence of nested base of neighbourhoods of 0. We write $E^{(i)} = \{k \in N : x_k - x_o \notin V_i\}$ for any positive integer i. Then for each i, we have $E^{(i+1)} \subset E^{(i)}$ and $E^{(i)} \in F(I)$. Choose $n(1)$ such that $k > n(1)$, then $E^{(1)} \neq \phi$. Then for each positive integer r such that $n(p+1) \leq r < n(2)$, choose $k'(r) \in E^{(p)}$, i.e., $x_{k'(r)} - x_0 \in V_1$. In general, choose $n(p+1) > n(p)$ such that $r > n(p+1)$, then $E^{(p+1)} \neq \phi$. Then for all r satisfying $n(p) \leq r < n(p+1)$, choose $k'(r) \in E^{(p)}$, i.e. $x_{k'(r)} - x_0 \in V$. Also for every neighbourhood V of 0, there is a symmetric neighbourhood W of 0 such that $W \cup W \subset V$. Then we get

$$\{k \in N : x_k - x_{k'(r)} \notin V\} \subseteq \{k \in N : x_k - x_0 \notin W\} \cup \{r \in N : x_{k'(r)} - x_0 \notin W\}.$$

Since $I\text{-}\lim_{k\to\infty} x_k = x_0$, therefore there is a neighbourhood W of 0 such that

$$\{k \in N : x_k - x_0 \notin W\} \in I$$

and $\lim_{r\to\infty} x_{k'(r)} = x_0$ implies $\{r \in N : x_{k'(r)} - x_0 \notin W\} \in I$.

Thus we have

$$\{k \in N : x_k - x_0 \notin V\} \in I$$

Next suppose for each neighbourhood V of 0 there exists a subsequence $(x_{k'(r)})$ of (x_k) such that $\lim_{r\to\infty} x_{k'(r)} = x_0$ and $\{k \in N : x_k - x_{k'(r)} \notin V\} \in I$.

Since V is a neighbourhood of 0, we may choose a symmetric neighbourhood W of 0 such that $W \cup W \subset V$. Then we have

$$\{k \in N : x_k - x_0 \notin V\} \subseteq \{k \in N : x_k - x_{k'(r)} \notin W\} \cup \{r \in N : x_{k'(r)} - x_0 \notin W\}.$$

Since both the sets on the right hand side of the above relation belongs to I. Therefore $\{k \in N : x_k - x_0 \notin V\} \in I$.
This completes the proof. □

Theorem 3.2. *Any B-sequentially closed subset of a B-sequentially compact subset of X is B-sequentially compact.*

Proof. Let A be a B-sequentially compact subset of X and E be a B-sequentially closed subset of A. Let $x = (x_k)$ be a sequence of points in E. Then x is a sequence of points in A. Since A is B-sequentially compact, there exists a subsequence $y = (y_r) = (x_{k_r})$ of the sequence (x_k) such that $B(y) \in A$. The subsequence (y_r) is also a sequence of points in E and E is B-sequentially closed, therefore $B(y) \in E$. Thus $x = (x_k)$ has a B-convergent subsequence with $B(y) \in E$, so E is B-sequentially compact. □

Theorem 3.3. *Let B be a regular subsequential method. Any B-sequentially compact subset of X is B-sequentially closed.*

Proof. Let A be any B-sequentially compact subset of X. For any $x_0 \in \bar{A}^B$, then there exists a sequence $x = (x_k)$ be a sequence of points in A such that $B(x) = x_0$. Since B is a subsequential method, there is a subsequence $y = (y_r) = (x_{k_r})$ of the sequence $x = (x_k)$ such that $I\text{-}\lim_r x_{k_r} = x_0$. Since B is regular, so $B(y) = x_0$. Since A is B-sequentially compact, there is a subsequence $z = (z_r)$ of the subsequence $y = (y_r)$ such that $B(z) = y_0 \in A$. Since $I\text{-}\lim_r z_r = x_0$ and B is regular, so $B(z) = x_0$. Then $x_0 = y_0$ and hence $x_0 \in A$. Thus A is B-sequentially closed. □

Theorem 3.4. *Let B be a regular subsequential method. Then a subset of X is B-sequentially compact if and only if it is B-sequentially countably compact.*

Proof. Let A be any B-sequentially compact subset of X and E be an infinite subset of A. Let $x = (x_k)$ be a sequence of different points of E. Since A is B-sequentially compact, so this implies that the sequence x has a convergent subsequence $y = (y_r) = (x_{k_r})$ with $B(y) = x_0$. Since B is subsequential method, y has a convergent subsequence $z = (z_r)$ of the subsequence y with $I\text{-}\lim_r z_r = x_0$. Since B is regular, we obtain that x_0 is a B-sequentially accumulation point of E. Then A is B-sequentially countably compact.

Next suppose A is any B-sequentially countably compact subset of X. Let $x = (x_k)$ be a sequence of different points in A. Put $G = \{x_k : k \in N\}$. If G is finite, then there is nothing to prove. If G is infinite, then G has a B-sequentially accumulation point in A. Also each set $G_n = \{x_n : n \geq k\}$, for each positive integer n, has a B-sequentially accumulation point in A. Therefore $\bigcap_{n=1}^{\infty} \bar{G}_n^B \neq \phi$. So there is an element $x_0 \in A$ such that $x_0 \in \bigcap_{n=1}^{\infty} \bar{G}_n^B$. Since B is a regular subsequential method, so $x_0 \in \bigcap_{n=1}^{\infty} \bar{G}_n$. Then there exists a subsequence $z = (z_r)$ of the sequence $x = (x_k)$ with $B(z) \in A$. This completes the proof. □

Theorem 3.5. *The B-sequential continuous image of any B-sequentially compact subset of X is B-sequentially compact.*

Proof. Let f be any B-sequentially continuous function on X and A be any B-sequentially compact subset of X. Let $y = (y_k) = (f(x_k))$ be a sequence of points in $f(A)$. Since A is B-sequentially compact, there exists a subsequence $z = (z_r) = (x_{k_r})$ of the sequence $x = (x_k)$ with $B(z) \in A$. Then the sequence $f(z) = (f(z_r)) = (f(x_{k_r}))$ is a subsequence of the sequence y. Since f is B-sequentially continuous, $B(f(z)) = f(x) \in f(A)$. Then $f(A)$ is B-sequentially compact. □

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