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A Note on Right Full k-Ideals of Seminearrings

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Abstract. This work extends the idea of k-ideals of semirings to seminearrings, the concept of k-ideals of seminearrings is introduced and investigated, which is an interesting for seminearrings and some interesting characterizations of k-ideals of seminearrings are obtained. Also, we prove that the set of all right full k-ideals of an additively inverse seminearring in which addition is commutative forms a complete lattice which is also modular in the same way as of the results of Sen and Adhikari.

1. Introduction and Preliminaries

The notion of semirings which is a generalization of rings introduced by Vandiver [13] in 1935, several researches have characterized the many type of ideals on the algebraic structures such as: In 1958, Is ki [7] considered and proved some theorems on quasi-ideals in semirings. In 1992, Sen and Adhikari [10] studied k-ideals in semirings. Moreover, Sen and Adhikari proved that the set of all full k-ideals of an additively inverse semiring in which addition is commutative forms a complete lattice which is also modular. In 1993, Sen and Adhikari [11] gave some characterizations of maximal k-ideals of semirings. In 1994, D nges [5] characterized quasi-ideals, regular semirings and regular elements of semirings using quasi-ideals. In 2000, Baik and Kim [2] characterized fuzzy k-ideals in semirings. In 2004, Shabir, Ali and Batool [12] gave some properties of quasi-ideals in semirings. In 2005, Fla ka, Kepka and aroch [6] gave some characterizations of bi-ideal-simple semirings. In 2008, Chinram [4] studied (m,n)-quasi-ideals of semirings. In this year, Atani and Atani [1] characterized some results on ideal theory of commutative semirings with non-zero identity analogues to commutative rings with non-zero identity. Moreover, they studied some essential properties of Noetherian and Artinian semirings. Now, the notion of seminearrings which is a

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generalization of semirings introduced and discussed by Rootselaar [9] in 1963. Therefore, we will study k-ideals of seminearrings in the same way as of k-ideals of semirings which was studied by Sen and Adhikari [10].

The purpose of this paper is threefold.

(i) To introduce the concept of (left, right) k-ideals of seminearrings.
(ii) To introduce the concept of (left, right) full k-ideals of additively inverse seminearrings.
(iii) To characterize the properties of (left, right) k-ideals of seminearrings, and (left, right) full k-ideals of additively inverse seminearrings.

For the sake of completeness, we state some definitions and notations that are introduced analogously to some definitions and notations in [10].

A *seminearring* [8] is a system consisting of a nonempty set S together with two binary operations on S called addition and multiplication such that

(i) S together with addition is a semigroup,
(ii) S together with multiplication is a semigroup, and
(iii) $(a+b)c = ac + bc$ for all $a, b, c \in S$.

We define a *subseminearring* A of a seminearring S to be a nonempty subset A of S such that when the seminearring operations of S is restricted to A, A is a seminearring in its own right. A seminearring S is said to be *additively commutative* if $a+b = b+a$ for all $a, b \in S$. A nonempty subset I of a seminearring S is called a *right(left) ideal* of S if

(i) $a+b \in I$ for all $a, b \in I$, and
(ii) $ar \in I$ $(ar \in I)$ for all $r \in S$ and $a \in I$.

A nonempty subset I of a seminearring S is called an *ideal* of S if it is both a left and a right ideal of S. A right(left) ideal I of a seminearring S is called a *right(left) k-ideal* of S if for any $a \in I$ and $x \in S$, $a+x \in I$ or $x+a \in I$ implies $x \in I$. A nonempty subset I of a seminearring S is called a *k-ideal* of S if it is both a left and a right k-ideal of S. A seminearring S is said to be *additively regular* if for any $a \in S$, there exists an element $b \in S$ such that $a = a+b+a$. A seminearring S is said to be *additively inverse* if for any $a \in S$, there exists a unique element $b \in S$ such that $a = a+b+a$ and $b = b+a+b$. In an additively inverse seminearring, the unique inverse b of an element a is usually denoted by a'. An element a of a seminearring S is called a *additive idempotent* of S if $a+a = a$ and the set of all additive idempotents of S denoted by E^+. A right(left) k-ideal I of an additively inverse seminearring S is called a *right(left) full k-ideal* of S if $E^+ \subseteq I$. A nonempty subset I of an additively inverse seminearring S is called a *full k-ideal* of S if it is both a left and a right full k-ideal of S. Let S be a seminearring and A a right ideal of S. Define the set

$$\overline{A} = \{a \in S \mid a + x \in A \text{ for some } x \in A\}.$$

Let S be an additively inverse seminearring. Define the set of all right full k-ideals of S by $I(S)$. An equivalence relation ρ on a seminearring S is called a *congruence* if for any $a,b,c \in S, (a,b) \in \rho$ implies

$$(c+a, c+b) \in \rho \quad \text{and} \quad (a+c, b+c) \in \rho$$

and

$$(ca, cb) \in \rho \quad \text{and} \quad (ac, bc) \in \rho.$$

We can easily prove that the set of all congruence classes S/ρ is a seminearring under addition and multiplication defined by

$$(a)_\rho + (b)_\rho = (a+b)_\rho \quad \text{and} \quad (a)_\rho (b)_\rho = (ab)_\rho$$

for all $a,b \in S$.

A lattice A is said to be *modular* [3] if for any $x,y,z \in A$, $y \le x$, $x \wedge z = y \wedge z$ and $x \vee z = y \vee z$ implies $x = y$.

2. **Lemmas**

Before the characterizations of k-ideals of seminearrings for the main results, we give some auxiliary results which are necessary in what follows. The following lemma is easy to verify.

Lemma 2.1. *Let S be a seminearring and I a right(left) ideal of S. Then I is a subseminearring of S.*

Corollary 2.2. *Let S be a seminearring and I an ideal of S. Then I is a subseminearring of S.*

Lemma 2.3. *Let S be an additively commutative seminearring, and A and B two right ideals of S. Then $A+B$ is a right ideal of S.*

Proof. Let $x,y \in A+B$ and $r \in S$. Then $x = a_1 + b_1$ and $y = a_2 + b_2$ for some $a_1, a_2 \in A$ and $b_1, b_2 \in B$. Thus

$$x + y = (a_1 + b_1) + (a_2 + b_2) = (a_1 + a_2) + (b_1 + b_2) \in A+B.$$

Since A and B are right ideals of S, we have

$$xr = (a_1 + b_1)r = a_1 r + b_1 r \in A+B.$$

Hence $A+B$ is a right ideal of S. □

Lemma 2.4. *Let S be a seminearring and $\mathscr{X} = \{J \mid J$ is a right(left) ideal of $S\}$. Then $\bigcap_{J\in\mathscr{X}} J$ is a right(left) ideal of S where $\bigcap_{J\in\mathscr{X}} J \neq \emptyset$.*

Proof. Let $x,y \in \bigcap_{J\in\mathscr{X}} J$ and $r \in S$. Then $x,y \in J$ for all $J \in \mathscr{X}$, so $x+y, xr \in J$ for all $J \in \mathscr{X}$. Thus $x+y, xr \in \bigcap_{J\in\mathscr{X}} J$. Hence $\bigcap_{J\in\mathscr{X}} J$ is a right ideal of S. □

Corollary 2.5. *Let S be a seminearring and $\mathscr{X} = \{J \mid J$ is an ideal of $S\}$. Then $\bigcap_{J\in\mathscr{X}} J$ is an ideal of S where $\bigcap_{J\in\mathscr{X}} J \neq \emptyset$.*

Lemma 2.6. *Let S be a seminearring and $\mathscr{X} = \{J \mid J$ is a right(left) k-ideal of $S\}$. Then $\bigcap_{J\in\mathscr{X}} J$ is a right(left) k-ideal of S where $\bigcap_{J\in\mathscr{X}} J \neq \emptyset$.*

Proof. By Lemma 2.4, we have $\bigcap_{J\in\mathscr{X}} J$ is a right ideal of S. Let $x \in \bigcap_{J\in\mathscr{X}} J$ and $r \in S$ be such that $x + r \in \bigcap_{J\in\mathscr{X}} J$. Then $x, x + r \in J$ for all $J \in \mathscr{X}$, so $r \in J$ for all $J \in \mathscr{X}$. Thus $r \in \bigcap_{J\in\mathscr{X}} J$. Hence $\bigcap_{J\in\mathscr{X}} J$ is a right k-ideal of S. $\square$

Corollary 2.7. *Let S be a seminearring and $\mathscr{X} = \{J \mid J$ is a k-ideal of $S\}$. Then $\bigcap_{J\in\mathscr{X}} J$ is a k-ideal of S where $\bigcap_{J\in\mathscr{X}} J \neq \emptyset$.*

Lemma 2.8. *Let S be a seminearring and $\mathscr{X} = \{J \mid J$ is a right(left) full k-ideal of $S\}$. Then $\bigcap_{J\in\mathscr{X}} J$ is a right(left) full k-ideal of S.*

Proof. By Lemma 2.6, we have $\bigcap_{J\in\mathscr{X}} J$ is a right k-ideal of S. Since $E^+ \subseteq J$ for all $J \in \mathscr{X}$, we have $E^+ \subseteq \bigcap_{J\in\mathscr{X}} J$. Hence $\bigcap_{J\in\mathscr{X}} J$ is a right full k-ideal of S. $\square$

Corollary 2.9. *Let S be a seminearring and $\mathscr{X} = \{J \mid J$ is a full k-ideal of $S\}$. Then $\bigcap_{J\in\mathscr{X}} J$ is a full k-ideal of S.*

Lemma 2.10. *Let S be a seminearring, and A and B two right k-ideals of S. If $A \subseteq B$, then $\overline{A} \subseteq \overline{B}$.*

Proof. Let $a \in \overline{A}$. Then $a + x \in A$ for some $x \in A$. Thus $a + x \in A \subseteq B$ for some $x \in A \subseteq B$, so $a \in \overline{B}$. Hence $\overline{A} \subseteq \overline{B}$. $\square$

Lemma 2.11. *Let S be an additively regular seminearring in which addition is commutative. Then E^+ is a right ideal of S.*

Proof. Let $x, y \in E^+$ and $r \in S$. Then $x = x + x$ and $y = y + y$. Thus $(x + y) + (x + y) = (x + x) + (y + y) = x + y$ and $xr + xr = (x + x)r = xr$, so $x + y, xr \in E^+$. Hence E^+ is a right ideal of S. $\square$

Lemma 2.12. *For an additively inverse seminearring S, $I(S)$ is a partially ordered set under inclusion. Moreover, if $\mathscr{X} = \{J \mid J \in I(S)\}$, then $\bigcap_{J\in\mathscr{X}} J$ is an infimum of $\mathscr{X}$.*

Proof. By Lemma 2.8, we have $\bigcap_{J\in\mathscr{X}} J \in I(S)$. Since $\bigcap_{J\in\mathscr{X}} J \subseteq J$ for all $J \in \mathscr{X}$, we have $\bigcap_{J\in\mathscr{X}} J$ is a lower bound of $\mathscr{X}$. Let C be a lower bound of $\mathscr{X}$. Then $C \subseteq J$ for all $J \in \mathscr{X}$, so $C \subseteq \bigcap_{J\in\mathscr{X}} J$. Hence $\bigcap_{J\in\mathscr{X}} J$ is an infimum of $\mathscr{X}$. $\square$

Lemma 2.13. *Let S be an additively commutative seminearring. If $e, f \in E^+$ and $r \in S$, then $e + f, er \in E^+$.*

Proof. Now, $(e+f)+(e+f) = (e+e)+(f+f) = e+f$ and $er+er = (e+e)r = er$. Hence $e + f, er \in E^+$ $\square$

3. **Main Results**

In this section, we give some characterizations of k-ideals of seminearrings. Finally, we prove that the set of all right full k-ideals of an additively inverse seminearring in which addition is commutative forms a complete lattice which is also modular.

Theorem 3.1. *Let S be an additively inverse seminearring. Then every right(left) k-ideal of S is an additively inverse subseminearring of S.*

Proof. Let I be a right k-ideal of S. By Lemma 2.1, we have I is a subseminearring of S. Let arbitrary $a \in I$. Since S is an additively inverse seminearring, we obtain $a + a' + a = a$ and $a' + a + a' = a'$. Now, $a + (a' + a) = a + a' + a = a \in I$. Since I is a right k-ideal of S, we have $a' + a \in I$. Again, $a' \in I$. Therefore I is an additively inverse subseminearring of S. □

Corollary 3.2. *Let S be an additively inverse seminearring. Then every k-ideal of S is an additively inverse subseminearring of S.*

Theorem 3.3. *Let S be an additively inverse seminearring in which addition is commutative and A a right ideal of S. Then*

$$\overline{A} = \{a \in S \mid a + x \in A \text{ for all } x \in A\}$$

is a right k-ideal of S such that $A \subseteq \overline{A}$.

Proof. Let $a, b \in \overline{A}$ and $r \in S$. Then $a + x, b + y \in A$ for some $x, y \in A$. Since $(a + b) + (x + y) = a + x + b + y \in A$ and $x + y \in A$, we have $a + b \in \overline{A}$. Since $ar + xr = (a + x)r \in A$ and $xr \in A$, we have $ar \in \overline{A}$. Hence $\overline{A}$ is a right ideal of S. Let $d \in S$ and $c \in \overline{A}$ be such that $c + d \in \overline{A}$. Then there exist $x, y \in A$ such that $c + x \in A$ and $c + d + y \in A$. Thus $d + (c + x + y) = (c + d + y) + x \in A$. Since $c + x + y \in A$, we have $d \in \overline{A}$. Therefore $\overline{A}$ is a right k-ideal of S. Let $a \in A$. Then $(a + a') + a = a \in A$, so $a + a' \in \overline{A}$. Suppose that $a \notin \overline{A}$. Since $a + a' \in \overline{A}$, we get $a' \notin \overline{A}$. Since $a' + (a + a) = a + a' + a = a \in A$, we have $a' \in \overline{A}$ that is a contradiction. Hence $a \in \overline{A}$ and so $A \subseteq \overline{A}$. □

Corollary 3.4. *Let S be an additively inverse seminearring in which addition is commutative and A a right ideal of S. Then $\overline{A}$ is an additively inverse subseminearring of S such that $A \subseteq \overline{A}$.*

Corollary 3.5. *Let S be an additively inverse seminearring in which addition is commutative and A a right ideal of S. Then $\overline{A} = A$ if and only if A is a right k-ideal of S.*

Proof. Assume that $\overline{A} = A$. Then, by Lemma 3.3, we have $\overline{A}$ is a right k-ideal of S. Hence A is a right k-ideal of S.

Conversely, assume that A is a right k-ideal of S. Then, by Lemma 3.3, we have $A \subseteq \overline{A}$. Let $x \in \overline{A}$. Then $x + y \in A$ for some $y \in A$. Since A is a right k-ideal of S, we have $x \in A$. Thus $\overline{A} \subseteq A$, so $\overline{A} = A$. □

Lemma 3.6. *Let S be an additively inverse seminearring in which addition is commutative, and A and B two right full k-ideals of S. Then $\overline{A+B}$ is a right full k-ideal of S such that $A \subseteq \overline{A+B}$ and $B \subseteq \overline{A+B}$.*

Proof. By Lemma 2.3, we have $A+B$ is a right ideal of S. By Lemma 3.3, we have $\overline{A+B}$ is a right k-ideal of S such that $A+B \subseteq \overline{A+B}$. Since A and B are right full k-ideals of S, we have $E^+ \subseteq A$ and $E^+ \subseteq B$. Now, let $x \in E^+$. Then $x \in A$ and $x \in B$, so $x = x + x \in A+B$. Thus $E^+ \subseteq A+B \subseteq \overline{A+B}$. Hence $\overline{A+B}$ is a right full k-ideal of S. Let $a \in A$. Then $a = a + a' + a$. We can show that $a' + a \in E^+$. Thus

$$a = a + a' + a = a + (a' + a) \in A + E^+ \subseteq A + B \subseteq \overline{A+B}.$$

Hence $A \subseteq \overline{A+B}$. We can prove in a similar manner that $B \subseteq \overline{A+B}$. This completes the proof. □

Theorem 3.7. *For an additively inverse seminearring S in which addition is commutative, $I(S)$ is a complete lattice which is also modular.*

Proof. By Lemma 2.12, we have $I(S)$ is a partially ordered set under inclusion. Let $A, B \in I(S)$. By Lemma 2.8, we have $A \cap B \in I(S)$. By Lemma 3.6, we have $\overline{A+B} \in I(S)$. Define

$$A \wedge B = A \cap B \quad \text{and} \quad A \vee B = \overline{A+B}.$$

Since $A \wedge B = A \cap B \subseteq A$ and $A \wedge B = A \cap B \subseteq B$, we have $A \wedge B$ is a lower bound of A and B. Let $C \in I(S)$ be such that $C \subseteq A$ and $C \subseteq B$. Then $C \subseteq A \cap B = A \wedge B$, so $A \cap B$ is an infimum of A and B. Since $A \vee B = \overline{A+B}$ and by Lemma 3.6, we have $A \subseteq \overline{A+B} = A \vee B$ and $B \subseteq \overline{A+B} = A \vee B$. Thus $\overline{A+B}$ is an upper bound of A and B. Let $D \in I(S)$ be such that $A \subseteq D$ and $B \subseteq D$. Then $A+B \subseteq D$. By Lemma 2.10, we have $\overline{A+B} \subseteq \overline{D}$. By Corollary 3.5, we have $\overline{D} = D$ and so $\overline{A+B} \subseteq D$. Thus $\overline{A+B}$ is a supremum of A and B. Hence $I(S)$ is a lattice. We shall show that $I(S)$ is a modular lattice. Let $A, B, C \in I(S)$ be such that $A \wedge B = A \wedge C$ and $A \vee B = A \vee C$ and $B \subseteq C$. Now, let $x \in C$. Then $x \in A \vee C = A \vee B = \overline{A+B}$. Thus there exists $a + b \in A+B$ such that $x + a + b \in A+B$, so $x + a + b = a_1 + b_1$ for some $a_1 \in A$ and $b_1 \in B$. This implies that $x + a + a' + b = x + a + b + a' = a_1 + b_1 + a'$. Since $x \in C, a + a' \in C$ and $b \in B \subseteq C$, we have $a_1 + b_1 + a' \in C$ but $b_1 \in C$. Thus $a_1 + a' \in C$. By Lemma 3.1, we have $a_1 + a' \in A$ and so $a_1 + a' \in A \cap C = A \cap B$. Thus $a_1 + a' \in B$. Since $x + a + b = a_1 + b_1$, we have $x + a + a' + b = a_1 + a' + b_1 \in B$. Since $(a + a') + b \in B$ and B is a right k-ideal of S, we have $x \in B$ and so $C \subseteq B$. Thus $B = C$. Therefore $I(S)$ is a modular lattice. By Lemma 2.12, we get that $I(S)$ is complete. □

In comparison our above results with results of k-ideals of semirings, we see that the set of all right full k-ideals of an additively inverse seminearring in which addition is commutative forms a complete lattice which is also modular which is an analogous result of full k-ideals of semirings.

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