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Mobius Graphs

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Abstract. The study of graphs on natural numbers as its vertex set and with adjacency defined using tools of number theoretic functions is interesting and may focus new light on structure of the number systems.

In this paper, we have studied the structure of finite graphs whose vertices are labeled with natural numbers and the adjacency is defined in terms of the well known Mobius function.

1. Introduction

The Mobius function in [1] is a well known function in Number theory and it is defined by

$\mu(1) = 1$ and

if $n > 1$ write $n = p_1^{a_1}, p_2^{a_2} \ldots p_r^{a_r}$ then

$$\mu(n) = \begin{cases} (-1)^r, & \text{if } a_1 = a_2 = \cdots = a_r = 1 \\ 0, & \text{otherwise.} \end{cases}$$

We have considered three types of graphs in this paper by defining the adjacency of two vertices labeled as a and b in the following manner:

(1) $\mu(ab) = 0$;
(2) $\mu(ab) = 1$;
(3) $\mu(ab) = -1$.

Several interesting properties of these graphs on numbers have been obtained.

The notations that are used in this paper are same as in [1] and [2].

2. Mobius graph

Using the Mobius function we have defined a finite graph on the first m natural numbers as its vertex set and two distinct vertices a, b are adjacent if and only if $\mu(ab) = 0$. This graph is called a Mobius graph and is denoted by M_0.

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Theorem 1. *If u $(\neq 1)$ is any vertex in the Mobius graph M_0 then*

$$\deg(u) = \begin{cases} m-1\,, & \textit{if } u \textit{ has a square factor} \\ (m-1) - \sum_{d|u} \mu(d) \left[\dfrac{m}{d}\right] + \sum\limits_{\substack{v=2 \\ (u,v)=1 \\ \mu(v)=0}}^{m} 1\,, & \textit{otherwise}\,. \end{cases}$$

Proof. If u has a square factor then for any vertex v other than u in the set of vertices $\{1, 2, \ldots, m\}$, uv also has a square factor. This implies $\mu(uv) = 0$. That is u is adjacent with all the other $m-1$ vertices. Therefore $\deg(u) = m-1$.

If u does not have a square factor then u is adjacent with a vertex v if (i) $(u, v) = d > 1$, (ii) $(u, v) = 1$ and v has a square factor. The number of such vertices is $(m-1)$ — the vertices in the set $\{1, 2, \ldots, m\}$ which are relatively prime to $u+$ the number of vertices for which $(u, v) = 1$ and v has a square factor. That is

$$\deg(u) = (m-1) - \sum_{d|u} \mu(d) \left[\frac{m}{d}\right] + \sum_{\substack{v=2 \\ (u,v)=1 \\ \mu(v)=0}}^{m} 1\,. \qquad \square$$

Remark. If $u = 1$ then u is adjacent to a vertex v in $\{2, 3, \ldots, m\}$ which has a square factor. The number of such vertices in M_0 is $\sum_{\substack{v=2 \\ \mu(v)=0}}^{m} 1$. That is $\deg(1) = \sum_{\substack{v=2 \\ \mu(v)=0}}^{m} 1$.

Theorem 2. *M_0 is a connected graph if and only if $m \geq 4$.*

Proof. If $m < 4$, M_0 is an empty graph, since $\mu(ab) \neq 0$ for any $a, b \in \{1, 2, 3\}$.

Conversely, if $m \geq 4$ the vertex 4 is adjacent to every vertex of the graph and hence M_0 is connected. $\square$

Note. Let $\pi(m)$ denote the number of primes $\leq m$.

Theorem 3. *The independence number of M_0 is $\pi(m) + 1$.*

Proof. The set of all primes together with 1 forms an independent set in M_0 as they are relatively prime to each other and hence no two of them are adjacent. Further this is the maximum independent set in M_0. For if v is any composite number included in this set then $\mu(vp) = 0$, where p is any prime factor of v and so the set is no more an independent set. If we form an independent set in any other way be including some composite numbers in such a way that μ of their product is not zero. Then clearly such a set is either a subset of this or of smaller size. Hence the independence number of M_0 is $\pi(m) + 1$. $\square$

Theorem 4. *The set of all square numbers together with 1 forms a clique in M_0.*

Proof. If u, v are any two square numbers then $\mu(uv) = 0$ and hence u and v are adjacent. Also 1 is adjacent with all these square numbers. Hence the set of all square numbers together with 1 forms a clique in M_0. $\square$

Theorem 5. *The set of all even numbers together with the set of all odd square numbers forms a maximum clique in M_0.*

Proof. The set of all even numbers forms a clique. For, if u, v are any two even numbers then $4|uv$ and hence $\mu(uv) = 0$, and therefore u, v are adjacent. The set of all odd square numbers is again a clique in M_0. Also since the set of all even numbers together with the set of all odd square numbers forms a clique in M_0. Further it can be easily seen that this is the maximum clique in M_0. □

3. **Positive Mobius graph**

Suppose G is a graph with its vertex set as $\{1, 2, \ldots, m\}$ where the two vertices a, b are adjacent if and only if $\mu(ab) = 1$. This graph is called a positive Mobius graph and is denoted by M_1.

We now establish an equivalent condition for the adjacency of a pair of vertices in the Positive Mobius graph.

Theorem 6. *Two vertices u, v are adjacent if and only if $v(u)$, $v(v)$ have the same parity, u, v being square free and $(u, v) = 1$ (where $v(m)$ denotes the number of distinct prime factors of m).*

Proof. (i) If u, v are adjacent then $\mu(uv) = 1$. So uv is a product of even number of distinct prime factors, which means $v(uv) =$ even, which implies if $v(u)$ is odd then $v(v)$ is also odd and if $v(u)$ is even then $v(v)$ must also be even. That is $v(u)$, $v(v)$ have the same parity.

(ii) u, v must also be square free, for if, any of them is not square free uv is also not square free which means $\mu(uv) = 0$, a contradiction.

(iii) If $(u, v) = d$ then $d^2|uv$, i.e. $\mu(uv) = 0$, contrary to hypothesis.

Hence $(u, v) = 1$.

Conversely, let $u = p_1 p_2 \ldots p_r$ and $v = q_1 q_2 \ldots q_s$ where $r + s$ is even and p's are distinct from q's.

Then

$$\begin{aligned}
\mu(uv) &= \mu(p_1 p_2 \ldots p_r q_1 q_2 \ldots q_s) \\
&= \prod_{i=1}^{r} \mu(p_i) \prod_{j=1}^{s} (q_j) \\
&= (1)^{r+s} \\
&= 1, \quad \text{since } r + s \text{ is even.}
\end{aligned}$$

Therefore u, v are adjacent. □

Remark. The set of all square numbers in the Positive Mobius graph are of degree zero.

Theorem 7. *The set of all primes forms a clique.*

Proof. If u, v are any two primes then $v(u) = u(v) = 1$ and $(u, v) = 1$ and also that u, v are square free. Therefore $\mu(uv) = (-1)^{1+1} = 1$ and hence u and v are adjacent. That is the set of all primes forms a clique. □

Remark. This is the maximum clique on $\pi(m)$ vertices.

Theorem 8. *The set of all even numbers together with the set of all odd square numbers forms a maximum independent set in M_1.*

Proof. The set of all even numbers $S_1 = \{2, 4, 6, \ldots\}$ is an independent set in M_1 and so also the set of all odd square numbers $S_2 = \{9, 18, 25, \ldots\}$. Now we claim that $S_1 \cup S_2$ forms an independent set in M_1.

For if $u, v \in S_1 \cup S_2$ then $u, v \in S_1$ or S_2.

Now

$$\begin{aligned}
u, v \in S_1 &\Rightarrow \mu(uv) = 0 \\
u \in S_1, v \in S_2 &\Rightarrow \mu(v) = 0 \Rightarrow \mu(uv) = 0 \\
u \in S_2, v \in S_1 &\Rightarrow \mu(u) = 0 \Rightarrow \mu(uv) = 0 \\
u, v \in S_2 &\Rightarrow \mu(uv) = 0
\end{aligned}$$

In all the above cases u, v are not adjacent. Hence $S_1 \cup S_2$ forms an independent set and it is obvious that it is a maximum independent set in M_1. □

4. Negative Mobius graph

Suppose G is a graph with its vertex set as $\{1, 2, \ldots, m\}$ where two vertices a, b are adjacent if an only if $\mu(ab) = -1$. This graph is called a Negative Mobius graph and is denoted by M_{-1}.

The following result establishes an equivalent condition for the adjacency of two vertices in a Negative Mobius graph M_{-1}.

Theorem 9. *u, v are adjacent if and only if $\nu(u)$, $\nu(v)$ have different party, $(u, v) = 1$ and also that u, v are square free.*

Proof. If u, v are adjacent then $\mu(uv) = -1$, which means uv is a product of odd number of distinct primes. That is if $\nu(u)$ is odd then $\nu(v)$ is even and vice-versa. Also u, v are square free and $(u, v) = 1$.

Conversely, if $\nu(u)$ and $\nu(v)$ are of different parity and if u, v are square free with $(u, v) = 1$ then uv is a product of an odd number of distinct primes and hence $\mu(uv) = -1$ so that u, v are adjacent. □

Remark. All square numbers are of degree zero.

Lemma 10. *M_{-1} does not contain a cycle if $m \leq 6$.*

Proof. If $m \geq 7$, then the vertices 1, 5, 6, 7 forms a cycle. Hence the lemma. □

Theorem 11. *M_{-1} has no triangles.*

Proof. If u, v, w are any three vertices in M_{-1} then we have to show that if uv, vw are edges in M_{-1} then uw is not an edge. One of the following holds good.

Either $(u,w)=1$ or $(u,w)\neq 1$.

(i) $(u,w)=1$.

By hypothesis $(u,v)=(v,w)=1$ and

$$\mu(uv)=\mu(vw)=-1\,.$$

Now

$$\mu(uv)=\mu(u)\cdot\mu(v)=-1 \tag{1}$$

and

$$\mu(vw)=\mu(v)\cdot\mu(w)=-1\,. \tag{2}$$

Form (1) and (2), it is clear that if $\mu(v)=1$ then $\mu(u)$ and $\mu(w)$ are both 1 and if $\mu(v)=-1$ then $\mu(u)$ and $\mu(w)$ are both 1. That is u,w have the same parity and hence u,w are not adjacent in M_{-1}.

(ii) $(u,w)\neq 1$
$\Rightarrow$ $(u,w)=d$
$\Rightarrow$ $d^2|uw$
$\Rightarrow$ $\mu(uw)=0$
$\Rightarrow$ u,w are not adjacent.

So M_{-1} has no triangles. □

Theorem 12. $M_1\cup M_{-1}=\overline{M}_0$.

Proof. M_1 contains the set of vertices for which $\mu(uv)=1$. M_{-1} contains the set of vertices for which $\mu(uv)=-1$. $M_1\cup M_{-1}$ contains the set of vertices for which $\mu(uv)=\pm 1$. In M_0, u, v are adjacent if $\mu(uv)=0$.

The properties $\mu(uv)=0$ and $\mu(uv)=\pm 1$ are complementary. Hence $M_1\cup M_{-1}$ and M_0 are complementary graphs. □

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References

[1] T. M. Apostol, *Introduction to Analytic Number Theory*, Springer International Student Edition, 1980.

[2] J. A. Bondy and U. S. R. Murty, *Graph Theory with Applications*, The Macmillan Press Ltd., 1976.

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