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Extensions of Steffensen's Inequality

W.T. Sulaiman

Abstract. Extension and new inequalities concerning Steffensen's inequality are presented.

1. Introduction

Steffensen's inequality reads as follows:

Theorem 1.1. *Assume that two integrable functions $f(t)$ and $g(t)$ are defined on the interval (a,b), that $f(t)$ non-increasing and that $0 \le g(t) \le 1$ in (a,b). Then*

$$\int_{a-\lambda}^{b} f(t)dt \le \int_a^b f(t)g(t)dt \le \int_a^{a+\lambda} f(t)dt, \tag{1}$$

where $\lambda = \int_a^b g(t)dt$.

Prcaric [6], however, through some modification, gives the following modification

Theorem 1.2. *Let $f : [0,1] \to \mathfrak{R}$ be nonnegative and non-increasing function and let $g : [0,1] \to \mathfrak{R}$ be an integrable function such that $0 \le g(t) \le 1$ for all $t \in [0,1]$. If $p \ge 1$, then*

$$\left(\int_0^1 f(t)g(t)dt\right)^p \le \int_0^{\lambda} f^p(t)dt, \tag{2}$$

where $\lambda = \left(\int_0^1 g(t)dt\right)^p$.
A mapping $\phi : \mathfrak{R} \to \mathfrak{R}$ is said to be convex on $[a,b]$ if

$$\phi(tx + (1-t)y) \le t\phi(x) + (1-t)\phi(y), \quad x, y \in [a,b], 0 \le t \le 1. \tag{3}$$

If (3) *reverses, then ϕ is called concave.*

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The aim of this paper is to give a generalization of Theorem 1.2, as well as other results including a reverse of Steffensen's inequality.

The following Lemma is needed

Lemma 1.3. *The mapping $\phi(x) = x^p$ is convex for $p \geq 1$, and concave for $0 \leq p \leq 1$.*

Proof. As

$$\phi''(x) = p(p-1)x^{p-2} \geq 0,$$

then ϕ is convex. Also for $0 < p \leq 1$, the concavity of ϕ follows from the inequality $\phi''(x) \leq 0$. □

2. Results

The following gives a generalization of Theorem 1.2.

Theorem 2.1. *Let $f, g, \phi \geq 0$, $0 \leq g \leq 1$, $p \geq 1$, $\phi(\lambda^{1/p}) \leq \lambda$, where $\lambda = \left(\int_0^1 g(t)dt\right)^p$, f is non-increasing. Then*

$$\int_0^\lambda \phi \circ f(x)dx \geq \lambda^{-1/p}\phi(\lambda^{1/p}) \int_0^1 \phi \circ f(x)g(x)dx\,. \tag{4}$$

Proof. As $\phi \geq 0$ and f is non-increasing, then $\phi \circ f$ is non-increasing. Also,

$$\lambda^{-1/p}\phi(\lambda^{1/p})g(x) \leq \lambda^{-1/p}\phi(\lambda^{1/p}) \leq \lambda^{1-1/p} \leq 1,$$

then

$$\int_0^\lambda \phi \circ f(x)dx - \lambda^{-1/p}\phi(\lambda^{1/p}) \int_0^1 \phi \circ f(x)g(x)dx$$

$$= \int_0^\lambda \phi \circ f(x)dx - (\lambda^{-1/p})\left(\int_0^\lambda + \int_\lambda^1\right)\phi \circ f(x)g(x)dx$$

$$= \int_0^\lambda \phi \circ f(x)(1 - \lambda^{-1/p}\phi(\lambda^{1/p})g(x))dx - \lambda^{-1/p}\phi(\lambda^{1/p}) \int_0^\lambda \phi \circ f(x)g(x)dx$$

$$\geq \phi \circ f(\lambda)\left(\int_0^\lambda (1 - \lambda^{-1/p}\phi(\lambda^{1/p})g(x))dx - \lambda^{-1/p}\phi(\lambda^{1/p}) \int_\lambda^1 g(x)dx\right)$$

$$= \phi \circ f(\lambda)\left(\lambda - \lambda^{-1/p}\phi(\lambda^{1/p})\left(\int_0^\lambda + \int_\lambda^1\right)g(x)dx\right)$$

$$= \phi \circ f(\lambda)\left(\lambda - \lambda^{-1/p}\phi(\lambda^{1/p}) \int_0^1 g(x)dx\right)$$

$$= \phi \circ f(\lambda)(\lambda - \phi(\lambda^{1/p})) \geq 0.$$ □

Remark 1. If we are putting $\phi(x) = x^p$, $p \geq 1$, we obtain the inequality (2) as follows

$$\int_0^\lambda f^p(x)dx \geq \left(\int_0^1 g(x)dx\right)^{p-1} \int_0^1 f^p(x)g(x)dx$$
$$\geq \left(\int_0^1 f(x)g(x)dx\right)^p.$$

The following result is dealing with Steffensen's inequality for $p > 0$.

Theorem 2.2. *Let $f, g, \phi \geq 0$, $0 \leq g \leq 1$, f is non-increasing. $\varphi(p) > 0$, $\lambda^{\varphi(p)} g \leq 1$, $\int_0^1 g(x)dx \leq \lambda^{1-\varphi(p)}$. Then*

$$\int_0^\lambda \phi \circ f(x)dx \geq \lambda^{\varphi(p)} \int_0^1 \phi \circ f(x)g(x)dx\,. \tag{5}$$

Proof. As $\phi \geq 0$ and f is non-increasing, then $\phi \circ f$ is non-increasing. Also,

$$\lambda^{\varphi(p)} g \leq 1 \quad \Rightarrow \quad \lambda^{\varphi(p)} \int_0^1 g(x)dx \leq 1$$
$$\Rightarrow \quad \lambda \leq 1\,.$$

Then, we have

$$= \int_0^\lambda \phi \circ f(x)dx - \lambda^{\varphi/p}\left(\int_0^\lambda + \int_\lambda^1\right)\phi \circ f(x)g(x)dx$$
$$= \int_0^\lambda \phi \circ f(x)(1 - \lambda^{\varphi(p)} g(x))dx - \lambda^{-1/p}\phi(\lambda^{1/p}) \int_\lambda^1 \phi \circ f(x)g(x)dx$$
$$\geq \phi \circ f(\lambda)\left(\int_0^\lambda (1 - \lambda^{\varphi(p)} g(x))dx - \lambda^{-1/p}\phi(\lambda^{1/p}) \int_\lambda^1 g(x)dx\right)$$
$$= \phi \circ f(\lambda)\left(\lambda - \lambda^{\varphi(p)}\left(\int_0^\lambda + \int_\lambda^1\right)g(x)dx\right)$$
$$= \phi \circ f(\lambda)\left(\lambda - \lambda^{\varphi/p} \int_0^1 g(x)dx\right)$$
$$= \phi \circ f(\lambda)(\lambda - \lambda) \geq 0.$$

□

Corollary 2.3. *Let $f, g, \phi \geq 0$, $\lambda^{1/p-1} g \leq 1$, f is non-increasing, $p > 0$, where $\lambda = \left(\int_0^1 g(t)dt\right)^{\frac{p}{2p-1}}$. Then*

$$\int_0^\lambda \phi \circ f(x)dx \geq \lambda^{1/p-1} \int_0^1 \phi \circ f(x)g(x)dx. \tag{6}$$

Proof. The proof follows from Theorem 2.2, by putting $\varphi(p) = 1/p - 1$, $0 < p < 1$. □

The following gives another extension of Theorem 2.1

Theorem 2.4. *Let $f, g, h, \phi \geq 0$, $0 \leq g \leq 1$, f is non-increasing, $p \geq 1$, $\phi(\lambda^{1/p}) \leq \int_0^\lambda h(x)dx$, $\lambda^{-1/p}\phi(\lambda^{1/p}) \leq h$, where $\lambda = \left(\int_0^1 g(t)dt\right)^p$. Then*

$$\int_0^\lambda \phi \circ f(x)h(x)dx - \lambda^{-1/p}\phi(\lambda^{1/p})\int_0^1 \phi \circ f(x)g(x)dx \tag{7}$$

Proof. As before, $\phi \circ f$ is non-increasing. Therefore

$$\int_0^\lambda \phi \circ f(x)h(x)dx - \lambda^{-1/p}\phi(\lambda^{1/p})\int_0^1 \phi \circ f(x)g(x)dx$$

$$= \int_0^\lambda \phi \circ f(x)h(x)dx - \lambda^{-1/p}\phi(\lambda^{1/p})\left(\int_0^\lambda + \int_\lambda^1\right)\phi \circ f(x)g(x)dx$$

$$= \int_0^\lambda \phi \circ f(x)(h(x) - \lambda^{-1(p)}\phi(\lambda^{1/p})g(x))dx$$

$$- \lambda^{-1/p}\phi(\lambda^{1/p})\int_\lambda^1 \phi \circ f(x)g(x)dx$$

$$= \phi \circ f(\lambda)\left(\int_0^\lambda (h(x) - \lambda^{-1/p}\phi(\lambda^{1/p})g(x))dx - \lambda^{-1/p}\phi(\lambda^{1/p})\int_\lambda^1 g(x)dx\right)$$

$$= \phi \circ f(\lambda)\left(\int_0^\lambda h(x)dx - \lambda^{-1(p)}\phi(\lambda^{1/p})\left(\int_0^\lambda + \int_\lambda^1\right)g(x)dx\right)$$

$$= \phi \circ f(\lambda)\left(\int_0^\lambda h(x)dx - \phi(\lambda^{1/p})\right) \geq 0.$$ □

The following gives a reverse inequality

Theorem 2.5. *Let $f, g, \phi \geq 0$, ϕ is concave with $\phi(0) = 0$, f is non-decreasing, $0 \leq g \leq 1$, $p \geq 1$ and $\lambda = \left(\int_0^1 g(t)dt\right)^p$. Then*

$$\int_0^\lambda \phi \circ f(x)dx \leq \phi\left(\lambda^{1-1/p}\int_0^1 f(x)g(x)dx\right). \tag{8}$$

Proof.

$$\int_0^\lambda \phi \circ f(x)dx - \phi\left(\lambda^{1-1/p}\int_0^1 f(X)g(x)dx\right)$$

$$
\begin{aligned}
&= \int_0^\lambda \phi \circ f(x)dx - \phi\left(\lambda \frac{1}{\lambda^{1/p}} \int_0^1 f(x)g(x)dx\right) \\
&\le \int_0^\lambda \phi \circ f(x)dx - \lambda\phi\left(\frac{1}{\lambda^{1/p}} \int_0^1 f(x)g(x)dx\right) \\
&\qquad\qquad\qquad\qquad \text{(as } \phi \text{ is concave with } \phi(0)=0) \\
&\le \int_0^\lambda \phi \circ f(x)dx - \lambda^{1-1/p}\left(\int_0^f \phi \circ f(x)g(x)dx\right) \\
&\qquad\qquad\qquad\qquad \text{(by Jensen's inequality)} \\
&= \int_0^\lambda \phi \circ f(x)dx - \lambda^{1-1/p}\left(\int_0^\lambda + \int_\lambda^1\right)\phi \circ f(x)g(X)dx \\
&= \int_0^\lambda \phi \circ f(x)(1-\lambda^{1-1/p}g(x))dx - \lambda^{1-1/p}\int_\lambda^1 \phi \circ f(x)g(x)dx \\
&\le \phi \circ f(\lambda)\left(\int_0^\lambda (1-\lambda^{1-1/p}g(x))dx - \lambda^{1-1/p}\int_\lambda^1 g(x)dx\right) \\
&= \phi \circ f(\lambda)\left(\lambda - \lambda^{1-1/p}\left(\int_0^\lambda + \int_\lambda^1\right)g(x)dx\right) \\
&= \phi \circ f(\lambda)\left(\lambda - \lambda^{1-1/p}\int_0^1 g(x)dx\right) \\
&= \phi \circ f(\lambda)(\lambda - \lambda) = 0. \qquad \square
\end{aligned}
$$

Corollary 2.6. *Let $f, g \ge 0$, f is non-decreasing, $0 \le g \le 1$, $p \ge 1$, $0 < q < 1$, and $\lambda = \left(\int_0^1 g(t)dt\right)^p$. Then*

$$\int_0^\lambda f^q(x)dx \le \lambda^{q-q/p}\left(\int_0^1 f(x)g(x)dx\right)^q. \tag{9}$$

Proof. The proof follows from Theorem 2.5 by pitting $\phi(x) = x^q$, $0 < q < 1$. □

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W.T. Sulaiman, *Department of Computer Engineering, College of Engineering, University of Mosul, Iraq*.
E-mail: `waadsulaiman@hotmail.com`

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