Journal of Informatics and Mathematical Sciences
Vol. 10, No. 4, pp. 599–611, 2018
ISSN 0975-5748 (online); 0974-875X (print)
Published by RGN Publications
http://www.rgnpublications.com
DOI: 10.26713/jims.v10i4.870

Research Article

Gracefulness of Some Arbitrary Supersubdivision of Cycle

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Abstract. A labeling of a graph is a mapping that carries some set of graph elements into numbers (usually positive integers). Graceful labeling of a graph with q edges is an injection from the set of its vertices to the sequence $\{0,1,2,\dots,q\}$ such that the values of edges are all integers from 1 to q, the value of an edge being absolute value of the difference between the integers attributed to its end vertices. In 2009, Sethuraman [8] posed a problem of labeling of arbitrary supersubdivision of graph. Fact that all cycles are not graceful in general, it was intresting to study same for arbitrary supersubdivisions of cycles. This inspiration lead us to some good results. In this paper, we prove that Even arbitrary supersubdivision of cycle is graceful. We also prove that Odd arbitrary supersubdivision of cycle C_n, for n even is graceful and for n odd is not graceful.

Keywords. Cycle; Subdivision of graphs; Supersubdivision of graphs; Arbitrary supersubdivision of a graph; Graceful labeling

MSC. 05C78

Received: March 24, 2018 **Accepted:** September 9, 2018

1. Introduction

We consider finite undirected graphs without loops and multiple edges. The class of graceful labeling was first introduced by A. Rosa, 1967 which he had named as β-valuation [7]. The name graceful labeling was later given by Solomon Golomb. A lot of research has been done on graceful graphs. Kathiresan [4] has proved that subdivision of ladder are graceful. Sethuraman and Selvaraju [8] have introduced supersubdivision of graphs and proved that there exists a graceful arbitrary supersubdivision of C_n, $n \geq 3$ with certain conditions. Later, in 2004, Kathiresan [3] also proved that arbitrary supersubdivision of stars are graceful. Ramchandran and Sekar [6] have given graceful labeling of supersubdivision of ladder.

First we understand few basic definitions which we need throughout the discussion. We also have to decide certain notations for various graphs for better reference. Then we proceed for main section of results which are already mentioned in abstract.

2. Definitions

Definition 2.1 (Cycle)**.** Cycle is a graph with an equal number of vertices and edges where vertices can be placed around circle so that two vertices are adjacent if and only if they appear consecutively along the circle. The cycle is denoted by C_n.

Definition 2.2 (Subdivision of a Graph)**.** Let G be a graph with p vertices and q edges. A graph H is said to be a subdivision of G if H is obtained by subdividing every edge of G exactly once. H is denoted by $S(G)$. Thus, $|V| = p + q$ and $|E| = 2q$.

Definition 2.3 (Supersubdivision of a Graph)**.** Let G be a graph with p vertices and q edges. A graph H is said to be a supersubdivision of G if it is obtained from G by replacing every edge e of G by a complete bipartite graph $K_{2,m}$. H is denoted by $SS(G)$. Thus, $|V| = p + mq$ and $|E| = 2mq$.

Definition 2.4 (Arbitrary Supersubdivision of a Graph)**.** Let G be a graph with p vertices and q edges. A graph H is said to be a arbitrary supersubdivision of G if it is obtained from G by replacing every edge e_i of G by a complete bipartite graph K_{2,m_i}, $i = 1,2,\dots,q$. H is denoted by $ASS(G)$. Thus, $|V| = p + \sum_{i=1}^{q} m_i$ and $|E| = \sum_{i=1}^{q} 2m_i$.

Definition 2.5 (Even Arbitrary Supersubdivision of a Graph)**.** Let G be a graph with p vertices and q edges. A graph H is said to be a even arbitrary supersubdivision of G if it is obtained from G by replacing every edge e_i of G by a complete bipartite graph K_{2,m_i}, where m_i is even, $i = 1,2,\dots,q$. H is denoted by $EASS(G)$. Thus, $|V| = p + \sum_{i=1}^{q} m_i$ and $|E| = \sum_{i=1}^{q} 2m_i$.

Definition 2.6 (Odd Arbitrary Supersubdivision of a Graph)**.** Let G be a graph with p vertices and q edges. A graph H is said to be a odd arbitrary supersubdivision of G if it is obtained from G by replacing every edge e_i of G by a complete bipartite graph K_{2,m_i}, where m_i is odd, $i = 1,2,\dots,q$. H is denoted by $OASS(G)$. Thus, $|V| = p + \sum_{i=1}^{q} m_i$ and $|E| = \sum_{i=1}^{q} 2m_i$.

Definition 2.7 (Graceful Labelling)**.** Let G be a graph with q edges. A graceful labeling of G is an injection from the set of its vertices to the sequence $\{0,1,2,\dots,q\}$ such that the values of edges are all integers from 1 to q, the value of an edge being the absolute value of the difference between the integers attributed to its end vertices.

3. Notations

In the complete bipartite graph $K_{2,m}$, we call the part consisting of two vertices, the 2-vertices part of $K_{2,m}$ and the part consisting of m vertices, the m-vertices part of $K_{2,m}$.

Let C_n be a cycle of length n. Let $c_1, c_2, \dots, c_n$ be the vertices of cycle. Let $c^k_{i,i+1}$, $k = 1,2,\dots,m_i$ be the vertices of the m_i-vertices part of K_{2,m_i} merged with the edge c_ic_{i+1} for $i = 1,2,\dots,n-1$ and $k = 1,2,\dots,m_i$. Let $c^k_{n,1}$, $k = 1,2,\dots,m_n$ be the vertices of the m_n-vertices part of K_{2,m_n} merged with the edge c_nc_1 and $k = 1,2,\dots,m_n$.

$$|V| = n + \sum_{i=1}^{n} m_i = n + m\,,$$

$$|E| = 2m_1 + 2m_2 + \dots + 2m_n = 2\left(\sum_{i=1}^{n} m_i\right) = 2m, \quad \text{where } \sum_{i=1}^{n} m_i = m\,.$$

For example, if $n = 7$ and $m_1 = 5$, $m_2 = 4$, $m_3 = 6$, $m_4 = 2$, $m_5 = 5$, $m_6 = 4$, $m_7 = 3$ then vertex labeling is as follows:

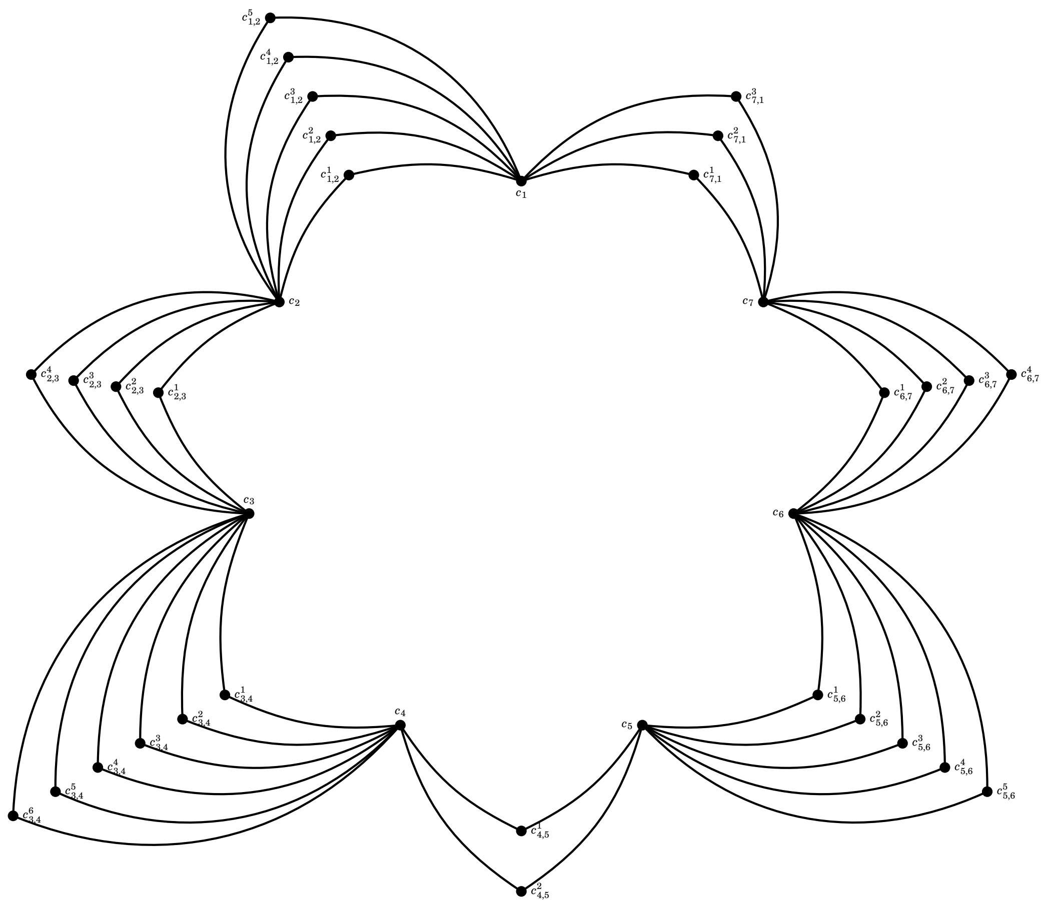

Figure 1. Graph with $n = 7$ with general vertex labels

4. Main Results

Theorem 4.1. *Even Arbitrary supersubdivision of C_n, i.e. EASS(C_n) is graceful.*

Proof. Define the following labeling $f : V \to \{0,1,2,\dots,2m\}$

$$f(c_1) = 0.$$

Case (i): If n is even.

$$f(c_r) = 2(r-1) \qquad \text{for } r = 2,3,\dots,\frac{n}{2}$$

$$f(c_{n-r}) = 1+2r \qquad \text{for } r = 0,1,2,\dots,\frac{n}{2}-2$$

$$f\left(c_{\frac{n}{2}+1}\right) = f\left(c_{\frac{n}{2}}\right)+1$$

Case (ii): If n is odd.

$$f(c_r) = 2(r-1) \qquad \text{for } r = 2,3,\dots,\frac{n+1}{2}$$

$$f(c_{n-r}) = 1+2r \qquad \text{for } r = 0,1,2,\dots,\frac{n-3}{2}$$

$$f\left(c_{n,1}^k\right) = 2m-2(k-1) \qquad k = 1,2,\dots,m_n.$$

$$\alpha_1 = f(c_1)+2m-2m_n$$

If n is even, for $i = 2,\dots,\frac{n-2}{2}$ and if n is odd, for $i = 2,\dots,\frac{n-1}{2}$

$$\alpha_i = f(c_i)+2m-2\left(\sum_{r=1}^{i-1} m_r\right)-2\left(\sum_{r=0}^{i-1} m_{n-r}\right).$$

For $k = 1,3,5,\dots,m_i-1$,

$$f\left(c_{i,i+1}^k\right) = \alpha_i - 4(l-1) = \alpha_i + 4 - 4l, \quad l = 1,2,\dots,\frac{m_i}{2}.$$

For $k = 2,4,6,\dots,m_i$,

$$f\left(c_{i,i+1}^k\right) = \alpha_i - 1 - 4(l-1) = \alpha_i + 3 - 4l, \quad l = 1,2,\dots,\frac{m_i}{2}.$$

If n is even, for $r = 1,2,\dots,\frac{n}{2}-1$ and if n is odd, for $r = 1,2,\dots,\frac{n-3}{2}$

$$\beta_r = f(c_{n-r+1})+2m-2\left(\sum_{i=0}^{r-1} m_{n-i}\right)-2\left(\sum_{i=1}^{r} m_i\right).$$

For $k = 1,3,5,\dots,m_{n-r}-1$,

$$f\left(c_{n-r,n-r+1}^k\right) = \beta_r - 4(l-1) = \beta_r + 4 - 4l, \quad l = 1,2,\dots,\frac{m_{n-r}}{2}.$$

For $k = 2,4,6,\dots,m_{n-r}$,

$$f\left(c_{n-r,n-r+1}^k\right) = \beta_r - 1 - 4(l-1) = \beta_r + 3 - 4l, \quad l = 1,2,\dots,\frac{m_{n-r}}{2}.$$

For n even,

$$f\left(c_{\frac{n}{2},\frac{n}{2}+1}^k\right) = f\left(c_{\frac{n}{2}}\right)+2k, \quad \text{where } k = 1,2,\dots,m_{\frac{n}{2}}.$$

For n odd,

$$f\left(c_{\frac{n+1}{2},\frac{n+3}{2}}^k\right) = f\left(c_{\frac{n+3}{2}}\right)+2k, \quad \text{where } k = 1,2,\dots,m_{\frac{n+1}{2}}.$$

From Table 1 and Table 2, labels of vertices, of m_r-part of K_{2,m_r}, where $r = 1$ to $\frac{n}{2}-1$, for n even and $r = 1$ to $\frac{n-1}{2}$, for n odd, are α_r to $\alpha_r - 2m_r + 3$ following arithmetic progressions A_l and B_l alternatively where

$$A_l = \alpha_r - 4(l-1) \text{ and } B_l = (\alpha_r - 1) - 4(l-1) \quad \text{for } l = 1,2,\dots,\frac{m_r}{2}$$

as

$$A_i = B_i + 1,\ A_i \neq B_j \quad \text{for all } i \text{ and } j.$$

Thus vertex labels of m_r-parts of K_{2,m_r} are distinct.

Table 1. Range of vertex label for n even

Graphs			
K_{2,m_n}	$2m$	to	$2m-2m_n+2$
K_{2,m_1}	$\alpha_1 = 2m-2m_n$	to	$\alpha_1 - 2m_1 + 3$
$K_{2,m_{n-1}}$	$\beta_1 = \alpha_1 - 2m_1 + 1$	to	$\beta_1 - 2m_n + 3$
K_{2,m_2}	$\alpha_2 = \beta_1 - 2m_n + 1$	to	$\alpha_2 - 2m_2 + 3$
$K_{2,m_{n-2}}$	$\beta_2 = \alpha_2 - 2m_2 + 1$	to	$\beta_2 - 2m_{n-1} + 3$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
K_{2,m_r}	$\alpha_r = \beta_{r-1} - 2m_{n-(r-2)} + 1$	to	$\alpha_r - 2m_r + 3$ $r = 2,3,\dots,\frac{n}{2}-1$
$K_{2,m_{n-r}}$	$\beta_r = \alpha_r - 2m_r + 1$	to	$\beta_r - 2m_{n-(r-1)} + 3$ $r = 1,2,\dots,\frac{n}{2}-2$
$K_{2,m_{\frac{n}{2}}}$	$\beta_{\frac{n}{2}-1} - 2m_{\frac{n}{2}+1} + 1$	to	$\beta_{\frac{n}{2}-1} - 2m_{\frac{n}{2}+1} - 2m_{\frac{n}{2}} + 3$

Table 2. Range of vertex label for n odd

Graphs			
K_{2,m_n}	$2m$	to	$2m-2m_n+2$
K_{2,m_1}	$\alpha_1 = 2m-2m_n$	to	$\alpha_1 - 2m_1 + 3$
$K_{2,m_{n-1}}$	$\beta_1 = \alpha_1 - 2m_1 + 1$	to	$\beta_1 - 2m_n + 3$
K_{2,m_2}	$\alpha_2 = \beta_1 - 2m_n + 1$	to	$\alpha_2 - 2m_2 + 3$
$K_{2,m_{n-2}}$	$\beta_2 = \alpha_2 - 2m_2 + 1$	to	$\beta_2 - 2m_{n-1} + 3$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
K_{2,m_r}	$\alpha_r = \beta_{r-1} - 2m_{n-(r-2)} + 1$	to	$\alpha_r - 2m_r + 3$ $r = 1,2,\dots,\frac{n-1}{2}$
$K_{2,m_{n-r}}$	$\beta_r = \alpha_r - 2m_r + 1$	to	$\beta_r - 2m_{n-(r-1)} + 3$ $r = 1,2,\dots,\frac{n-3}{2}$
$K_{2,m_{\frac{n+1}{2}}}$	$\alpha_{\frac{n-1}{2}} - 2m_{\frac{n-1}{2}} + 1$	to	$\alpha_{\frac{n-1}{2}} - 2m_{\frac{n-1}{2}} - 2m_{\frac{n+1}{2}} + 3$

Also, labels of vertices, of m_{n-r}-part of $K_{2,m_{n-r}}$, where $r = 1$ to $\frac{n}{2}-1$, for n even and $r = 1$ to $\frac{n-3}{2}$, for n odd, are β_r to $\beta_r - 2m_{n-r} + 3$ following arithmetic progressions A_l and B_l alternatively, where

$$A_l = \beta_r - 4(l-1) \text{ and } B_l = \big(\beta_r - 1\big) - 4(l-1) \quad \text{for } l = 1,2,\dots,\frac{m_{n-r}}{2}$$

as

$$A_i = B_i + 1,\ A_i \neq B_j \quad \text{for all } i \text{ and } j.$$

Thus vertex labels of m_{n-r}-parts of $K_{2,m_{n-r}}$ are distinct.

For n even,

$$f(c_r) = 2(r-1), \qquad r = 1,2,3,\dots,\frac{n}{2}: \text{ i.e. } 2,4,\dots,(n-2).$$

$$f(c_{n-r}) = 1+2r, \qquad r = 0,1,\dots,\frac{n}{2}-2: \text{ i.e. } 1,3,\dots,(n-3).$$

$$f\big(c_{\frac{n}{2}+1}\big) = n-1$$

For n odd,

$$f(c_r) = 2(r-1), \qquad r = 1,2,3,\dots,\tfrac{n+1}{2} : \text{ i.e. } 2,4,\dots,(n-1)$$

$$f(c_{n-r}) = 1+2r, \qquad r = 0,1,\dots,\tfrac{n-3}{2} : \text{ i.e. } 1,3,\dots,(n-2)$$

and, we know that

$$f\left(c^{m_{\frac{n}{2}}+1}_{\frac{n}{2}+1,\frac{n}{2}+2}\right) = f\left(c_{\frac{n}{2}}\right) + 2 = n-2+2 = n\,.$$

Thus, labels of vertices on cycle are in any case less than least label of outer vertex in any way. Thus vertex labels are distinct.

In Table 3 and in Table 4, the edge weights covered are given if n is even and odd, respectively. It can be clearly seen that all edge weights are distinctly covered.

Table 3. Edge weights of cycle with n even, for $r = 1,2,\dots,\frac{n-2}{2}$ and $s = 1,2,\dots,\frac{n}{2}-2$

Graphs			
K_{2,m_1}	$2m-2m_n$	to	$2m-(2m_n+2m_1)+1$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
K_{2,m_r}	$2m-\left(\sum_{i=0}^{r-1} 2m_{n-i}+\sum_{i=0}^{r-1} 2m_i\right)$	to	$2m-\left(\sum_{i=0}^{r-1} 2m_{n-i}+\sum_{i=0}^{r} 2m_i\right)+1$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$K_{2,m_{\frac{n}{2}}}$	$2m_{\frac{n}{2}}$	to	1
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$K_{2,m_{n-s}}$	$2m-\left(\sum_{i=0}^{s-1} 2m_{n-i}+\sum_{i=0}^{s} 2m_i\right)$	to	$2m-\left(\sum_{i=0}^{s} 2m_{n-i}+\sum_{i=0}^{s} 2m_i\right)+1$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
K_{2,m_n}	$2m$	to	$2m-2m_n+1$

Table 4. Edge weights of cycle with n odd, for $r = 1,2,\dots,\frac{n-1}{2}$ and $s = 1,2,\dots,\frac{n-3}{2}$

Graphs			
K_{2,m_1}	$2m-2m_n$	to	$2m-(2m_n+2m_1)+1$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
K_{2,m_r}	$2m-\left(\sum_{i=0}^{r-1} 2m_{n-i}+\sum_{i=0}^{r-1} 2m_i\right)$	to	$2m-\left(\sum_{i=0}^{r-1} 2m_{n-i}+\sum_{i=0}^{r} 2m_i\right)+1$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$K_{2,m_{\frac{n+1}{2}}}$	$2m_{\frac{n+1}{2}}$	to	1
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$K_{2,m_{n-s}}$	$2m-\left(\sum_{i=0}^{s-1} 2m_{n-i}+\sum_{i=0}^{s} 2m_i\right)$	to	$2m-\left(\sum_{i=0}^{s} 2m_{n-i}+\sum_{i=0}^{s} 2m_i\right)+1$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
K_{2,m_n}	$2m$	to	$2m-2m_n+1$

□

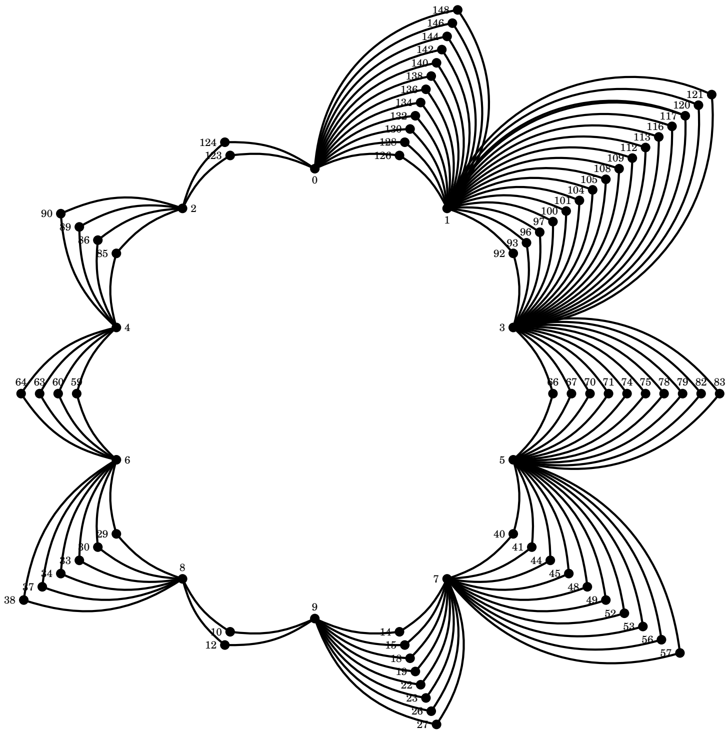

Figure 2. Graceful $EASS(C_{10})$ with $m_1 = m_5 = 2$, $m_2 = m_3 = 4$, $m_4 = 6$, $m_6 = 8$, $m_7 = m_8 = 10$, $m_9 = 16$, $m_{10} = 12$

We draw $EASS(C_7)$ with $m_1 = 4$, $m_2 = 6$, $m_3 = 4$, $m_4 = 10$, $m_5 = 8$, $m_6 = 2$, $m_7 = 8$.

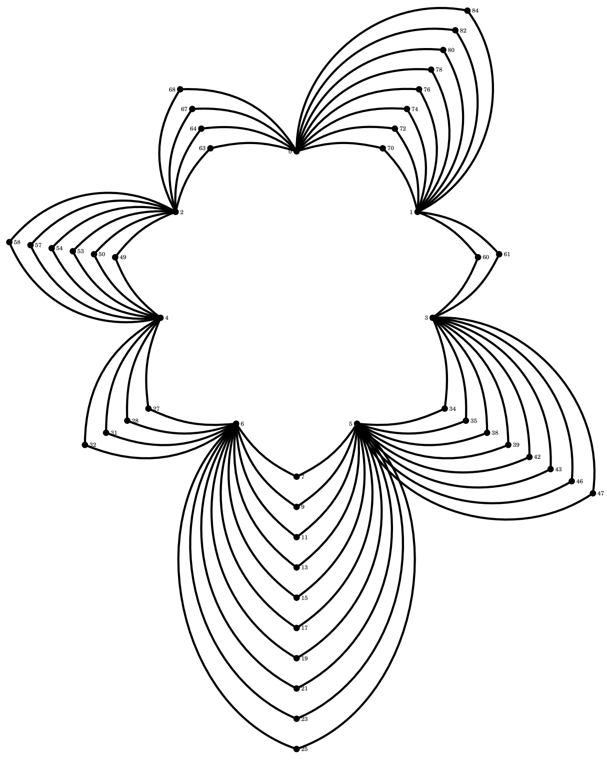

Figure 3. Graceful $EASS(C_7)$ with $m_1 = m_3 = 4$, $m_2 = 6$, $m_4 = 10$, $m_5 = 8$, $m_6 = 2$, $m_7 = 8$

Theorem 4.2. *Odd arbitrary supersubdivision of cycle of odd length C_n, i.e. $OASS(C_n)$, is not graceful.*

Proof. Consider a cycle of odd length C_n, where $n = 2t+1$. Let $m_i = 2(r_i)+1$, for $i = 1,2,\dots n$. Let $c_1, c_2, \dots, c_n$ be the vertices of cycle. Let $c^k_{i,i+1}$, $k = 1,2,\dots,m_i$ be the vertices of the m_i-vertices part of K_{2,m_i} merged with the edge $c_i c_{i+1}$ for $i = 1,2,\dots,n-1$ and $k = 1,2,\dots,m_i$. Let $c^k_{n,1}$, $k = 1,2,\dots,m_n$ be the vertices of the m_n-vertices part of K_{2,m_n} merged with the edge $c_n c_1$ and $k = 1,2,\dots,m_n$.
After arbitrary supersubdivision of cycle C_n by K_{2,m_i}, number of edges = $|E| = 2m_1 + 2m_2 + 2m_3 + \dots + 2m_n$

$$\begin{aligned}
|E| &= 2m_1 + 2m_2 + 2m_3 + \dots + 2m_n \\
&= 2(m_1 + m_2 + m_3 + \dots + m_n) \\
&= 2(2r_1 + 2r_2 + 2r_3 + \dots + 2r_n + n) \\
&= 4(r_1 + r_2 + \dots + r_n) + 4t + 2 \\
&= 4(r_1 + r_2 + \dots + r_n + t) + 2 \\
&\equiv 2 (\text{mod } 4).
\end{aligned}$$

Every vertex of the type $c^k_{i,i+1}$ is of degree 2 i.e. even.
Every vertex c_i is of degree $m_{i-1} + m_i$, which is even, where $i = 2,3,\dots n-1$.
Also, the vertex c_1 has degree $m_1 + m_n$, the vertex c_n has degree $m_{n-1} + m_n$, both even.
Therefore, every vertex is of even degree, as all m_i's are odd. Hence $OASS(C_n)$ is an Eularian graph, with size 2(mod 4).
Thus, $OASS(C_n)$ is not graceful for n odd by Rosa's theorem, an Eularian graph with number of edges $q \equiv 1 (\text{mod } 4)$ or $q \equiv 2 (\text{mod } 4)$ can not be graceful [7]. □

Theorem 4.3. *Odd Arbitrary supersubdivision of C_n, i.e. $OASS(C_n)$ for n even is graceful if $m_i \geq 3$.*

Proof. Define the following labeling $f : V \to \{0,1,2,\dots,2m\}$

$$\begin{aligned}
&f(c_1) = 0 \\
&f(c_r) = 2(r-1) && \text{for } r = 2,3,\dots,\tfrac{n}{2} \\
&f(c_{n-r}) = 1 + 2r && \text{for } r = 0,1,2,\dots,\tfrac{n}{2} - 2 \\
&f\left(c^k_{n,1}\right) = 2m - 2(k-1) && k = 1,2,\dots,m_n. \\
&\alpha_1 = 2m - 2m_n
\end{aligned}$$

For $i = 2,\dots,\frac{n}{2} - 1$

$$\alpha_i = f(c_i) + 2m - 2\left(\sum_{r=1}^{i-1} m_r\right) - 2\left(\sum_{r=0}^{i-1} m_{n-r}\right).$$

For $k = 1,3,5,\dots,m_i - 2$,

$$f\left(c^k_{i,i+1}\right) = \alpha_i - 4(l-1) = \alpha_i + 4 - 4l \qquad l = 1,2,\dots,\frac{m_i - 1}{2}.$$

For $k = 2,4,6,\dots,m_i - 1$,

$$f\left(c^k_{i,i+1}\right) = \alpha_i - 1 - 4(l-1) = \alpha_i + 3 - 4l \qquad l = 1,2,\dots,\frac{m_i - 1}{2}.$$

For $k = m_i$,

$$f\left(c^{m_i}_{i,i+1}\right) = \beta_i - 2m_{n-i}.$$

For $i = 1,2,\dots,\frac{n}{2} - 1$

$$\beta_i = f(c_i) + 2m - 2\left(\sum_{r=1}^{i} m_r\right) - 2\left(\sum_{r=0}^{i-1} m_{n-r}\right) + 3.$$

For $k = 1,3,5,\dots,m_{n-i}$,

$$f\left(c^k_{n-i,n-i+1}\right) = \beta_i - 4(l-1) = \beta_i + 4 - 4l, \quad l = 1,2,\dots,\frac{m_{n-i} - 1}{2} + 1.$$

For $k = 2,4,6,\dots,m_{n-i} - 1$,

$$f\left(c^k_{n-i,n-i+1}\right) = \beta_i - 1 - 4(l-1) = \beta_i + 3 - 4l \qquad l = 1,2,\dots,\frac{m_{n-i} - 1}{2}$$

$$f\left(c^k_{\frac{n}{2},\frac{n}{2}+1}\right) = f\left(c_{\frac{n}{2}}\right) + 2k, \qquad \text{where } k = 1,2,\dots,m_{\frac{n}{2}}.$$

From Table 5, labels of vertices, of m_i-part of K_{2,m_i} are α_i to $\alpha_i - 2m_i + 5$, following arithmetic progressions A_l and B_l alternatively, where

$$A_l = \alpha_i - 4(l-1) \quad \text{and} \quad B_l = (\alpha_i - 1) - 4(l-1), \quad \text{for } l = 1,2,\dots,\frac{m_i - 1}{2}$$

as $A_i = B_i + 1$, $A_i \neq B_j$ for all i and j.

Also, labels of vertices, of m_{n-i}-part of $K_{2,m_{n-i}}$, where $i = 1$ to $\frac{n}{2} - 1$, are β_i to $\beta_i - 2m_{n-i} + 2$, following arithmetic progressions A_l and B_l alternatively where

$$A_l = \beta_i - 4(l-1) \quad \text{and} \quad B_l = \left(\beta_i - 1\right) - 4(l-1), \quad \text{for } l = 1,2,\dots,\frac{m_{n-i} + 1}{2}.$$

as

$$A_i = B_i + 1,\ A_i \neq B_j \text{ for all } i \text{ and } j.$$

Thus vertex labels of m_{n-i}-parts of $K_{2,m_{n-i}}$ are distinct.

Table 5. Range of vertex labels

Graphs		Range of vertex labels	
K_{2,m_n}	$2m$	to	$2m - 2m_n + 2$
K_{2,m_1}	$\alpha_1 = 2m - 2m_n$	to	$\alpha_1 - 2m_1 + 5, \beta_1 - 2m_{n-1}$
$K_{2,m_{n-1}}$	$\beta_1 = \alpha_1 - 2m_1 + 3$	to	$\beta_1 - 2m_{n-1} + 2$
K_{2,m_2}	$\alpha_2 = \beta_1 - 2m_{n-1} - 1$	to	$\alpha_2 - 2m_2 + 5, \beta_2 - 2m_{n-2}$
$K_{2,m_{n-2}}$ $\vdots$	$\beta_2 = \alpha_2 - 2m_2 + 3$ $\vdots$	to $\vdots$	$\beta_2 - 2m_{n-2} + 2$ $\vdots$
K_{2,m_i}	$\alpha_i = \beta_{i-1} - 2m_{n-(i-1)} - 1$	to	$\alpha_i - 2m_i + 5, \beta_i - 2m_{n-i}$, $i = 2,3,\dots,\frac{n}{2} - 1$
$K_{2,m_{n-i}}$	$\beta_i = \alpha_i - 2m_i + 3$	to	$\beta_i - 2m_{n-i} + 2$, $i = 1,2,\dots,\frac{n}{2} - 2$
$K_{2,m_{\frac{n}{2}}}$	$\beta_{\frac{n}{2}-1} - 2m_{\frac{n}{2}+1} - 1$	to	$\beta_{\frac{n}{2}-1} - 2m_{\frac{n}{2}+1} - 2m_{\frac{n}{2}} + 1$

Table 6. Edge weights of cycle for $i = 1, 2, \ldots, \frac{n}{2} - 1$

Graphs	Range of Edge Weights
K_{2,m_n}, $j = 1$ to m_n $c_n c^j_{n,1}$, $c_1 c^j_{n,1}$	$2m$ to $2m - 2m_n + 1$
K_{2,m_1}, $j = 1$ to $m_1 - 1$ $c_2 c^j_{1,2}$, $c_1 c^j_{1,2}$	$2m - 2m_n$ to $2m - 2(m_n + m_1 - 1) + 1$
$K_{2,m_{n-1}}$, $j = 1$ to m_{n-1} $c_n c^j_{n-1,n}$, $c_{n-1} c^j_{n-1,n}$	$2m - 2(m_n + m_1 - 1)$ to $2m - 2(m_n + (m_1 - 1) + (m_{n-1} - 1)) + 1$
$K_{2,m_{n-1}}$, $c_{n-1} c^{m_{n-1}}_{n-1,n}$, $c_n c^{m_{n-1}}_{n-1,n}$	$2m - 2(m_n + (m_1 - 1) + (m_{n-1} - 1))$ and $2m - 2(m_n + (m_1 - 1) + (m_{n-1} - 1)) - 2$
K_{2,m_1}, $c_2 c^{m_1}_{1,2}$, $c_1 c^{m_1}_{1,2}$	$2m - 2(m_n + (m_1 - 1) + (m_{n-1} - 1)) - 1$ and $2m - 2(m_n + (m_1 - 1) + (m_{n-1} - 1)) - 3$
$\vdots$	$\vdots$
K_{2,m_i}, $j = 1$ $to m_i - 1$ $c_{i+1} c^j_{i,i+1}$, $c_i c^j_{i,i+1}$	$2m - 2\left(m_n + \sum_{r=1}^{i-1}(m_r - 1) + \sum_{r=1}^{i-1}(m_{n-r} - 1)\right) - 4(i-1)$ to $2m - 2\left(m_n + \sum_{r=1}^{i}(m_r - 1) + \sum_{r=1}^{i-1}(m_{n-r} - 1)\right) - 4(i-1) + 1$
$K_{2,m_{n-i}}$, $j = 1$ to $m_{n-i} - 1$ $c_{i+1} c^j_{i,i+1}$, $c_i c^j_{i,i+1}$	$2m - 2\left(m_n + \sum_{r=1}^{i}(m_r - 1) + \sum_{r=1}^{i-1}(m_{n-r} - 1)\right) - 4(i-1)$ to $2m - 2\left(m_n + \sum_{r=1}^{i}(m_r - 1) + \sum_{r=1}^{i}(m_{n-r} - 1)\right) - 4(i-1) + 1$
$K_{2,m_{n-i}}$, $c_{i+1} c^{m_{n-i}}_{i,i+1}$, $c_i c^{m_{n-i}}_{i,i+1}$	$2m - 2\left(m_n + \sum_{r=1}^{i}(m_r - 1) + \sum_{r=1}^{i}(m_{n-r} - 1)\right) - 4(i-1)$ and $2m - 2\left(m_n + \sum_{r=1}^{i}(m_r - 1) + \sum_{r=1}^{i}(m_{n-r} - 1)\right) - 4(i-1) - 2$
K_{2,m_i}, $c_{i+1} c^{m_i}_{i,i+1}$, $c_i c^{m_i}_{i,i+1}$	$2m - 2\left(m_n + \sum_{r=1}^{i}(m_r - 1) + \sum_{r=1}^{i}(m_{n-r} - 1)\right) - 4(i-1) - 1$ and $2m - 2\left(m_n + \sum_{r=1}^{i}(m_r - 1) + \sum_{r=1}^{i}(m_{n-r} - 1)\right) - 4(i-1) - 3$
$\vdots$	$\vdots$
$K_{2,m_{\frac{n}{2}}}$, $j = 1$ to $m_{\frac{n}{2}}$ $c_{\frac{n}{2}+1} c^j_{\frac{n}{2},\frac{n}{2}+1}, c_{\frac{n}{2}} c^j_{\frac{n}{2},\frac{n}{2}+1}$	$2m_{\frac{n}{2}}$ to 1

The lable of vertex $c^{m_i}_{i,i+1}$ is $\beta_i - 2m_{n-i} = \alpha_{i+1} + 1$ giving vertex labels of m_i-parts of K_{2,m_i} as distinct.

$$f(c_i) = 2(i-1), \qquad i = 1,2,3,\dots,\frac{n}{2} : \text{i.e. } 2,4,\dots,(n-2).$$

$$f(c_{n-i}) = 1 + 2i, \qquad i = 0,1,\dots,\frac{n}{2} - 2 : \text{i.e. } 1,3,\dots,(n-3).$$

$$f\left(c_{\frac{n}{2}+1}\right) = n - 1$$

and, we know that

$$f\left(c^{m_{\frac{n}{2}}+1}_{\frac{n}{2}+1,\frac{n}{2}+2}\right) = f\left(c_{\frac{n}{2}}\right) + 2 = n - 2 + 2 = n.$$

Thus, labels of vertices on cycle are in any case less than least label of outer vertex in any way. Thus vertex labels are distinct.
In Table 6, the edge weights covered are given. It can be clearly seen that all edge weights are distinctly covered. □

5. Illustrations

(I): We draw $SS(C_{12})$ with $m = 8$.

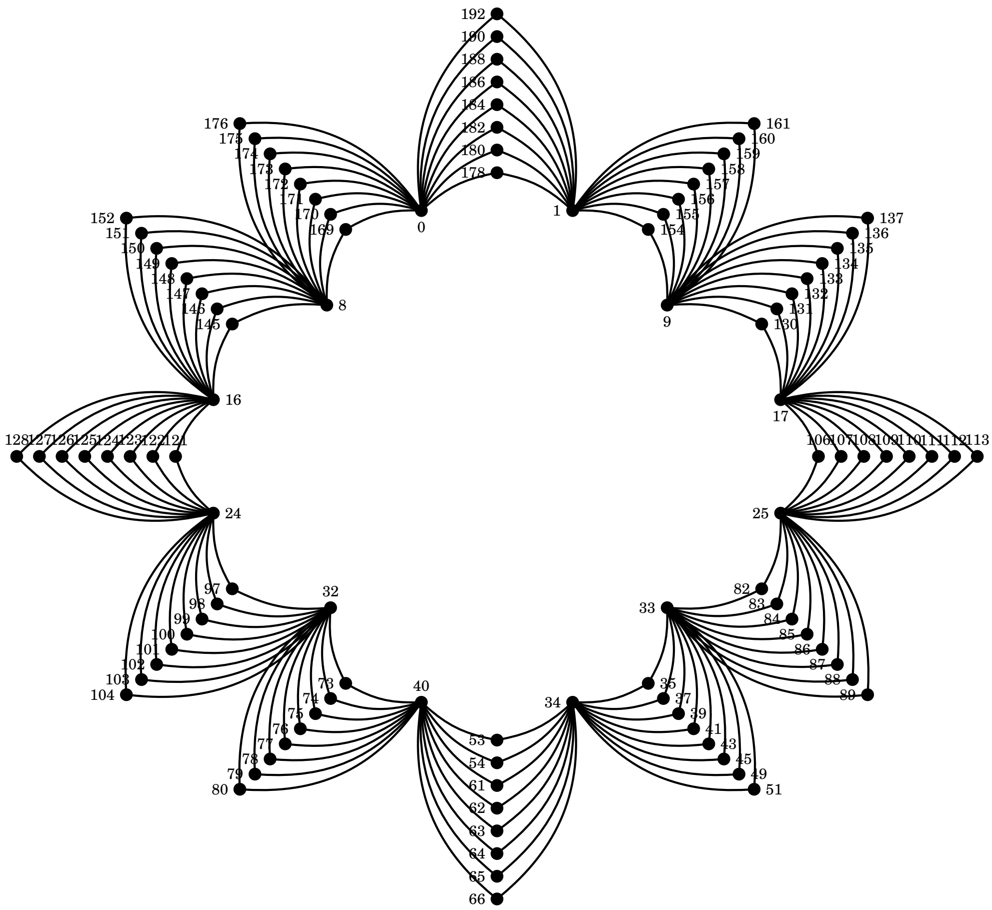

Figure 4. Graceful $SS(C_{12})$ with $m = 8$

(II): We draw $SS(C_{12})$ with $m = 9$.

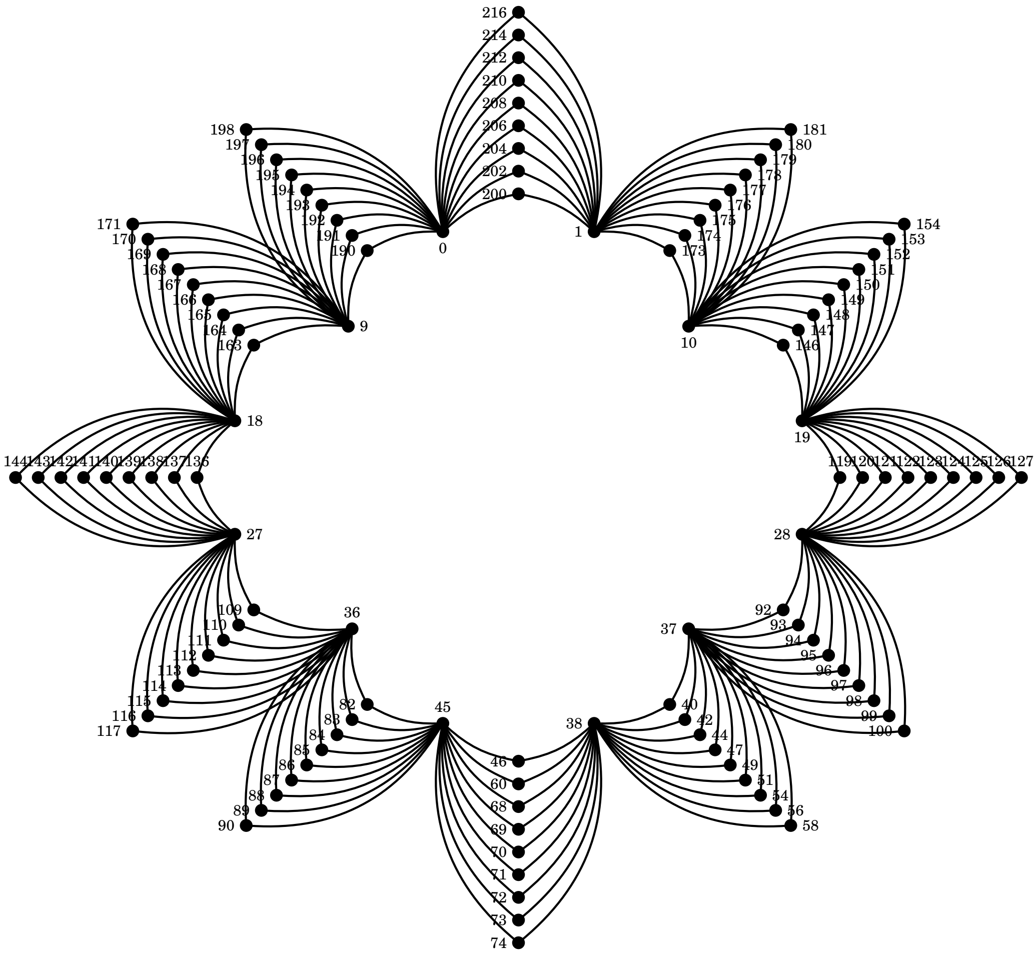

Figure 5. Graceful $SS(C_{12})$ with $m = 9$

6. Conclusion

Rosa [7] proved that the n-cycle C_n is graceful if and only if $n \equiv 0$ or $3(\text{mod } 4)$ whereas we proved $OASS(C_n)$ and $EASS(C_n)$ are graceful even if $n \equiv 1$ or $2(\text{mod } 4)$. We have proved gracefulness of even arbitrary supersubdivision of C_n for any n and odd arbitrary supersubdivision of C_n for n even with $m_i \geq 3$. We also have shown nongracefulness of $OASS(C_n)$ when n is odd. But it is still left to be discussed what happens if few m_i's are even and few m_i's are odd. One can head towards the solution of 'Is $ASS(C_n)$ graceful, in general?'

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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