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On the Absolute Summability Factors of Infinite Series involving Quasi-f-power Increasing Sequence

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Abstract. In this note we improve a result concerning absolute summability factor of an infinite series via quasi β-power increasing sequence achieved by Sevli and Leindler [1].

1. Introduction

A positive sequence (b_n) is said to be almost increasing if exist a positive increasing sequence (c_n) and two positive constants A and B such that $Ac_n \le b_n \le Bc_n$.

A positive sequence $a = (a_n)$ is said to be quasi β-power increasing if there exists a constant $K = k(\beta, a) \ge 1$ such that

$$Kn^\beta a_n \ge m^\beta a_m \tag{1.0}$$

holds for all $n \ge m$. If (1.0) stays with $\beta = 0$ then a is called a quasi increasing sequence. It should be noted that every almost increasing sequence is a quasi β-power increasing sequence for any nonnegative β, but the converse need not be true as can be seen by taking $a_n = n^{-\beta}$.

A positive sequence $\alpha = (\alpha_n)$ is said to be a quasi-f-power increasing sequence, $f = (f_n)$, if there exits a constant $K = K(\alpha, f)$ such that

$$Kf_n\alpha_n \ge f_m\alpha_m$$

holds for $n \ge m \ge 1$ (see [3]). Clearly if α is quasi-f-power increasing sequence, then αf is quasi increasing sequence.

Let T be a lower triangular matrix, (s_n) a sequence, and

$$T_n := \sum_{\nu=0}^{n} t_{n\nu} s_\nu. \tag{1.1}$$

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A series $\sum a_n$ is said to be summable $|T|_k$, $k \geq 1$ if

$$\sum_{n=1}^{\infty} n^{k-1}|\Delta T_{n-1}|^k < \infty. \tag{1.2}$$

Given any lower triangular matrix T one can associate the matrices $\bar{T}$ and $\widehat{T}$, with entries defined by

$$\bar{t}_{nv} = \sum_{t=v}^{n} t_{ni}, \quad n, i = 0, 1, 2 \ldots, \ \widehat{t}_{nv} = \bar{t}_{nv} - \bar{t}_{n-1,v}$$

respectively. With $s_n = \sum_{i=0}^{n} a_i \lambda_i$,

$$t_n = \sum_{v=0}^{n} t_{nv} s_v = \sum_{v=0}^{n} t_{nv} \sum_{i=0}^{v} a_i \lambda_i = \sum_{i=0}^{n} a_i \lambda_i \sum_{v=i}^{n} t_{nv} = \sum_{i=0}^{n} \bar{t}_{ni} a_i \lambda_i, \tag{1.3}$$

$$Y_n := t_n - t_{n-1} = \sum_{t=0}^{n} \bar{t}_{ni} a_i \lambda_i - \sum_{i=0}^{n-1} \bar{t}_{n-1,i} a_i \lambda_i = \sum_{i=0}^{n} \widehat{t}_{ni} a_i \lambda_i, \quad \text{as } \bar{t}_{n-1,n} = 0. \tag{1.4}$$

We call T a triangle if T is lower triangular and $t_{nn} \neq 0$ for all n. We designate $A = (a_{nv})$ to be of class Ω if the following holds

(i) is lower triangular

(ii) $a_{nv} \geq 0$, $n, v = 0, 1, \ldots$

(iii) $a_{n-1,v} \geq a_{nv}$, for $n \geq v + 1$

(iv) $a_{n0} = 1$, $n = 0, 1, \ldots$

By t_n we denote the nth $(C, 1)$ mean of the sequence (na_n), that is $t_n = \frac{1}{n+1} \sum_{v=1}^{n} v a_v$.

Very recently, Selvi and Leindler [1] proved the following result

Theorem 1.1. *Let $A \in \Omega$ satisfying*

$$na_{nn} = O(1), \quad n \to \infty \tag{1.5}$$

and let (λ_n) be a sequence of real numbers satisfying

$$\sum_{n=1}^{m} \lambda_n = o(m), \quad m \to \infty \tag{1.6}$$

and

$$\sum_{n=1}^{m} |\Delta \lambda_n| = o(m), \quad m \to \infty. \tag{1.7}$$

If (X_n) is a quasi-f-increasing sequence and the conditions

$$\sum_{n=1}^{m} \frac{1}{n} |t_n|^k = O(X_m), \quad m \to \infty, \tag{1.8}$$

$$\sum_{n=1}^{\infty} nX_n(\beta,\mu)|\Delta|\Delta\lambda_n|| < \infty, \tag{1.9}$$

are satisfied then the series $\sum a_n\lambda_n$ *is summable* $|A|_k \geq 1$, *where* $(f_n) = (n^{\beta}\log^{\mu} n)$, $\mu \geq 0$, $0 \leq \beta < 1$ *and* $X_n(\beta,\mu) = \max\{n^{\beta}(\log n)^{\mu}, \log n\}$.

2. **Main Result**

The purpose of this paper is to give the following

Theorem 2.1. *Let* $A \in \Omega$, *Let* (X_n) *be a quasi-f-power increasing sequence and let* (λ_n) *be sequence of real numbers all are satisfying* (1.5) *and*

$$\lambda_n = o(1), \quad n \to \infty, \tag{2.1}$$

$$|\lambda_n| X_n = O(1), \quad n \to \infty, \tag{2.2}$$

$$\sum_{n=1}^{\infty} nX_n|\Delta|\Delta\lambda_n|| < \infty, \tag{2.3}$$

$$\sum_{n=1}^{m} \frac{|t_n|^k}{nX_n^{K-1}} = O(X_m), \tag{2.4}$$

$$\sum_{v=1}^{n-1} \widehat{a}_{n,v+1} a_{vv} = O(a_{nn}), \quad n \to \infty, \tag{2.5}$$

then the series $\sum a_n\lambda_n$ *is summable* $|A|_k$, $k \geq 1$, *where* $f = (f_n)$, $f_n = n^{\beta}\left(\prod_{\mu=1}^{N} \log^{\mu} n\right)^{P\mu}$, $0 \leq \beta < 1$, $1 \leq N < \infty$, $P_{\mu} > 0$, *and* $\log^{\mu} n = \log(\log^{\mu-1} n)$.

The goodness and advantage of Theorem 2.1 follows from the following remark.

Remark. (1) Although the condition (2.5) is added (in comparing with Theorem 1.1) but this condition is trivial. As an example if we are putting $a_{nv} = p_v/P_n$ (in order to have the $|\bar{N}, p_n|_k$ summability), (2.5) is obvious by the following

$$\sum_{v=1}^{n-1} \widehat{a}_{n,v+1} a_{vv} = \sum_{v=1}^{n-1} \frac{p_n P_v}{P_n P_{n-1}} \frac{p_v}{P_v} = \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} P_v = \frac{p_n}{P_n} = a_{nn} = O(a_{nn}).$$

(2) Let us define the following two groups of conditions

group $I = \{(1.6), (1.7), (1.9)\}$,

group $J = \{(2.1), (2.2), (2.3)\}$.

It is clear that (see [1]) $I \Rightarrow J$, but not the converse. Therefore the conditions of Theorem 3.1 are weaker than those of Theorem 1.1.

(3) The condition (1.8) in using impose us to loose powers of estimation. For example in the proof of Theorem 1.1, through the estimations of I_1 and I_2, the power $\lambda_n|^{k-1}$ has been lost through the estimation when it has been substituted by $(|\lambda_n|^{k-1} = O(1))$, while we have not this case on using (2.4).

(4) The sequence (X_n) used in Theorem 2.1 is more general and in some sense make some condition is weaker than (X_n) defined in Theorem 1.1.

3. **Lemmas**

Lemma 3.1. *The conditions* (1.6)*,* (1.7) *and* (1.9) *implies* $\lambda_n = O(1)$*, as* $n \to \infty$*. For the proof see* [1].

Lemma 3.2. *Let* $A \in \Omega$*. Then*

$$\sum_{v=1}^{n-1} |\Delta_n a_{n-1m,v}| \le a_{nn},$$

$$\sum_{n=v+1}^{m+1} |\Delta_n a_{n-1m,v}| \le a_{vv}.$$

Lemma 3.3. *Let* $A \in \Omega$*. Then*

$$\widehat{a}_{n+1,v} \le a_{nn} \qquad \text{for } n \ge v+1,$$

$$\sum_{n=v+1}^{m+1} \widehat{a}_{n+1,v} \le 1, \quad v = 0, 1, \dots .$$

Lemma 3.4. *Let* (X_n) *be as defined in Theorem* 2.1*. Then conditions* (2.1) *and* (2.3) *implies*

$$nX_n|\Delta\lambda_n| = O(1), \qquad \text{as } n \to \infty, \tag{3.1}$$

$$\sum_{n=1}^{\infty} X|\Delta\lambda_n| < \infty. \tag{3.2}$$

Proof. Let f_n be as defined in Theorem 2.1. Then by (2.1), $\Delta|\Delta\lambda_n| \to 0$, and hence

$$\begin{aligned} nX_n|\Delta\lambda_n| &= nX_n\left|\sum_{v=n}^{\infty}\Delta|\Delta\lambda_v|\right| \\ &\le nX_n\sum_{v=n}^{\infty}|\Delta|\Delta\lambda_v|| = O(1)nf_n^{-1}f_nX_n\sum_{v=n}^{\infty}|\Delta|\Delta\lambda_v|| \\ &= O(1)nf_n^{-1}\sum_{v=n}^{\infty}f_vX_v|\Delta|\Delta\lambda_v|| \\ &= O(1)\sum_{v=n}^{\infty}vf_v^{-1}f_vX_v|\Delta|\Delta\lambda_v|| \end{aligned}$$

$$= O(1)\sum_{v=n}^{\infty} vX_v|\Delta|\Delta\lambda_v||$$

$$= O(1),$$

$$\sum_{n=1}^{\infty} X_n|\Delta\lambda_n| \le \sum_{n=1}^{\infty} X_n \sum_{v=n}^{\infty}|\Delta\lambda_v|| = \sum_{n=1}^{\infty} f_n^{-1} f_n X_n \sum_{v=n}^{\infty}|\Delta|\Delta\lambda_v||$$

$$= O(1)\sum_{n=1}^{\infty} f_n^{-1}\sum_{v=n}^{\infty} f_v X_v|\Delta|\Delta\lambda_v||$$

$$= O(1)\sum_{v=1}^{\infty} f_v X_v|\Delta|\Delta\lambda_v||\sum_{n=1}^{v} f_n^{-1}.$$

Now, as

$$\sum_{n=1}^{v} f_n^{-1} = \sum_{n=1}^{v} n^{\beta+\epsilon} f_n^{-1} n^{-\beta-\epsilon}, \quad 0 < \beta+\epsilon < 1$$

$$\le v^{\beta+\epsilon} f_v^{-1}\sum_{n=1}^{v} n^{-\beta-\epsilon} = O(1)v^{\beta+\epsilon} f_v^{-1}\int_0^v x^{-\beta-\epsilon}dx = O(1)v f_v^{-1}.$$

Then, we have

$$\sum_{n=1}^{\infty} X_n|\Delta\lambda_n| = O(1)\sum_{n=1}^{\infty} nX_n|\Delta|\Delta\lambda_n|| < \infty. \qquad \square$$

4. **Proof of Theorem 1.2**

Let y_n denotes the nth term of the A-transform of the series $\sum a_n\lambda_n$. Then by (1.4), we have, for $n \ge 1$

$$Y_n = y_n - y_{n-1}$$

$$= \sum_{v=0}^{n} \widehat{a}_{nv} a_v \lambda_v$$

$$= \sum_{v=1}^{n-1}\Delta\left(\frac{\widehat{a}_{nv}\lambda_v}{v}\right)\sum_{r=1}^{v} r a_r + \frac{a_{nn}\lambda_n}{n}\sum_{v=1}^{n} v a_v$$

$$= \frac{(n+1)a_{nn}t_n\lambda_n}{n} - \sum_{v=1}^{n-1}\Delta_n a_{n-1,v}\lambda_v t_v \frac{v+1}{v} + \sum_{v=1}^{n-1}\widehat{a}_{n,v+1}\lambda_v t_v \frac{1}{v} + \sum_{v=1}^{n-1}\widehat{a}_{n,v+1}\Delta\lambda_v t_v$$

$$= Y_{n1} + Y_{n2} + Y_{n3} + Y_4.$$

To complete the proof, by Minkowski's inequality, it is sufficient to show that

$$\sum_{n-1}^{\infty} n^{k-1}|Y_{nv}|^k < \infty, \quad v = 1,2,3,4.$$

Applying Holder's inequality,

$$\begin{aligned}
\sum_{n=1}^{m+1} n^{k-1}|Y_{n1}|^k &= \sum_{n=1}^{m+1} n^{k-1}\left|\frac{(n+1)a_{nn}t_n\lambda_n}{n}\right|^k \\
&= O(1)\sum_{n=1}^{m+1}(na_n n)^{k-1}\frac{|t_n|^k\lambda_n|(|\lambda_n|X_n)^{k-1}}{nX_n^{k-1}} \\
&= O(1)\sum_{v=1}^{m}\frac{|\lambda_v|\,|t_v|^k}{vX_v^{k-1}} \\
&= O(1)\sum_{v=1}^{m-1}\Delta|\lambda_v|\sum_{r=0}^{v}\frac{|t_r|^k}{rX_r^{k-1}} + O(1)|\lambda_m|\sum_{v=1}^{m}\frac{|t_v|^k}{vX_v^{k-1}} \\
&= O(1)\sum_{v=1}^{m}|\Delta\lambda_v|X_v + O(1)|\lambda_m|X_m \\
&= O(1),
\end{aligned}$$

$$\begin{aligned}
\sum_{n=1}^{m+1} n^{k-1}|Y_{n2}|^k &= \sum_{n=1}^{m+1} n^{k-1}\left|\sum_{v=1}^{n-1}\Delta_n a_{n-1,v}\lambda_v\frac{v+1}{v}\right|^k \\
&= O(1)\sum_{n=1}^{m+1} n^{k-1}\left|\sum_{v=1}^{n-1}\Delta_n a_{n-1,v},\lambda_v t_v\frac{v+1}{v}\right|^k \\
&= O(1)\sum_{n=1}^{m+1} n^{k-1}\left(\sum_{v=1}^{n-1}|\Delta_n a_{n-1,v}|\,|\lambda_v|\,|t_V|\right)^k \\
&= O(1)\sum_{n=1}^{m+1} n^{k-1}\sum_{v=1}^{n-1}|\Delta_n a_{n-1,v}|\,|\lambda_v|^k|t_V|^k\left(\sum_{v=1}^{n-1}\Delta_n a_{n-1,v}|\right)^{k-1} \\
&= O(1)\sum_{n=1}^{m+1}(na_{nn})^{k-1}\sum_{v=1}^{n-1}|\Delta_n a_{n-1,v}|\,|\lambda_v|^k|t_v|^k \\
&= O(1)\sum_{v=1}^{m}|\lambda_v|^k|t_v|^k\sum_{n=v+1}^{m+1}|\Delta_n, a_{n-1,v}| \\
&= O(1)\sum_{v=1}^{m}a_{vv}|\lambda_v|^k|t_v|^k \\
&= O(1)\sum_{v=1}^{m}\frac{|\lambda_v|\,|t_v|^k}{vX_v^{k-1}} \\
&= O(1), \quad \text{as in the case of } Y_{n1}.
\end{aligned}$$

$$\sum_{n=1}^{m+1} n^{k-1}|Y_{n3}|^k = \sum_{n=1}^{m+1} n^{k-1}\left|\sum_{v=1}^{n-1}\widehat{a}_{n,v+1}\lambda_v t_v \frac{1}{v}\right|^k$$

$$= O(1)\sum_{n=1}^{m+1} n^{k-1}\left(\sum_{v=1}^{n-1}\frac{\widehat{a}_{n,v+1}|\lambda_v||t_v|}{v}\right)^k$$

$$= O(1)\sum_{n=1}^{m+1} n^{k+1}\sum_{v=1}^{n-1} a_{vv}|\widehat{a}_{n,v+1}||\lambda_v|^k|t_v|^k\left(\sum_{v=1}^{n-1} a_{vv}\widehat{a}_{n,v+1}\right)^{k-1}$$

$$= O(1)\sum_{n=1}^{m+1}(na_{nn})^{k-1}\sum_{v=1}^{n-1} a_{vv}\widehat{a}_{n,v+1}|\lambda_v|^k|t_v|^k$$

$$= O(1)\sum_{v=1}^{m}\frac{|\lambda_v||t_v|^K}{vX_V^{k-1}}\sum_{n=v+1}^{m+1}\widehat{a}_{n,v+1} = O(1)\sum_{v=1}^{m}\frac{|\lambda_v||t_v|^k}{Vx_v^{k-1}}$$

$$= O(1), \quad \text{as in the case of } Y_{n1}.$$

$$\sum_{n=1}^{m+1} n^{k-1}|Y_{n4}|^k = \sum_{n=1}^{m+1} n^{k-1}\left|\sum_{v=1}^{n-1}\widehat{a}_{n,v+1}\Delta\lambda_v t_v\right|^k$$

$$= O(1)\sum_{n=1}^{m+1} n^{k-1}\left(\sum_{v=1}^{n-1}\widehat{a}_{n,v+1}|\Delta\lambda_v||t_v|\right)^k$$

$$= O(1)\sum_{n=1}^{m+1} n^{k-1}\sum_{v=1}^{n-1}(\widehat{a}_{n,v+1})^k\frac{|\Delta\lambda_v|}{X_v^{k-1}}|t_v|^k\left(\sum_{v=1}^{n-1} X_v|\Delta\lambda_v|\right)^{k-1}$$

$$= O(1)\sum_{n=1}^{m+1}(na_{nn})^{k-1}\sum_{v=1}^{n-1}\widehat{a}_{n,v+1}\frac{|\Delta\lambda_v|}{X_v^{k-1}}|t_v|^k$$

$$= O(1)\sum_{v=1}^{m}\frac{|\Delta\lambda_v|}{X_v^{k-1}}|t_v|^k\sum_{n=v+1}^{m+1}\widehat{a}_{n,v+1}$$

$$= O(1)\sum_{v=1}^{m}\frac{v|\Delta\lambda_v|}{vX_v^{k-1}}|t_v|^k$$

$$= O(1)\sum_{v=1}^{m-1}\Delta(v|\Delta\lambda_v|)\sum_{r=1}^{v}\frac{|t_r|^k}{rX_r^{k-1}} + O(1)m|\Delta\lambda_m|X_m$$

$$= O(1)\sum_{v=1}|\Delta\lambda_v|X_v + O(1)\sum_{v=1}^{m} v|\Delta|\Delta\lambda_v||X_v + O(1)m|\Delta\lambda_m|X_m$$

$$= O(1).$$

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