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Monotonicity Properties of the First Eigenvalue of the Laplacian Operator on Ricci Solitons

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Abstract. In this paper, we deal with the monotonicity properties of the first eigenvalue of the Laplacian operator on Ricci solitons. Firstly, by using the monotonicity formula of the $\mathscr{F}$ functional, we derive a monotonicity formula of the first eigenvalue of the Laplacian operator on Ricci solitons. Based on this, we also prove an exponential decreasing property of the first eigenvalue.

1. Introduction and Main Results

Let (M^n, g) be an n-dim C^∞ complete Riemannian manifold, and Δ denote the Laplacian operator. For the compact manifold, it is well known that the eigenvalue problem $-\Delta\varphi = \lambda\varphi$ has discrete eigenvalues, which are listed as

$$0 = \lambda_0(g) < \lambda_1(g) \leq \lambda_2(g) \leq \cdots .$$

Moreover we call $\lambda_i(g)$ the ith eigenvalue and call a function φ_i satisfying $\Delta\varphi_i = -\lambda_i(g)\varphi_i$ the ith eigenfunction.

Recall that the first eigenvalue $\lambda_1(g)$ for the closed Riemannian manifold M^n is defined:

$$\lambda_1(g) = \inf_{\phi\in\Omega} \frac{\int_{M^n} \left|\nabla\phi\right|_g^2 d\mu_g}{\int_{M^n} \phi^2 d\mu_g}, \tag{1}$$

where Ω is the completing Hilbert space of

$$\Omega_0 = \left\{\phi \in C^\infty(M^n) \,\middle|\, \int_{M^n} \phi\, d\mu_g = 0\right\}$$

under the norm $\|\phi\|_1^2 = \int_{M^n} \phi^2 d\mu_g + \int_{M^n} |\nabla\phi|_g^2 d\mu_g$.

We denote the geodesic ball with center p and radius r by $B(p,r)$ in the n-dim manifold M^n. Then the first Dirichlet eigenvalue $\lambda_1^D(B(p,r))$ of Laplacian operator

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can be denoted as:

$$\lambda_1^D(B(p,r)) = \inf_{\phi \in H_0^2(B(p,r))} \frac{\int_{B(p,r)} |\nabla\phi|^2 d\mu}{\int_{B(p,r)} \phi^2 d\mu}, \tag{2}$$

where $H_0^2(B(p,r))$ is the completing Hilbert space of $C_0^\infty(B(p,r))$ under the norm

$$\|\phi\|_1^2 = \int_{B(p,r)} \phi^2 d\mu + \int_{B(p,r)} |\nabla\phi|^2 d\mu.$$

The upper bound for the first eigenvalue of Laplacian operator are very useful in geometry analysis and PDE. In [1], S. Y. Cheng used the approach of Jacobi fields to obtain an upper bound for the first eigenvalue $\lambda_1(g)$ of Laplacian operator as follows:

Theorem 1.1 (Cheng). *Let (M^n, g) be a compact Riemannian manifold satisfying $Rc \geq 0$, then*

$$\lambda_1(g) \leq \frac{C_n}{d_{M^n}^2},$$

where $C_n = 2n(n+4)$ and d_{M^n} is the diameter of M^n.

On the other hand, for studying the Ricci flow, in [2] Perelman defined a new functional named as $\mathscr{F}$ functional

$$\mathscr{F}(g,f) = \int_{M^n} (R + |\nabla f|^2) e^{-f} d\mu,$$

and proved a monotonicity property of the $\mathscr{F}$ functional along the Ricci flow.

Theorem 1.2 (Perelman). *Let $(M^n, g(t))$, $t \in [0, T)$, be a solution to the Ricci flow on closed manifold M^n*

$$\frac{\partial}{\partial t} g_{ij} = -2R_{ij},$$

and the function f satisfies

$$\frac{\partial}{\partial t} f = -R - \Delta f + |\nabla f|^2, \tag{3}$$

then we have the following monotonicity property

$$\frac{d}{dt}\mathscr{F}(g(t), f(t)) = 2\int_{M^n} |Rc + \nabla\nabla f|^2 e^{-f} d\mu \geq 0, \tag{4}$$

and if the equality in (4) *holds then $(M^n, g(t))$ is a steady gradient Ricci soliton such that $Rc(g(t)) + \nabla^{g(t)}\nabla^{g(t)} f = 0$.*

Remark 1. Theorem 1.2 states that the manifold $(M^n, g(t))$ being a steady gradient Ricci soliton is a necessary condition of $\frac{d}{dt}\mathscr{F}(g(t), f(t)) = 0$. Furthermore, in this paper, we will prove that it is actually a sufficient condition.

Theorem 1.3. *Let $(M^n, g(t))$ be a steady gradient Ricci soliton which is a special solution to the Ricci flow on closed manifold M^n*

$$Rc(g(t)) + \nabla^{g(t)}\nabla^{g(t)} f = 0, \tag{5}$$

then

$$\frac{d}{dt}\mathscr{F}(g(t), f(t)) = 2\int_{M^n} |Rc + \nabla\nabla f|^2 e^{-f} d\mu = 0.$$

Then by using Theorem 1.3, we present a monotonicity property of the first eigenvalue of the Laplacian operator on Ricci solitons by using the monotonicity formula of the $\mathscr{F}$ functional as follows:

Theorem 1.4. *Let $(M^n, g(t))$ be a steady gradient Ricci soliton on closed manifold M^n*

$$Rc(g(t)) + \nabla^{g(t)}\nabla^{g(t)} f = 0$$

with positive scalar curvature. Then there exists a time T_0, such that when $t > T_0$ we have $\lambda_1(g(t)) \le \lambda_1(g(0))$.

Corollary 1.5. *Let $(M^n, g(t))$ be a steady gradient Ricci soliton* (5) *with positive scalar curvature, which is a special solution to the Ricci flow on closed manifold M^n. Then the time of blowing up for the Ricci flow satisfies*

$$T_{Blow} \le \frac{n}{2}\left(\frac{1}{\inf\limits_{x\in M^n} R(g(0))} - \frac{1}{4\lambda_1(g(0)) + \sup\limits_{x\in M^n} R(g(0))}\right).$$

Then recall that the 2-positive curvature operator is defined as follows:

Definition 1.6 (2-positive curvature operator)**.** A Riemannian manifold (M^n, g) has 2-positive curvature operator if

$$\mu_\alpha(\mathscr{R}) + \mu_\beta(\mathscr{R}) > 0 \tag{6}$$

for arbitrary $\alpha \neq \beta$.

In [3], B hm and Wilking derived a convergence result of 2-positive curvature along the Ricci flow:

Theorem 1.7 (B hm-Wilking)**.** *On a compact manifold the normalized Ricci flow*

$$\frac{\partial}{\partial \tilde{t}}\tilde{g}_{ij} = -2\tilde{R}_{ij} + \frac{2}{n}\tilde{r}\tilde{g}_{ij}, \tag{7}$$

evolves a Riemannian metric with 2-positive curvature operator to a limit metric with constant sectional curvature.

Then by using Theorem 1.4 and 1.7, we can prove the following comparison theorem for the first eigenvalue of the Laplacian operator.

Theorem 1.8. *Let $(M^n, g(t))$ be a steady gradient Ricci soliton* (5) *on closed manifold M^n, and suppose that the curvature operator $\mathscr{R}(g(0))$ is a 2-positive curvature operator. Then along the Ricci flow, we have*

$$\lambda_1(g_{M_K^n}) \leq \exp\left(\frac{2}{n}\int_0^\infty r(\tau)d\tau\right)\lambda_1(g(0))$$

and

$$\frac{1}{2}\lambda_1(g_{M_K^n})\exp\left(-\frac{2}{n}\int_0^t r(\tau)d\tau\right) \leq \lambda_1(g(t)) \leq \frac{3}{2}\lambda_1(g_{M_K^n})\exp\left(-\frac{2}{n}\int_0^t r(\tau)d\tau\right),$$

where $r(\tau)$ is the average scalar curvature of the metric $g(\tau)$ and M_K^n is the space form with the constant curvature K.

The paper is organized as follows: In section 2, we prove Theorem 1.3 and 1.4 by using the monotonicity formula of the $\mathscr{F}$ functional. Based on this, in section 3, we prove Theorem 1.8.

2. **Proof of Theorem 1.3 and 1.4**

***Proof of Theorem* 1.3.** For the Ricci soliton (5), we have (see [4])

$$\frac{\partial}{\partial t}f(t) = |\nabla^{g(t)}f(t)|^2_{g(t)}.$$

Furthermore, it follows from (5) that $R(g(t)) + \Delta_{g(t)}f(t) = 0$, which implies that

$$\frac{\partial}{\partial t}f(t) = -R(g(t)) - \Delta_{g(t)}f(t) + |\nabla^{g(t)}f(t)|^2_{g(t)}.$$

Hence the potential function f in the equation of Ricci soliton satisfies (3), then we can using the monotonicity formula (4) to the Ricci soliton (5) and derive that

$$\frac{d}{dt}\mathscr{F}(g(t), f(t)) = 2\int_{M^n} |Rc + \nabla\nabla f|^2 e^{-f} d\mu = 0.$$

This implies that for the steady gradient Ricci soliton (5), we have that $\mathscr{F}(g(t), f(t)) \equiv C$ for any time t, where C is a constant independent on the time t. □

Then by using Theorem 1.3 and the monotonicity formula of the $\mathscr{F}$ functional, we can prove Theorem 1.4.

***Proof of Theorem* 1.4.** Let $\varphi = e^{-\frac{f}{2}}$, where f is the corresponding function in Theorem 1.2. Then we can rewrite the $\mathscr{F}$ functional as follows:

$$\mathscr{F}(g, f) = \int_{M^n} (Re^{-f} + 4|\nabla e^{-\frac{f}{2}}|^2)d\mu = \int_{M^n} (R\varphi^2 + 4|\nabla\varphi|^2)d\mu$$
$$= \int_{M^n} (-4\Delta\varphi + R\varphi)\varphi d\mu.$$

Then by using Theorem 1.3 and the definition of the λ functional, we have

$$\begin{aligned}\lambda(g(t)) &= \inf\left\{\mathscr{F}(g(t),f(t))\,\middle|\, f\in C^\infty(M^n), \int_{M^n} e^{-f}d\mu = 1\right\}\\ &= \inf\left\{\int_{M^n}(-4\Delta\varphi + R\varphi)\varphi\, d\mu\,\middle|\, \varphi\in C^\infty(M^n), \int_{M^n}\varphi^2 d\mu = 1\right\}.\end{aligned}$$

According to the definition, we know that the λ functional is actually the first eigenvalue of the operator $-4\Delta + R$. Furthermore, for any fixed time t, we know that the infimum of $\lambda(g(t))$ can be attained by a function φ_t according to the proof of Perelman [2].

For any $\varepsilon > 0$ we consider the geodesic ball $B(p,\varepsilon)$ and $M^n\backslash B(p,\varepsilon)$ in the n-dim manifold M^n. We denote u_ε and v_ε as the first Dirichlet eigenfunctions of Laplacian operator corresponding to $B(p,\varepsilon)$ and $M^n\backslash B(p,\varepsilon)$, and define the following two functions:

$$\widetilde{u}_\varepsilon(x) = \begin{cases} u_\varepsilon(x), & x\in B(p,\varepsilon)\\ 0, & x\in M^n\backslash B(p,\varepsilon)\end{cases}, \qquad \widetilde{v}_\varepsilon(x) = \begin{cases} v_\varepsilon(x), & x\in M^n\backslash B(p,\varepsilon)\\ 0, & x\in B(p,\varepsilon)\end{cases}$$

This implies that the volume

$$Vol(\sup p\widetilde{u}_\varepsilon(x)\cap \sup p\widetilde{v}_\varepsilon(x)) = 0. \tag{8}$$

Thus

$$\frac{\int_{M^n}|\nabla\widetilde{u}_\varepsilon|^2 d\mu}{\int_{M^n}\widetilde{u}_\varepsilon^2 d\mu} = \frac{\int_{B(p,\varepsilon)}|\nabla u_\varepsilon|^2 d\mu}{\int_{B(p,\varepsilon)}u_\varepsilon^2 d\mu} = \lambda_1^D(B(p,\varepsilon)) \tag{9}$$

$$\frac{\int_{M^n}|\nabla\widetilde{v}_\varepsilon|^2 d\mu}{\int_{M^n}\widetilde{v}_\varepsilon^2 d\mu} = \frac{\int_{M^n\backslash B(p,\varepsilon)}|\nabla v_\varepsilon|^2 d\mu}{\int_{M^n\backslash B(p,\varepsilon)}v_\varepsilon^2 d\mu} = \lambda_1^D(M^n\backslash B(p,\varepsilon)). \tag{10}$$

Then we choose a constant C such that $\int_{M^n}(\widetilde{u}_\varepsilon + C\widetilde{v}_\varepsilon)d\mu = 0$. Thus by the definition of the first eigenvalue of Laplacian operator $\lambda_1(g)$ we have

$$\begin{aligned}\lambda_1(g) &\le \frac{\int_{M^n}|\nabla(\widetilde{u}_\varepsilon + C\widetilde{v}_\varepsilon)|^2 d\mu}{\int_{M^n}(\widetilde{u}_\varepsilon + C\widetilde{v}_\varepsilon)^2 d\mu} = \frac{\int_{M^n}|\nabla\widetilde{u}_\varepsilon|^2 d\mu + C^2\int_{M^n}|\nabla\widetilde{v}_\varepsilon|^2 d\mu}{\int_{M^n}\widetilde{u}_\varepsilon^2 d\mu + C^2\int_{M^n}\widetilde{v}_\varepsilon^2 d\mu}\\ &= \frac{\int_{B(p,\varepsilon)}|\nabla u_\varepsilon|^2 d\mu + C^2\int_{M^n\backslash B(p,\varepsilon)}|\nabla v_\varepsilon|^2 d\mu}{\int_{B(p,\varepsilon)}u_\varepsilon^2 d\mu + C^2\int_{M^n\backslash B(p,\varepsilon)}v_\varepsilon^2 d\mu},\end{aligned}$$

where the first equality we use (8). Then let $\varepsilon\to 0$, we have

$$\begin{aligned}\lambda_1(g) &\le \lim_{\varepsilon\to 0}\frac{\int_{B(p,\varepsilon)}|\nabla u_\varepsilon|^2 d\mu + C^2\int_{M^n\backslash B(p,\varepsilon)}|\nabla v_\varepsilon|^2 d\mu}{\int_{B(p,\varepsilon)}u_\varepsilon^2 d\mu + C^2\int_{M^n\backslash B(p,\varepsilon)}v_\varepsilon^2 d\mu}\\ &= \lim_{\varepsilon\to 0}\frac{\int_{M^n\backslash B(p,\varepsilon)}|\nabla v_\varepsilon|^2 d\mu}{\int_{M^n\backslash B(p,\varepsilon)}v_\varepsilon^2 d\mu}\end{aligned}$$

$$= \lim_{\varepsilon\to 0}\left(\inf\left\{\int_{M^n\backslash B(p,\varepsilon)}|\nabla\phi|^2 d\mu \,\middle|\, \int_{M^n\backslash B(p,\varepsilon)}\phi^2 d\mu = 1\right\}\right)$$
$$= \inf\left\{\int_{M^n}|\nabla\phi|^2 d\mu | \phi\in C^\infty(M^n), \int_{M^n}\phi^2 d\mu = 1\right\}.$$

where the second equality we use the fact that (10) satisfies for any $\varepsilon > 0$.

Let $\lambda_1(g(t))$ denote the first eigenvalue of the Laplacian operator for the time t, and φ_0 be the eigenfunction corresponding to the first eigenvalue $\lambda_1(g(0))$. Then the combination of Theorem 1.2, 1.3, the definition of $\lambda(g(t))$ functional and the fact

$$\lambda_1(g(t)) \le \inf\left\{\int_{M^n}|\nabla\phi|^2_{g(t)} d\mu_{g(t)} \,\middle|\, \phi\in C^\infty(M^n), \int_{M^n}\phi^2 d\mu_{g(t)} = 1\right\}$$

leads to

$$\begin{aligned}
&4\lambda_1(g(t)) + \int_{M^n} R(g(t))\varphi_t^2 d\mu_{g(t)}\\
&= \inf_{\int_{M^n}\phi^2 d\mu = 1}\int_{M^n} -4\phi\Delta_{g(t)}\phi d\mu_{g(t)} + \int_{M^n} R(g(t))\varphi_t^2 d\mu_{g(t)}\\
&\le \int_{M^n}(-4\Delta_{g(t)}\varphi_t + R(g(t))\varphi_t)\varphi_t d\mu_{g(t)}\\
&= \lambda(g(0))\\
&\le \int_{M^n}(-4\Delta_{g(0)}\varphi_0 + R(g(0))\varphi_0)\varphi_0 d\mu_{g(0)}\\
&= 4\lambda_1(g(0)) + \int_{M^n} R(g(0))\varphi_0^2 d\mu_{g(0)},
\end{aligned}$$

where we use the fact that φ_t is the eigenfunction corresponding to $\lambda(g(t))$.

On the other hand, by using the maximum principle and the following evolution equation $\frac{\partial R}{\partial t} = \Delta R + 2|Rc|^2 \ge \Delta R + \frac{2}{n}R^2$, we have that

$$R(g(t)) \ge \inf_{x\in M^n} R(g(t)) \ge \frac{n \inf\limits_{x\in M^n} R(g(0))}{n - 2t \inf\limits_{x\in M^n} R(g(0))}. \tag{11}$$

The combination of the above results we have

$$\begin{aligned}
4\lambda_1(g(t)) + \frac{n \inf\limits_{x\in M^n} R(g(0))}{n - 2t \inf\limits_{x\in M^n} R(g(0))} &\le 4\lambda_1(g(t)) + \int_{M^n} R(g(t))\phi_t^2 d\mu_{g(t)}\\
&= 4\lambda_1(g(0)) + \int_{M^n} R(g(0))\phi_0^2 d\mu_{g(0)}\\
&\le 4\lambda_1(g(0)) + \sup_{x\in M^n} R(g(0))
\end{aligned}$$

Since M^n is a closed manifold with positive scalar curvature, we can choose the time

$$T_0 = \frac{n}{2}\left(\frac{1}{\inf\limits_{x\in M^n} R(g(0))} - \frac{1}{\sup\limits_{x\in M^n} R(g(0))}\right)$$

such that when $t \geq T_0$, we have

$$\frac{n \inf\limits_{x\in M^n} R(g(0))}{n - 2t \inf\limits_{x\in M^n} R(g(0))} \geq \sup_{x\in M^n} R(g(0)).$$

Hence when $t \geq T_0$, it follows that $\lambda_1(g(t)) \leq \lambda_1(g(0))$. □

By using the proof of Theorem 1.4, we can actually estimate the time of blowing up exactly and prove the Corollary 1.5.

***Proof of Corollary* 1.5.** Firstly by (11), we derive that the time of blowing up satisfies

$$T_{Blow} \leq \frac{n}{2 \inf\limits_{x\in M^n} R(g(0))}, \tag{12}$$

which gives a upper bound of T_{Blow}. Furthermore, noting the proof of Theorem 1.4, we have

$$4\lambda_1(g(t)) + \frac{n \inf\limits_{x\in M^n} R(g(0))}{n - 2t \inf\limits_{x\in M^n} R(g(0))} \leq 4\lambda_1(g(0)) + \sup_{x\in M^n} R(g(0)).$$

Since the first eigenvalue of the Laplacian operator $\lambda_1(g(t)) > 0$ for any time t, we have

$$\frac{n \inf\limits_{x\in M^n} R(g(0))}{n - 2t \inf\limits_{x\in M^n} R(g(0))} \leq 4\lambda_1(g(0)) + \sup_{x\in M^n} R(g(0)).$$

Thus the time of blowing up satisfies

$$T_{Blow} \leq \frac{n}{2}\left(\frac{1}{\inf\limits_{x\in M^n} R(g(0))} - \frac{1}{4\lambda_1(g(0)) + \sup\limits_{x\in M^n} R(g(0))}\right),$$

which is an improvement version of (12). □

3. **Proof of Theorem 1.8**

In this section, we prove Theorem 1.8 by using Theorem 1.4 and 1.7.

***Proof of Theorem* 1.8.** For using the convergence result in Theorem 1.7, we deal with the normalized Ricci flow. Given a solution $g(t)$ of Ricci flow, the metrics $\widetilde{g}(\widetilde{t}) = c(t)g(t)$, where

$$c(t) = \exp\left(\frac{2}{n}\int_0^t r(\tau)d\tau\right)$$

and $\widetilde{t}(t)=\int_0^t c(\tau)d\tau$, are the solution of the normalized Ricci flow

$$\frac{\partial}{\partial\widetilde{t}}\widetilde{g}_{ij}=-2\widetilde{R}_{ij}+\frac{2}{n}\widetilde{r}\widetilde{g}_{ij},$$

with $\widetilde{g}(0)=g(0)$, where

$$\widetilde{r}(\widetilde{g}(\widetilde{t}))=\frac{\int_{M^n}R(\widetilde{g}(\widetilde{t}))d\mu_{\widetilde{g}(\widetilde{t})}}{\int_{M^n}d\mu_{\widetilde{g}(\widetilde{t})}}$$

is the average scalar curvature.

Hence, the relation between the first eigenvalue of Laplacian operator along the Ricci flow $\lambda_1(g(t))$ and the one along the normalized Ricci flow $\lambda_1(\widetilde{g}(\widetilde{t}))$ satisfies

$$\begin{aligned}
\lambda_1(\widetilde{g}(\widetilde{t}))&=\inf\left\{\frac{\int_{M^n}|\nabla\phi|^2_{\widetilde{g}(\widetilde{t})}d\mu_{\widetilde{g}(\widetilde{t})}}{\int_{M^n}\phi^2d\mu_{\widetilde{g}(\widetilde{t})}}\,\middle|\,\phi\in C^\infty(M^n),\int_{M^n}\phi d\mu_{\widetilde{g}(\widetilde{t})}=0\right\}\\
&=\inf\left\{\frac{\int_{M^n}|\nabla\phi|^2_{c(t)g(t)}d\mu_{c(t)g(t)}}{\int_{M^n}\phi^2d\mu_{c(t)g(t)}}\,\middle|\,\phi\in C^\infty(M^n),\int_{M^n}\phi d\mu_{c(t)g(t)}=0\right\}\\
&=\inf\left\{\frac{c(t)\int_{M^n}|\nabla\phi|^2_{g(t)}d\mu_{g(t)}}{\int_{M^n}\phi^2d\mu_{g(t)}}\,\middle|\,\phi\in C^\infty(M^n),\int_{M^n}\phi d\mu_{g(t)}=0\right\}\\
&=c(t)\lambda_1(g(t)),
\end{aligned}$$

where we use the fact that

$$\int_{M^n}\phi d\mu_{\widetilde{g}(\widetilde{t})}=c(t)^{\frac{n}{2}}\int_{M^n}\phi d\mu_{g(t)}=0$$

if and only if $\int_{M^n}\phi d\mu_{g(t)}=0$.

Thus by using Theorem 1.4, we have

$$\frac{\lambda_1(\widetilde{g}(\widetilde{t}))}{c(t)}\le\frac{\lambda_1(\widetilde{g}(0))}{c(0)}=\lambda_1(g(0)).$$

Since the curvature operator $\mathscr{R}(g(0))$ is a 2-positive curvature operator, then let $t\to\infty$, by Theorem 1.7 we have the metrics $g(t)$ convergent to a space form M^n_K with constant positive curvature K. Thus

$$\frac{\lambda_1(g_{M^m_K})}{\exp\left(\frac{2}{n}\int_0^\infty r(\tau)d\tau\right)}=\frac{\lim\limits_{\widetilde{t}\to\infty}\lambda_1(\widetilde{g}(\widetilde{t}))}{\lim\limits_{t\to\infty}c(t)}=\lim_{t\to\infty}\frac{\lambda_1(\widetilde{g}(\widetilde{t}))}{c(t)}\le\lambda_1(g(0)),$$

which implies that

$$\lambda_1(g_{M^n_K})\le\exp\left(\frac{2}{n}\int_0^\infty r(\tau)d\tau\right)\lambda_1(g(0)).$$

On the other hand, when t is large enough, we have

$$\frac{1}{2}\lambda_1(g_{M^n_K})\le\lambda_1(\widetilde{g}(\widetilde{t}))\le\frac{3}{2}\lambda_1(g_{M^n_K}),$$

which implies that

$$\frac{1}{2}\lambda_1(g_{M_K^n}) \leq c(t)\lambda_1(g(t)) \leq \frac{3}{2}\lambda_1(g_{M_K^n}).$$

Thus

$$\begin{aligned}\frac{1}{2}\lambda_1(g_{M_K^n})\exp\left(-\frac{2}{n}\int_0^t r(\tau)d\tau\right) &\leq \lambda_1(g(t)) \\ &\leq \frac{3}{2}\lambda_1(g_{M_K^n})\exp\left(-\frac{2}{n}\int_0^t r(\tau)d\tau\right).\end{aligned}$$

□

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