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Optimizing the Sum of Linear Absolute Value Functions on An Interval

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Abstract. In this paper we give a new result for solving the problem of optimizing the sum of absolute values in the form $|x - a_r|$ over any interval.

1. Introduction

Consider the following problem

Optimize

$$f(x) = \sum_{r=1}^{n} |x - a_r|, \text{ where } a_{r-1} < a_r \text{ for each } 2 \le r \le n$$

Over any given interval I.

This problem has different applications in different aspects such as digital communication and approximation techniques, see [2]. Also, Han-Lin and Chian-Son [1] solved obtained minimized this sum over the set of all real numbers using so-called goal programming. In [3], we obtained an explicit formula that gives the minimum of this sum over the set of all real numbers. In this paper we introduce and prove a theorem which directly gives the optimum value of $f(x)$ over any given interval. Our proof depends on rewriting f as a piecewise linear function. We do so by generalizing the case when $n = 2$, that is; $f(x) = |x - a_1| + |x - a_2|$, $a_1 < a_2$, to the case when n is any positive integer, that is;

$$f(x) = \sum_{r=1}^{n} |x - a_r|, \text{ where } a_{r-1} < a_r \text{ for each } 2 \le r \le n.$$

For the case when $n = 2$; if $f(x) = |x - a_1| + |x - a_2|$, $a_1 < a_2$ then

$$f(x) = \begin{cases} -(x - a_1); & x \le a_1 \\ x - a_1; & x > a_1 \end{cases} + \begin{cases} -(x - a_2); & x \le a_2 \\ x - a_2; & x > a_2 \end{cases}$$

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and hence

$$f(x)=\begin{cases}-(x-a_1)-(x-a_2); & x\le a_1\\ (x-a_1)-(x-a_2); & a_1<x\le a_2\\ (x-a_1)+(x-a_2); & x>a_2\end{cases}.$$

2. **The main results**

We start this section with the solution of the proposed problem when the interval I is of the form $[b_1,b_2]$, where $b_1<b_2$.

Theorem 2.1. *Consider the function* $f(x)=\sum_{r=1}^{n}|x-a_r|$ *over* $[b_1,b_2]$ *where* $a_{r-1}<a_r$ *for each* $2\le r\le n$, $b_1,b_2\in\mathbb{R}$. *Then*

A. *If* n *is odd, then* $f(x)$ *has an absolute maximum value at*

$$\begin{cases}x=b_1 & \text{if } b_2\le a_{\frac{n+1}{2}} \text{ or } (b_1<a_{\frac{n+1}{2}}<b_2 \text{ and } f(b_1)\ge f(b_2))\\ x=b_2 & \text{if } b_1\ge a_{\frac{n+1}{2}} \text{ or } (b_1<a_{\frac{n+1}{2}}<b_2 \text{ and } f(b_1)\le f(b_2))\end{cases}$$

and $f(x)$ *has an absolute minimum value at*

$$\begin{cases}x=b_1 & \text{if } b_1\ge a_{\frac{n+1}{2}}\\ x=b_2 & \text{if } b_2\le a_{\frac{n+1}{2}}\\ x=a_{\frac{n+1}{2}} & \text{if } b_1<a_{\frac{n+1}{2}}<b_2\end{cases}$$

B. *If* n *is even, then* $f(x)$ *has an absolute maximum value at*

$$\begin{cases}x=b_1 & \text{if } b_2\le a_{\frac{n}{2}} \text{ or } (b_1<a_{\frac{n}{2}} \text{ and } a_{\frac{n}{2}}<b_2\le a_{\frac{n}{2}+1})\\ & \text{or } (b_1<a_{\frac{n}{2}} \text{ and } b_2>a_{\frac{n}{2}+1} \text{ and } f(b_1)\ge f(b_2))\\ x=b_2 & \text{if } b_1\ge a_{\frac{n}{2}+1} \text{ or } (a_{\frac{n}{2}}\le b_1<a_{\frac{n}{2}+1} \text{ and } b_2>a_{\frac{n}{2}+1})\\ & \text{or } (b_1<a_{\frac{n}{2}} \text{ and } b_2>a_{\frac{n}{2}+1} \text{ and } f(b_1)\le f(b_2))\end{cases}$$

and $f(x)$ *has an absolute minimum value at*

$$\begin{cases}x=b_1 & \text{if } b_1\ge a_{\frac{n}{2}+1}\\ x=b_2 & \text{if } b_2\le a_{\frac{n}{2}}\\ x=t\ \forall\ t\in[a_{\frac{n}{2}},b_2] & \text{if } b_1\le a_{\frac{n}{2}} \text{ and } a_{\frac{n}{2}}<b_2\le a_{\frac{n}{2}+1}\\ x=t\ \forall\ t\in[b_1,a_{\frac{n}{2}+1}] & \text{if } a_{\frac{n}{2}}\le b_1<a_{\frac{n}{2}+1} \text{ and } b_2\ge a_{\frac{n}{2}+1}\\ x=t\ \forall\ t\in[a_{\frac{n}{2}},a_{\frac{n}{2}+1}] & \text{if } b_1\le a_{\frac{n}{2}} \text{ and } b_2\ge a_{\frac{n}{2}+1}\end{cases}$$

and $f(x)$ *is constant if* $a_{\frac{n}{2}}\le b_1<a_{\frac{n}{2}+1}$ *and* $a_{\frac{n}{2}}<b_2\le a_{\frac{n}{2}+1}$.

Proof. Our goal is to show that f is convex on $\mathbb{R}$ in both cases, either n is odd or n is even, and we will see that f has an absolute minimum value at $x=a_{\frac{n+1}{2}}$ when n is odd and it has an absolute minimum value at $x=t\ \forall\ t\in[a_{\frac{n}{2}},a_{\frac{n}{2}+1}]$ when n is even. After that we will restrict the natural domain of f to be the closed bounded interval $[b_1,b_2]$, and then we will discuss all possible situations of b_1, b_2

in relation with $a_{\frac{n+1}{2}}$ when n is odd and in relation with $a_{\frac{n}{2}}, a_{\frac{n}{2}+1}$ when n is even. First, we rewrite the function f as a piecewise linear function as follows:

$$f(x) = \begin{cases} -\sum_{r=1}^{n}(x - a_r) = g_1(x); & x \leq a_1 \\ \sum_{r=1}^{i}(x - a_r) - \sum_{r=i+1}^{n}(x - a_r) = g_{i+1}(x); & a_i < x \leq a_{i+1},\ i = 1, \dots, n-1 \\ \sum_{r=1}^{n}(x - a_r) = g_{n+1}(x); & x > a_n \end{cases}$$

Now, we consider the cases when n is odd and when n is even:

A. Let n be odd. Then the functions $g_1, \dots, g_{\frac{n+1}{2}}$ are strictly decreasing linear functions (each of them has x's with negative sign more than x's with positive sign). On the other hand, the functions $g_{\frac{n+3}{2}}, \dots, g_{n+1}$ are strictly increasing linear functions (each of them has x's with positive sign more than x's with negative sign). Since f is continuous on $\mathbb{R}$ (sum of continuous functions), then we can conclude that f is strictly decreasing over $(-\infty, a_{\frac{n+1}{2}}]$ and strictly increasing over $[a_{\frac{n+1}{2}}, \infty)$. This implies that $\min(f) = f(a_{\frac{n+1}{2}})$, that is; f has an absolute minimum value at $x = a_{\frac{n+1}{2}}$. We can see that f is convex on $\mathbb{R}$, and the general shape of f when n is odd appears in Figure 1. Now, let $x \in [b_1, b_2]$. When $b_2 \leq a_{\frac{n+1}{2}}$ then f is strictly decreasing over $[b_1, b_2]$, implies that f has an absolute maximum value at $x = b_1$ and has an absolute minimum value at $x = b_2$. When $b_1 < a_{\frac{n+1}{2}} < b_2$ then f is strictly decreasing over $[b_1, a_{\frac{n+1}{2}}]$, strictly increasing over $[a_{\frac{n+1}{2}}, b_2]$, which implies that f has an absolute maximum value at $x = b_1$ if $f(b_1) \geq f(b_2)$, and f has an absolute maximum value at $x = b_2$ if $f(b_1) \leq f(b_2)$, and moreover f has an absolute minimum value at $x = a_{\frac{n+1}{2}}$. When $b_1 \geq a_{\frac{n+1}{2}}$ then f is strictly increasing over $[b_1, b_2]$, which implies that f has an absolute maximum value at $x = b_2$ and has an absolute minimum value at $x = b_1$.

B. Let n be even. Then the functions $g_1, \dots, g_{\frac{n}{2}}$ are strictly decreasing linear functions, $g_{\frac{n}{2}+1}$ is a constant function, and the functions $g_{\frac{n}{2}+2}, \dots, g_{n+1}$ are strictly increasing linear functions. Since f is continuous on $\mathbb{R}$, then we can conclude that f is strictly decreasing over $(-\infty, a_{\frac{n}{2}}]$, is a constant over $[a_{\frac{n}{2}}, a_{\frac{n}{2}+1}]$, and is strictly increasing over $[a_{\frac{n}{2}+1}, \infty)$, this implies that $\min(f) = f(t)\, \forall t \in [a_{\frac{n}{2}}, a_{\frac{n}{2}+1}]$. We can see that f is convex on $\mathbb{R}$, and the general shape of f when n is even appears in Figure 2. Now, let $x \in [b_1, b_2]$. When $b_2 \leq a_{\frac{n}{2}}$ then f is strictly decreasing over $[b_1, b_2]$, implies that f has an absolute maximum value at $x = b_1$ and has an absolute minimum value at $x = b_2$. When $b_1 \geq a_{\frac{n}{2}+1}$ then f is strictly increasing over $[b_1, b_2]$, which implies that f has an absolute maximum value at $x = b_2$ and has an absolute minimum value at $x = b_1$. When $b_1 < a_{\frac{n}{2}}$, $a_{\frac{n}{2}} < b_2 \leq a_{\frac{n}{2}+1}$ then f is strictly decreasing over $[b_1, a_{\frac{n}{2}}]$ and constant over $[a_{\frac{n}{2}}, b_2]$, implies that f has an absolute maximum value at $x = b_1$ and has an absolute minimum value at $x = t\, \forall t \in [a_{\frac{n}{2}}, b_2]$. When $a_{\frac{n}{2}} \leq b_1 < a_{\frac{n}{2}+1}$, $a_{\frac{n}{2}} < b_2 \leq a_{\frac{n}{2}+1}$ then

$b_1, b_2 \in [a_{\frac{n}{2}}, a_{\frac{n}{2}+1}]$, since f is constant over $[a_{\frac{n}{2}}, a_{\frac{n}{2}+1}]$ when $x \in \mathbb{R}$ then f is constant when $x \in [b_1, b_2]$. In addition, when $a_{\frac{n}{2}} \leq b_1 < a_{\frac{n}{2}+1}$, $b_2 > a_{\frac{n}{2}+1}$ then f is constant over $[b_1, a_{\frac{n}{2}+1}]$ and strictly increasing over $[a_{\frac{n}{2}+1}, b_2]$, implies that f has an absolute maximum value at $x = b_2$ and has an absolute minimum value at $x = t\ \forall t \in [b_1, a_{\frac{n}{2}+1}]$. □

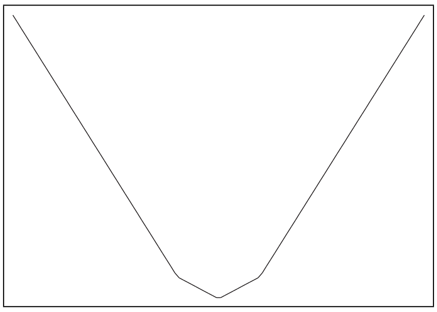

Figure 1
The general shape of f when n is odd

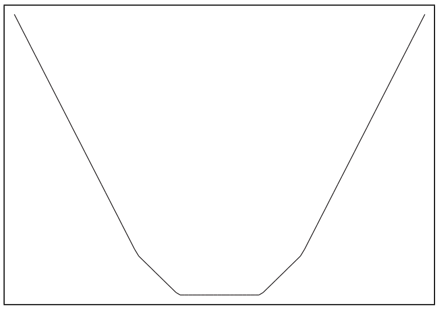

Figure 2
The general shape of f when n is even

Remark 2.2. The solution of the proposed problem is summarized in the following four tables for all other forms of the interval I. The proof of each one of them is similar to the proof of the previous theorem.

Table 1. n is odd and I is a finite interval

Interval	Conditions	Absolute max(f) at	Absolute min(f) at
$x \in (b_1, b_2]$	$b_2 \leq a_{\frac{n+1}{2}}$	None	$x = b_2$
	$b_1 < a_{\frac{n+1}{2}} < b_2$ and $f(b_1) \leq f(b_2)$	$x = b_2$	$x = a_{\frac{n+1}{2}}$
	$b_1 < a_{\frac{n+1}{2}} < b_2$ and $f(b_1) > f(b_2)$	None	$x = a_{\frac{n+1}{2}}$
	$b_1 \geq a_{\frac{n+1}{2}}$	$x = b_2$	None
$x \in [b_1, b_2)$	$b_2 \leq a_{\frac{n+1}{2}}$	$x = b_1$	None
	$b_1 < a_{\frac{n+1}{2}} < b_2$ and $f(b_1) < f(b_2)$	None	$x = a_{\frac{n+1}{2}}$
	$b_1 < a_{\frac{n+1}{2}} < b_2$ and $f(b_1) \geq f(b_2)$	$x = b_1$	$x = a_{\frac{n+1}{2}}$
	$b_1 \geq a_{\frac{n+1}{2}}$	None	$x = b_1$
$x \in I = (b_1, b_2)$	$a_{\frac{n+1}{2}} \in I$	None	$x = a_{\frac{n+1}{2}}$
	$a_{\frac{n+1}{2}} \notin I$	None	None

Table 2. n is odd and I is an infinite interval

Interval	Conditions	Absolute max(f) at	Absolute min(f) at
$(-\infty,\infty)$		None	$x=a_{\frac{n+1}{2}}$
$x\in(-\infty,b]$	$b\le a_{\frac{n+1}{2}}$	None	$x=b$
	$b>a_{\frac{n+1}{2}}$	None	$x=a_{\frac{n+1}{2}}$
$x\in[b,\infty)$	$b<a_{\frac{n+1}{2}}$	None	$x=a_{\frac{n+1}{2}}$
	$b\ge a_{\frac{n+1}{2}}$	None	$x=b$
$x\in I=(-\infty,b)$ or (b,∞)	$a_{\frac{n+1}{2}}\in I$	None	$x=a_{\frac{n+1}{2}}$
	$a_{\frac{n+1}{2}}\notin I$	None	None

Table 3. n is even and I is a finite interval

Interval	Conditions	Absolute max(f) at	Absolute min(f) at
$x\in(b_1,b_2]$	$b_2\le a_{\frac{n}{2}}$	None	$x=b_2$
	$b_1<a_{\frac{n}{2}}$ and $a_{\frac{n}{2}}<b_2\le a_{\frac{n}{2}+1}$	None	$x=t\ \forall t\in[a_{\frac{n}{2}},b_2]$
	$b_1<a_{\frac{n}{2}},b_2>a_{\frac{n}{2}+1}$ and $f(b_1)>f(b_2)$	None	$x=t\ \forall t\in[a_{\frac{n}{2}},a_{\frac{n}{2}+1}]$
	$b_1<a_{\frac{n}{2}},b_2>a_{\frac{n}{2}+1}$ and $f(b_1)\le f(b_2)$	$x=b_2$	$x=t\ \forall t\in[a_{\frac{n}{2}},a_{\frac{n}{2}+1}]$
	$b_1\ge a_{\frac{n}{2}}$ and $b_2\le a_{\frac{n}{2}+1}$	$x=t\ \forall t\in(b_1,b_2]$	$x=t\ \forall t\in(b_1,b_2]$
	$a_{\frac{n}{2}}\le b_1<a_{\frac{n}{2}+1}$ and $b_2>a_{\frac{n}{2}+1}$	$x=b_2$	$x=t\ \forall t\in(b_1,a_{\frac{n}{2}+1}]$
	$b_1\ge a_{\frac{n}{2}+1}$	$x=b_2$	None
$x\in[b_1,b_2)$	$b_2\le a_{\frac{n}{2}}$	$x=b_1$	None
	$b_1<a_{\frac{n}{2}}$ and $a_{\frac{n}{2}}<b_2\le a_{\frac{n}{2}+1}$	$x=b_1$	$x=t\ \forall t\in[a_{\frac{n}{2}},b_2)$
	$b_1<a_{\frac{n}{2}},b_2>a_{\frac{n}{2}+1}$ and $f(b_1)\ge f(b_2)$	$x=b_1$	$x=t\ \forall t\in[a_{\frac{n}{2}},a_{\frac{n}{2}+1}]$
	$b_1<a_{\frac{n}{2}},b_2>a_{\frac{n}{2}+1}$ and $f(b_1)<f(b_2)$	None	$x=t\ \forall t\in[a_{\frac{n}{2}},a_{\frac{n}{2}+1}]$
	$b_1\ge a_{\frac{n}{2}}$ and $b_2\le a_{\frac{n}{2}+1}$	$x=t\forall t\in[b_1,b_2)$, f is constant	$x=t\forall t\in[b_1,b_2)$
	$a_{\frac{n}{2}}\le b_1<a_{\frac{n}{2}+1}$ and $b_2>a_{\frac{n}{2}+1}$	None	$x=t\ \forall t\in[b_1,a_{\frac{n}{2}+1}]$
	$b_1\ge a_{\frac{n}{2}+1}$	None	$x=b_1$
$x\in(b_1,b_2)$	$b_2\le a_{\frac{n}{2}}$	None	None
	$b_1<a_{\frac{n}{2}}$ and $a_{\frac{n}{2}}<b_2\le a_{\frac{n}{2}+1}$	None	$x=t\ \forall t\in[a_{\frac{n}{2}},b_2)$
	$b_1<a_{\frac{n}{2}}$ and $b_2>a_{\frac{n}{2}+1}$	None	$x=t\forall t\in[a_{\frac{n}{2}},a_{\frac{n}{2}+1}]$
	$b_1\ge a_{\frac{n}{2}}$ and $b_2\le a_{\frac{n}{2}+1}$	$x=t\forall t\in(b_1,b_2)$, f is constant	$x=t\forall t\in(b_1,b_2)$
	$a_{\frac{n}{2}}\le b_1<a_{\frac{n}{2}+1}$ and $b_2>a_{\frac{n}{2}+1}$	None	$x=t\forall t\in(b_1,a_{\frac{n}{2}+1}]$
	$b_1\ge a_{\frac{n}{2}+1}$	None	None

Table 4. n is even and I is an infinite interval

Interval	Conditions	Absolute max(f) at	Absolute min(f) at
$(-\infty,\infty)$		None	$x=t\ \forall t\in[a_{\frac{n}{2}},a_{\frac{n}{2}+1}]$
$x\in(-\infty,b]$	$b\le a_{\frac{n}{2}}$	None	$x=b$
	$a_{\frac{n}{2}}<b\le a_{\frac{n}{2}+1}$	None	$x=t\ \forall t\in[a_{\frac{n}{2}},b]$
	$b>a_{\frac{n}{2}+1}$	None	$x=t\ \forall t\in[a_{\frac{n}{2}},a_{\frac{n}{2}+1}]$
$x\in[b,\infty)$	$b<a_{\frac{n}{2}}$	None	$x=t\ \forall t\in[a_{\frac{n}{2}},a_{\frac{n}{2}+1}]$
	$a_{\frac{n}{2}}\le b<a_{\frac{n}{2}+1}$	None	$x=t\ \forall t\in[b,a_{\frac{n}{2}+1}]$
	$b\ge a_{\frac{n}{2}+1}$	None	$x=b$
$x\in[-\infty,b)$	$b\le a_{\frac{n}{2}}$	None	None
	$a_{\frac{n}{2}}<b\le a_{\frac{n}{2}+1}$	None	$x=t\ \forall t\in[a_{\frac{n}{2}},b)$
	$b>a_{\frac{n}{2}+1}$	None	$x=t\ \forall t\in[a_{\frac{n}{2}},a_{\frac{n}{2}+1}]$
$x\in(b,\infty)$	$b<a_{\frac{n}{2}}$	None	$x=t\ \forall t\in[a_{\frac{n}{2}},a_{\frac{n}{2}+1}]$
	$a_{\frac{n}{2}}\le b<a_{\frac{n}{2}+1}$	None	$x=t\ \forall t\in(b,a_{\frac{n}{2}+1}]$
	$b\ge a_{\frac{n}{2}+1}$	None	None

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