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Research Article

Solution of a Class of Fourth Order Singular Singularly Perturbed Boundary Value Problems by Haar Wavelets Method and Quintic B-Spline Method

Kailash Yadav* and J.P. Jaiswal

Department of Mathematics, Maulana Azad National Institute of Technology, Bhopal 462003, India
***Corresponding author:** kailashyadaviitr@gmail.com

Abstract. In this paper, Haar wavelet method is described for solving a class of fourth order singular singularly perturbed boundary value problems. Its efficiency is tested by solving two examples for which the exact solution is known. Numerical comparisons have been carried out to demonstrate the efficiency and the performance of the proposed method.

Keywords. Boundary value problem, Haar wavelet; Quartic B-spline; Collocation point; Grid point

MSC. 65L10; 65L11; 65T60

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1. Introduction

The current paper is devoted to discuss Haar wavelet technique for finding the numerical solutions of fourth order singular singularly perturbed boundary value problems, given by

$$Ly(x) = \epsilon y^4(x) + \frac{a}{x}y'''(x) + \frac{b}{x}y''(x) + \frac{c}{x}y'(x) + \frac{d}{x}y(x) = f(x), \quad x \in (0,1), \tag{1.1}$$

$$y(0) = p, \quad y(1) = r, \quad y''(0) = q, \quad y''(1) = s, \quad p,\, q,\, r,\, s \in R, \tag{1.2}$$

where ϵ is a small positive parameter ($0 < \epsilon \ll 1$) premultiplying the higher derivative of the differential equation and $f(x)$ is sufficiently smooth function and a, b, c and d are constants and p, q, r, s are known constants. Singularly perturbed problems are classified on the basis that how the order of the original differential equation is affected if one sets $\epsilon \to 0$. Singularly perturbed problems are very popular in the field of science and engineering e.g., quantum mechanics, optimal control, chemical reactor theory, fluid dynamics etc. (one can follow [2],[7],[6],[10],[11],[16],[17],[21],[20],[22]). There are three standard approaches to solve singularly perturbed boundary value problems numerically the ideal equation (1.1), namely, finite difference method ([1],[2],[10],[11],[17],[22]), finite element method [5] and spline approximation ([4],[15],[18],[19]). In this paper, we have used the technique of Haar wavelet method to approximate highest derivative appearing in the differential equation by Haar series and other derivatives are obtained through integration of Haar series. The integration of Haar wavelets is preferred because the differentiation of Haar wavelet always results impulse functions. Through integration we can expand differential equation into Haar Matrix H with Haar coefficient matrix of $2M \times 2M$ order on collocation points. The main idea of this technique is to convert a differential equation into algebraic one. In order to approximate the solution of differential equation, we collocate the algebraic equations at collocation points. The benefits of Haar wavelets transform are sparse matrix of representations than other existing method. In this article, the error analysis is mentioned that shows high order convergence can be achieved on increasing the value of M to obtain the required approximation.

The Haar wavelet family for $x \in [0,1)$, defined as ([8],[9],[12],[14]) The scaling function for the family of Haar wavelets is defined on the interval $[0,1)$ and is given as follows:

$$h_1(x) = \begin{cases} 1 & \text{for } x \in [0,1), \\ 0 & \text{elsewhere.} \end{cases} \tag{1.3}$$

All other functions in Haar wavelet family are defined on subintervals of $[0,1)$ and are given as follows:

$$h_i(x) = \begin{cases} 1 & \text{for } x \in [\xi_1, \xi_2), \\ -1 & \text{for } x \in [\xi_2, \xi_3), \\ 0 & \text{otherwise,} \end{cases} \tag{1.4}$$

where

$$\xi_1 = \frac{k}{m}, \quad \xi_2 = \frac{k+0.5}{m}, \quad \xi_3 = \frac{k+1}{m}, \quad i = 2,3,\ldots,2M. \tag{1.5}$$

The integer $m = 2^j$, where $j = 0, 1, \ldots, J$, and $M = 2^J$, and integer $k = 0, 1, \ldots, m-1$. The relation between i, m and k is given by $i = m + k + 1$. The integer k is translation parameter and j indicates the level of the wavelet. The maximal level of resolution is the integer J.

The function $h_2(x)$ is called the mother wavelet, and all other functions in the Haar wavelet family except the scaling function are generated from the mother wavelet by the operations of dilation and translation.

The Haar wavelet functions are orthogonal to each other because

$$\int_0^1 h_i(x)h_l(x)dx = \begin{cases} 2^{-j} & \text{for } l = i, \\ 0 & \text{for } l \neq i. \end{cases} \tag{1.6}$$

Any function $f(x)$ which is square integrable in the interval (0,1) can be expressed as an infinite sum of Haar wavelets in the form

$$f(x) = \sum_{i=1}^{\infty} a_i h_i(x). \tag{1.7}$$

The above series terminates at finite terms if $f(x)$ is piecewise constant or can be approximated as piecewise constant during each subinterval.

These integrals can be evaluated using (1.4), by doing it first four of them are given by

$$p_{i,1}(x) = \begin{cases} x - \xi_1 & \text{for } x \in [\xi_1, \xi_2), \\ \xi_3 - x & \text{for } x \in [\xi_2, \xi_3), \\ 0 & \text{otherwise}, \end{cases} \tag{1.8}$$

$$p_{i,2}(x) = \begin{cases} \frac{1}{2}(x - \xi_1)^2 & \text{for } x \in [\xi_1, \xi_2), \\ \frac{1}{4m^2} - \frac{1}{2}(\xi_3 - x)^2 & \text{for } x \in [\xi_2, \xi_3), \\ \frac{1}{4m^2} & \text{for } x \in [\xi_3, 1), \\ 0 & \text{otherwise}, \end{cases} \tag{1.9}$$

$$p_{i,3}(x) = \begin{cases} \frac{1}{6}(x - \xi_1)^3 & \text{for } x \in [\xi_1, \xi_2), \\ \frac{1}{4m^2}(x - \xi_2) + \frac{1}{6}(\xi_3 - x)^3 & \text{for } x \in [\xi_2, \xi_3), \\ \frac{1}{4m^2}(x - \xi_2) & for\ x \in [\xi_3, 0), \\ 0 & \text{otherwise}, \end{cases} \tag{1.10}$$

$$p_{i,4}(x) = \begin{cases} \frac{1}{24}(x - \xi_1)^4 & \text{for } x \in [\xi_1, \xi_2), \\ \frac{1}{8m^2}(x - \xi_2)^2 - \frac{1}{24}(\xi_3 - x)^4 + \frac{1}{192m^4} & \text{for } x \in [\xi_2, \xi_3), \\ \frac{1}{8m^2}(x - \xi_2)^2 + \frac{1}{192m^4} & for\ x \in [\xi_3, 0) \\ 0 & \text{otherwise}. \end{cases} \tag{1.11}$$

We also consider the following notation:

$$C_{i,v} = \int_0^1 p_{i,v}(x)dx, \quad v = 1, 2, \ldots. \tag{1.12}$$

The rest of the paper is organized as follows: Section 2, general formulation of the numerical technique based on Haar wavelets. Just before final section, we consider two numerical problems for comparison with existing methods. Finally, in the last section, we give the concluding remarks.

2. Haar Wavelet Method for Solving Fourth Order Differential Equations

To apply Haar wavelet method for problem (1.1) we approximate highest order derivative $y^{(4)}(x)$ using Haar wavelet series as follows

$$y^{(4)}(x) = \sum_{i=1}^{2M} a_i h_i(x). \tag{2.1}$$

On integrating (2.1) and using the boundary conditions (1.2) with $a = 0$, $b = 1$, we can get $y'''(x)$, $y''(x)$, $y'(x)$ and finally $y(x)$ can be expanded in form of Haar wavelet series.

$$y'''(x) = y'''(0) + \sum_{i=1}^{2M} a_i p_{i,1}(x), \tag{2.2}$$

$$y''(x) = q + xy'''(0) + \sum_{i=1}^{2M} a_i p_{i,2}(x), \tag{2.3}$$

$$y'(x) = y'(0) + qx + \frac{x^2}{2} y'''(0) + \sum_{i=1}^{2M} a_i p_{i,3}(x), \tag{2.4}$$

$$y(x) = p + xy'(0) + q\frac{x^2}{2} + \frac{x^3}{6} y'''(0) + \frac{x^4}{24} y^{(4)}(0) + \sum_{i=1}^{2M} a_i p_{i,4}(x), \tag{2.5}$$

where $p_{i,1}$, $p_{i,2}$, $p_{i,3}$ and $p_{i,4}$ are defined in equations (1.8), (1.9), (1.10) and (1.11), respectively. The presence of two integration constants allow us the additional of two more equations which can be done by using particulars on the above equation and boundary conditions at both ends of the rule. Discretization using collocation points. $x_j = \frac{j-0.5}{2M}$, $j = 1, 2, \dots, 2M$ of the equations (2.1)-(2.5) can be reduced into the following matrix form

$$y^{(4)} = \begin{bmatrix} h_1(x_1) & \dots & h_{2M}(x_1) & 0 & 0 \\ h_1(x_2) & \dots & h_{2M}(x_2) & 0 & 0 \\ \vdots & \vdots & \vdots & & \\ h_1(x_{2M}) & \dots & h_{2M}(x_{2M}) & 0 & 0 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_{2M} \\ y'(0) \\ y'''(0) \end{bmatrix}, \tag{2.6}$$

$$y''' = \begin{bmatrix} p_{1,1}(x_1) & \dots & p_{2M,1}(x_1) & 0 & 1 \\ p_{1,1}(x_2) & \dots & p_{2M,1}(x_2) & 0 & 1 \\ \vdots & \vdots & \vdots & & \\ p_{1,1}(x_{2M}) & \dots & p_{2M,1}(x_{2M}) & 0 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_{2M} \\ y'(0) \\ y'''(0) \end{bmatrix}, \tag{2.7}$$

$$y'' = \begin{bmatrix} p_{1,2}(x_1) & \dots & p_{2M,2}(x_1) & 0 & x_1 \\ p_{1,2}(x_2) & \dots & p_{2M,2}(x_2) & 0 & x_2 \\ \vdots & \vdots & \vdots & & \\ p_{1,2}(x_{2M}) & \dots & p_{2M,2}(x_{2M}) & 0 & x_{2M} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_{2M} \\ y'(0) \\ y^{(3)}(0) \end{bmatrix} + \begin{bmatrix} q \\ q \\ \vdots \\ q \end{bmatrix}, \tag{2.8}$$

$$y' = \begin{bmatrix} p_{1,3}(x_1) & \dots & p_{2M,3}(x_1) & 1 & x_1^2/2 \\ p_{1,3}(x_2) & \dots & p_{2M,3}(x_2) & 1 & x_2^2/2 \\ \vdots & \vdots & \vdots & & \\ p_{1,3}(x_{2M}) & \dots & p_{2M,3}(x_{2M}) & 1 & x_{2M}^2/2 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_{2M} \\ y'(0) \\ y'''(0) \end{bmatrix} + \begin{bmatrix} qx_1 \\ qx_2 \\ \vdots \\ qx_{2M} \end{bmatrix}, \tag{2.9}$$

and

$$y = \begin{bmatrix} p_{1,4}(x_1) & \dots & p_{2M,4}(x_1) & x_1 & x_1^3/2 \\ p_{1,4}(x_2) & \dots & p_{2M,4}(x_2) & x_2 & x_2^3/2 \\ \vdots & \vdots & \vdots & & \\ p_{1,4}(x_{2M}) & \dots & p_{2M,4}(x_{2M}) & x_{2M} & x_{2M}^3/2 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_{2M} \\ y'(0) \\ y'''(0) \end{bmatrix} + \begin{bmatrix} p+qx_1^2/2 \\ p+qx_2^2/2 \\ \vdots \\ p+qx_{2M}^2/2 \end{bmatrix}. \tag{2.10}$$

The value of unknown term $y'''(0)$ and $y'(0)$ be calculated by integrating equation (2.2) and (2.4) from 0 to 1 and is given by

$$y'''(0) = s - q - \sum_{i=1}^{2M} a_i C_{i,1} \tag{2.11}$$

and

$$y'(0) = r - p - \frac{1}{3}q - \frac{1}{6}s + \sum_{i=1}^{2M} a_i \left(\frac{1}{6}C_{i,1} - C_{i,3}\right). \tag{2.12}$$

These values are substituted in the expressions (2.3), (2.5) in order to obtain system of equations whose and solution gives us the Haar coefficients. Babolian and Shahsavaran [3] have shown that the error bound is inversely proportional to the level of resolution of Haar wavelet. This ensures the convergence of Haar wavelet approximation when M is increased.

3. Numerical Examples

To demonstrate the applicability of the method, we consider the two linear singular perturbed problems, which have been widely discussed in the approximate and exact solutions are available for comparison. The computer characteristic is Microsoft Windows 10 Intel(R) Core(TM) i3 CPU M 380@ 2.53 GHz with 3.00 GB of RAM, 64-bit operating system throughout this paper. Here we use the software MATLAB R2014a, for numerical computing.

Example 3.1. Consider the following 4th order singular perturbation problem [13]:

$$-\epsilon y^{(4)} - \frac{1}{x}y(x) = f(x), \tag{3.1}$$

with boundary conditions

$$y(0) = 0,\ y(1) = 0,\ y''(0) = 0,\ y''(1) = 0, \tag{3.2}$$

where $f(x) = e^x\left\{\epsilon(8+7x+x^2)-(1-x)\right\} + \frac{2}{3}\epsilon(1-x^2)$.

The exact solution for the above example is

$$y(x) = x(1-x)e^x - \frac{2}{3}ex(1-x^2). \tag{3.3}$$

The numerical results for the Example 3.1 are presented in Table 1. Table 1 shows the maximum absolute errors for different values of M and ϵ. Tables 3-6 provide a comparison of maximum absolute errors along with quintic B-spline (QBSM) method discussed in [13] and it is concluded that the present method gives better results than QBSM. Figure 1, compares the exact and numerical solution of the Example 3.1 for $\epsilon = 0.0001$ and $M = 32$.

Example 3.2. Consider the another 4th order singular perturbation problem [13]:

$$\epsilon y^{(4)} + \frac{1}{x}y'' + \frac{1}{x}y(x) = f(x), \tag{3.4}$$

with boundary conditions

$$y(0) = 0,\ y(1) = 0,\ y''(0) = 0,\ y''(1) = 0, \tag{3.5}$$

where $f(x) = e^x\left\{\epsilon(x+4)+2+\frac{2}{x}\right\} - \frac{2}{x} + \frac{8}{3} - \frac{7}{2}e - x + (\frac{1}{3}-\frac{1}{2}e)x^2$.

The exact solution is given by

$$y(x) = xe^x + \left(\frac{2}{3}-\frac{1}{2}e\right)x - x^2 + \left(\frac{1}{3}-\frac{1}{2}e\right)x^3. \tag{3.6}$$

Table 6 shows the maximum absolute error of the Example 3.2 for different values of ϵ and M.

Tables 7-10 show the maximum absolute errors at collocation points along with the existing method QBSM and it is concluded that the present method gives better results than QBSM. Figure 2, compares the exact and numerical solutions of the Example 3.2 for $\epsilon = 0.0001$ and $M = 32$.

Table 1. Maximum absolute error of Example 3.1 for different value of M and various small value of ϵ.

$\epsilon = 10^{-K}$	$M = 8$	$M = 16$	$M = 32$	$M = 64$	$M = 128$	$M = 256$	$M = 512$
$K = 0$	4.12E-04	1.05E-04	2.65E-05	6.63E-06	1.66E-06	4.14E-07	1.04E-07
$K = 1$	3.40E-04	8.68E-05	2.18E-05	5.45E-06	1.36E-06	3.41E-07	8.52E-08
$K = 2$	1.30E-04	3.28E-05	8.25E-06	2.06E-06	5.16E-07	1.29E-07	3.23E-08
$K = 3$	2.44E-05	6.14E-06	1.55E-06	3.89E-07	9.72E-08	2.43E-08	6.08E-09
$K = 4$	3.52E-06	9.21E-07	2.32E-07	5.81E-08	1.45E-08	3.63E-09	9.08E-10
$K = 5$	4.48E-07	1.16E-07	2.95E-08	7.40E-09	1.85E-09	4.63E-10	1.16E-10
$K = 6$	4.81E-08	1.36E-08	3.39E-09	8.57E-10	2.14E-10	5.37E-11	1.34E-11
$K = 7$	4.88E-09	1.48E-09	3.66E-10	9.34E-11	2.34E-11	5.86E-12	1.47E-12
$K = 8$	4.88E-10	1.51E-10	4.06E-11	9.69E-12	2.45E-12	6.17E-13	1.54E-13
$K = 9$	4.88E-11	1.51E-11	4.19E-12	1.06E-12	2.54E-13	6.35E-14	1.59E-14
$K = 10$	4.88E-12	1.52E-12	4.21E-13	1.10E-13	2.66E-14	6.43E-15	1.68E-15

Table 2. Comparison of maximum absolute error of Example 3.1 for different value of M and $\epsilon = 10^{-K}$.

$\epsilon = 10^{-K}$	$M = 8$		$M = 16$	
	Lodhi et al. [13]	Present method	Lodhi et al. [13]	Present method
$K = 0$	8.51E-04	4.12E-04	2.21E-04	1.05E-04
$K = 1$	6.99E-04	3.40E-04	1.74E-04	8.68E-05
$K = 2$	2.62E-04	1.30E-04	6.62E-05	3.28E-05
$K = 3$	4.90E-05	2.44E-05	1.25E-05	6.14E-06
$K = 4$	7.52E-06	3.52E-06	1.86E-06	9.21E-07
$K = 5$	1.07E-06	4.48E-07	2.43E-07	1.16E-07
$K = 6$	1.30E-07	4.81E-08	2.87E-08	1.36E-08
$K = 7$	1.34E-08	4.88E-09	3.62E-09	1.48E-09
$K = 8$	1.35E-09	4.88E-10	3.82E-10	1.51E-10
$K = 9$	1.35E-10	4.88E-11	3.84E-11	1.51E-11
$K = 10$	1.35E-11	4.88E-12	3.84E-12	1.52E-12

Table 3. Comparison of maximum absolute error of Example 3.1 for different value of M and $\epsilon = 10^{-K}$.

$\epsilon = 10^{-K}$	$M = 32$		$M = 64$	
	Lodhi et al. [13]	Present method	Lodhi et al. [13]	Present method
$K = 0$	5.30E-05	2.65E-05	1.33E-05	6.62E-06
$K = 1$	4.36E-05	2.18E-05	1.09E-05	5.45E-06
$K = 2$	1.65E-05	8.25E-06	4.13E-06	2.06E-06
$K = 3$	3.11E-06	1.55E-06	7.78E-07	3.89E-07
$K = 4$	4.65E-07	2.32E-07	1.16E-07	5.81E-08
$K = 5$	5.94E-08	2.95E-08	1.48E-08	7.40E-08
$K = 6$	6.92E-09	3.39E-09	1.72E-09	8.57E-10
$K = 7$	7.59E-10	3.66E-10	1.89E-10	9.34E-11
$K = 8$	9.40E-10	4.06E-11	2.04E-11	9.69E-12
$K = 9$	1.01E-11	4.19E-12	2.43E-12	1.05E-12
$K = 10$	1.02E-12	4.21E-13	2.60E-13	1.10E-13

Table 4. Comparison of maximum absolute error of Example 3.1 for different value of M and $\epsilon = 10^{-K}$.

$\epsilon = 10^{-K}$	$M = 128$		$M = 256$	
	Lodhi et al. [13]	Present method	Lodhi et al. [13]	Present method
$K = 0$	3.31E-06	1.66E-06	8.29E-07	4.14E-07
$K = 1$	2.73E-06	1.36E-06	6.82E-07	3.41E-07
$K = 2$	1.03E-06	5.16E-07	2.57E-07	1.29E-07
$K = 3$	1.94E-07	9.72E-08	4.86E-08	2.43E-08
$K = 4$	2.90E-08	1.45E-08	7.24E-09	3.63E-09
$K = 5$	3.70E-09	1.85E-09	9.25E-10	4.63E-10
$K = 6$	4.30E-10	2.14E-10	1.07E-10	5.37E-11
$K = 7$	4.70E-11	2.34E-11	1.17E-11	5.86E-12
$K = 8$	4.98E-12	2.45E-12	1.24E-12	6.17E-13
$K = 9$	5.21E-13	2.54E-13	1.28E-13	6.35E-14
$K = 10$	5.70E-14	2.66E-14	3.50E-14	6.42E-15

Table 5. Comparison of maximum absolute error of Example 3.1 for different value of M and $\epsilon = 10^{-K}$.

$\epsilon = 10^{-K}$	$M = 512$	
	Lodhi et al. [13]	Present method
$K = 0$	1.99E-07	1.04E-07
$K = 1$	1.76E-07	8.52E-08
$K = 2$	4.39E-08	3.23E-08
$K = 3$	1.22E-08	6.08E-09
$K = 4$	1.28E-09	9.08E-10
$K = 5$	2.13E-10	1.16E-10
$K = 6$	2.65E-11	1.34E-11
$K = 7$	2.92E-12	1.47E-12
$K = 8$	2.94E-13	1.54E-13
$K = 9$	3.12E-14	1.59E-14
$K = 10$	3.25E-15	1.68E-15

Table 6. Maximum absolute error of Example 5.2 for different value of M and various small value of ϵ.

$\epsilon = 10^{-K}$	$M = 8$	$M = 16$	$M = 32$	$M = 64$	$M = 128$	$M = 256$	$M = 512$
$K = 0$	2.98E-05	7.51E-06	1.88E-06	4.70E-07	1.18E-07	2.94E-08	7.34E-09
$K = 1$	8.68E-06	2.12E-06	5.27E-07	1.31E-07	3.28E-08	8.21E-09	2.05E-09
$K = 2$	1.74E-06	7.14E-07	1.40E-07	3.47E-08	8.67E-09	2.17E-09	5.42E-10
$K = 3$	6.41E-08	3.14E-08	8.79E-09	2.27E-09	5.68E-10	1.43E-10	3.57E-11
$K = 4$	7.21E-08	1.95E-09	5.72E-10	2.05E-10	5.50E-11	1.40E-11	3.51E-12
$K = 5$	8.47E-08	5.03E-09	2.49E-10	3.04E-12	4.33E-12	1.32E-12	3.46E-13
$K = 6$	8.59E-08	5.35E-09	3.28E-10	1.88E-11	7.69E-13	6.11E-14	3.00E-14
$K = 7$	8.60E-08	5.38E-09	3.36E-10	2.08E-11	1.26E-12	6.87E-14	2.37E-15
$K = 8$	8.61E-08	5.38E-09	3.36E-10	2.10E-11	1.31E-12	8.12E-14	5.47E-15
$K = 9$	8.61E-08	5.38E-09	3.36E-10	2.10E-11	1.31E-12	8.24E-14	5.69E-15
$K = 10$	8.60E-08	5.38E-09	3.36E-10	2.10E-11	1.31E-12	8.23E-14	5.94E-15
$K = 10$	4.88E-12	1.52E-12	4.21E-13	1.10E-13	2.66E-14	6.43E-15	1.68E-15

Table 7. Comparison of maximum absolute error of Example 3.2 for different value of M and $\epsilon = 10^{-K}$.

$\epsilon = 10^{-K}$	$M = 8$		$M = 16$	
	Lodhi et al. [13]	Present method	Lodhi et al. [13]	Present method
$K = 0$	2.41E-04	2.95E-05	6.02E-05	7.51E-06
$K = 1$	8.12E-05	8.68E-06	1.77E-05	2.12E-06
$K = 2$	1.40E-05	1.74E-06	5.01E-05	7.14E-07
$K = 3$	6.50E-07	6.41E-08	2.58E-07	3.14E-08
$K = 4$	4.26E-07	7.21E-08	6.36E-09	1.95E-09
$K = 5$	5.26E-07	8.47E-08	3.02E-08	5.03E-09
$K = 6$	5.36E-07	8.59E-08	3.28E-08	5.35E-09
$K = 7$	5.37E-07	8.60E-08	3.30E-08	5.38E-09
$K = 8$	5.38E-07	8.61E-08	3.30E-08	5.38E-09
$K = 9$	5.38E-07	8.61E-08	3.30E-08	5.38E-09
$K = 10$	5.37E-07	8.60E-08	3.30E-08	5.38E-09

Table 8. Comparison of maximum absolute error of Example 3.2 for different value of M and $\epsilon = 10^{-K}$.

$\epsilon = 10^{-K}$	$M = 32$		$M = 64$	
	Lodhi et al. [13]	Present method	Lodhi et al. [13]	Present method
$K = 0$	1.50E-05	1.88E-06	3.76E-06	4.70E-07
$K = 1$	4.26E-06	5.27E-07	1.05E-06	1.31E-07
$K = 2$	1.10E-06	1.40E-07	2.73E-07	3.47E-08
$K = 3$	7.09E-08	8.79E-09	1.82E-08	2.27E-09
$K = 4$	5.20E-09	5.72E-10	1.68E-09	2.05E-10
$K = 5$	1.35E-09	2.49E-10	5.72E-11	3.04E-12
$K = 6$	1.98E-09	3.27E-10	1.11E-10	1.88E-11
$K = 7$	2.05E-09	3.36E-10	1.26E-10	2.08E-11
$K = 8$	2.05E-09	3.36E-10	1.28E-10	2.10E-11
$K = 9$	2.05E-09	3.36E-10	1.28E-10	2.10E-11
$K = 10$	2.05E-09	3.36E-10	1.28E-10	2.10E-11

Table 9. Comparison of maximum absolute error of Example 3.2 for different value of M and $\epsilon = 10^{-K}$.

$\epsilon = 10^{-K}$	$M = 128$		$M = 256$	
	Lodhi et al. [13]	Present method	Lodhi et al. [13]	Present method
$K = 0$	9.40E-07	1.77E-07	2.35E-07	2.94E-08
$K = 1$	2.63E-07	1.18E-07	6.56E-08	8.21E-09
$K = 2$	6.91E-08	8.67E-09	1.73E-08	2.17E-09
$K = 3$	4.56E-09	5.68E-10	1.14E-09	1.43E-10
$K = 4$	4.49E-10	5.50E-11	1.11E-10	1.40E-11
$K = 5$	3.71E-11	4.33E-12	1.07E-11	1.32E-12
$K = 6$	3.68E-12	7.68E-13	6.23E-13	6.11E-14
$K = 7$	7.55E-12	1.26E-12	3.80E-13	6.87E-14
$K = 8$	7.97E-12	1.31E-12	5.28E-13	8.12E-14
$K = 9$	7.99E-12	1.31E-12	5.37E-13	8.24E-14
$K = 10$	8.02E-12	1.31E-12	4.75E-13	8.25E-14

Table 10. Comparison of the absolute error for Example 3.2 for $M = 512$.

$\epsilon = 10^{-K}$	$M = 512$	
	Lodhi et al. [13]	Present method
$K = 0$	5.34E-08	7.34E-09
$K = 1$	1.50E-08	2.05E-09
$K = 2$	3.39E-09	5.42E-10
$K = 3$	2.81E-10	3.57E-11
$K = 4$	7.18E-12	3.51E-12
$K = 5$	1.08E-12	3.45E-13
$K = 6$	2.59E-13	3.00E-14
$K = 7$	4.90E-14	2.37E-15
$K = 8$	1.34E-14	5.47E-15
$K = 9$	9.71E-14	5.69E-15
$K = 10$	1.77E-15	5.94E-15

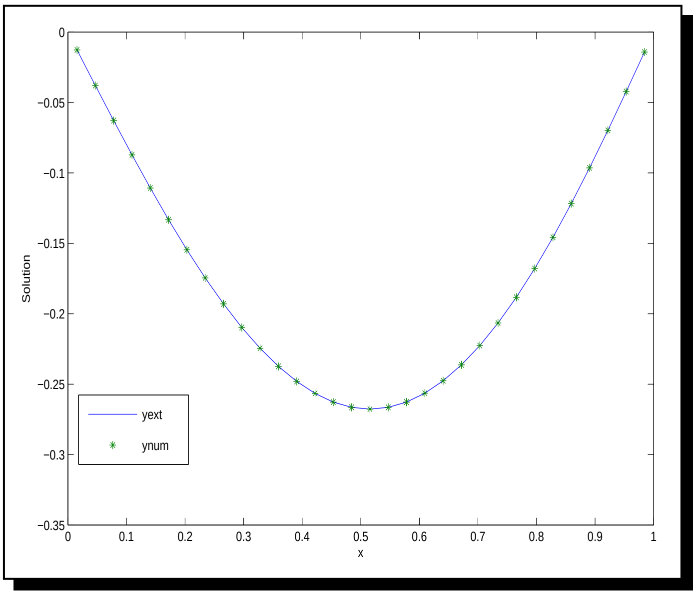

Figure 1. Physical behavior of exact and approximate solutions at collocation points of Example 3.1 for $\epsilon = 0.0001$ and $M = 32$.

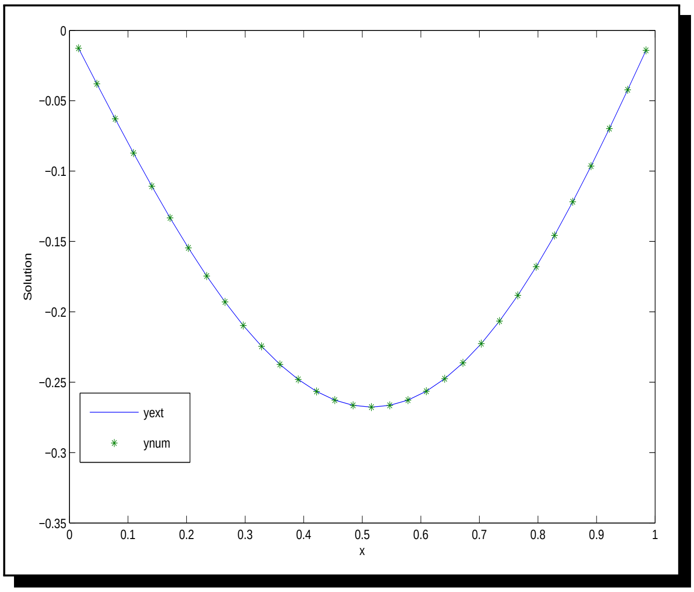

Figure 2. Physical behavior of exact and approximate solutions at collocation points of Example 3.2 for $\epsilon = 0.0001$ and $M = 32$.

4. Conclusion

In the present study, a numerical algorithm is developed using Haar wavelet method for solution of a class of fourth order singular singularly perturbed boundary value problems. The developed method has been utilized to improve a Solution of a class of fourth order singular singularly perturbed boundary value problems by quintic B-spline method. The proposed method is computationally efficient and the algorithm can be easily implemented on computer.

The Haar solution are very good in agreement with exact solutions available in the literature. The comparison with analytical solution shows that Haar wavelets gives better results with less computational cost. It is due to the sparsity of the transformation matrix and the small number of the wavelets coefficients.

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Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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