Journal of Informatics and Mathematical Sciences
Vol. 9, No. 3, pp. 657–663, 2017
ISSN 0975-5748 (online); 0974-875X (print)
Published by RGN Publications

http://www.rgnpublications.com

Proceedings of the Conference
Current Scenario in Pure and Applied Mathematics
December 22-23, 2016
Kongunadu Arts and Science College (Autonomous)
Coimbatore, Tamil Nadu, India

Research Article

On Achromatic Coloring of Corona Graphs

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Abstract. Let $G = (V(G), E(G))$ be a simple graph and an achromatic coloring of G is a proper vertex coloring of G in which every pair of colors appears on at least one pair of adjacent vertices. The achromatic number of G denoted by $\psi(G)$, is the greatest number of colors in an achromatic coloring of G. In this paper, we find out the achromatic number for Corona graph of Cycle with Path graphs on the same order n, Path with Cycle graphs on the same order n, Path with Complete graphs on the same order n, Path of order n with Star graph on order $n+1$, Path with Wheel graphs on the same order n and Ladder graph with Path graph on the same order n.

Keywords. Achromatic coloring; Achromatic number; Corona graph

MSC. 05C15

Received: January 7, 2017 **Accepted:** March 4, 2017

1. Introduction

Let G be a finite undirected graph with no loops and multiple edges. A coloring of a graph G is a partitioning of the vertex set V into color classes. In graph theory, coloring of graphs are very extended areas of research. A coloring of a graph can be described by a function that maps elements of a graph which are vertex-vertex coloring, edge-edge coloring or total coloring into some set of numbers which are called colors such that some property is satisfied. A proper coloring of the vertices of a graph G assigns a color to each vertex of G in such a way that no two adjacent vertices have the same color. The chromatic number $\chi(G)$ is the minimum number of color required in any proper coloring of G. An achromatic coloring [2] of a graph is a proper vertex coloring such that each pair of color classes is adjacent by at least one edge. The achromatic number was defined and studied by Harary, Hedetniemi and Prins [3]. They shown that, for every complete n-coloring τ of a graph G, there exists a complete homomorphism ϕ of G onto K_n and conversely. They considered the largest possible number of colors in an achromatic coloring is called the achromatic number and is denoted by ψ. The greatest number of colors used in a complete coloring of G is the achromatic number $\alpha(G)$ of G. It is clear that $\chi(G) \leq \alpha(G) \leq \psi(G)$.

Computing the achromatic number of a general graph was proved to be NP complete by Yannakakis and Gavril [10] in 1980. In 1976, Hell and Miller [9] who found the achromatic number of Paths and Cycles.

Graph coloring problem is expected to have wide variety of applications such as scheduling, frequency assignment in cellular networks, timetabling, etc.,

2. Preliminaries

Graph products are interesting and useful in many situations [7]. Let G_1 and G_2 be two graphs on disjoint sets of n_1 and n_2 vertices respectively. The Corona $G_1 \circ G_2$ of G_1 and G_2 is defined as the graph obtained by taking one copy of G_1 and n_1 copies of G_2 and then joining the ith vertex of G_1 to every vertex in the ith copy of G_2. The Corona of two graphs was first introduced by Frucht and Harary in 1970 [8].

In this paper, we find the achromatic number for the Corona graph of Cycle with Path graph on the same order $C_n \circ P_n$, Path with Cycle on the same order $P_n \circ C_n$, Path with Complete graph on the same order $P_n \circ K_n$, Path graph on order n with Star graph on the order $n+1$ say $P_n \circ K_{1,n}$ and Path graph with Wheel graph on the same order $P_n \circ W_n$ and Ladder graph with Path on the same order $L_n \circ P_n$.

3. Achromatic Number on Corona Graphs

Theorem 3.1. *For $n \geq 5$, the achromatic number of Corona of C_n with P_n is $n+4$.*
i.e., $\psi(C_n \circ P_n) = n+4$.

Proof. Let $v_1, v_2, \dots, v_n$ be the vertices of Cycle graph C_n and $u_1, u_2, \dots, u_n$ be the vertices of Path graph P_n.
i.e., $V(C_n) = \{v_i : 1 \leq i \leq n\}$ and $V(P_n) = \{u_1, u_2, \dots, u_n\}$.

By the definition of Corona graph, each vertex of C_n is adjacent to every vertex of a copy of P_n ie., every vertex of $V(C_n)$ is adjacent to every vertex from the set $V(P_n)$. Thus the Corona of two graphs can be defined as

$$V(C_n \circ P_n) = \{v_1, v_2, \dots, v_n\} \cup \{u_{ij} : 1 \le i \le n; 1 \le j \le n\}.$$

Let $P_n^{(1)}, P_n^{(2)}, \dots, P_n^{(n)}$ be the n-copies of the Path graph P_n.

Now assign a proper vertex coloring as follows:

Consider the color class $C = \{c_1, c_2, \dots, c_{n+4}\}$.

- For $1 \le i \le n$, assign the color c_i to v_i.
- For $P_n^{(i)}$, $i = 1,2,3$, $j = 2,3,\dots,n$, assign the color c_{n+j-1} to u_{ij}.
- Assign the colors c_{i+2} to u_{ij} for $i = 1,2,3$.
- For $V(P_n^i)$ for $i = 5,6,\dots,n$, let us assign the colors to contribute some pairs as follows:

  Assign c_{i+j+1} colors to all the vertices of $V(P_n^i)$ for $i = 5$ and $1 \le j \le n-2$.

  Assign c_{i+j+1} colors to all the vertices of $V(P_n^i)$ for $i = 6$ and $1 \le j \le n-3$.

  $\vdots \qquad \vdots \qquad \vdots$

  Assign c_{i+j+1} colors to all the vertices of $V(P_n^i)$ for $i = n$ and $1 \le j \le n-(n-3)$.

  For the remaining vertices of $V(P_n^i)$ for $i = 5,6\dots,n$, assign the colors c_1 and c_2 alternatively.

- In $V(P_n^i)$ for $i = 4$ of every copy, assign the colors as follows:

  Assign the colors c_{n+3}, c_{n+1}, c_{n+4} and c_{n+2} to u_{4j} $(1 \le j \le 4)$ and assign the colors c_1 and c_2 alternatively to the remaining vertices of V_4 to make the coloring as achromatic.

Now the coloring makes the non-adjacency condition is possible. Thus by the procedure of achromatic coloring, the coloring accommodates all the pairs of the color class and hence it is maximal. An easy check shows that the above said coloring is achromatic.

Hence, $\psi(C_n \circ P_n) = n+4$, for $n \ge 5$. □

Theorem 3.2. *For $n \ge 5$, the achromatic coloring of Corona of Path graph P_n with Cycle graph C_n is $n+4$.*
i.e., $\psi(P_n \circ C_n) = n+4$.

Proof. The proof is same as Theorem 3.1. □

Theorem 3.3. *For $n \ge 2$, the achromatic coloring of Corona of P_n with K_n is $2n$.*
i.e., $\psi(P_n \circ K_n) = 2n$.

Proof. Let $v_1, v_2, \dots, v_n$ be the vertices of Path graph P_n and $u_1, u_2, \dots, u_n$ be the vertices of Complete graph K_n.

i.e., $V(P_n) = \{v_1, v_2, \dots, v_n\}$ and $V(K_n) = \{u_1, u_2, \dots, u_n\}$.

Let $V(P_n \circ K_n) = \{v_i : 1 \le i \le n\} U \{u_{ij} : 1 \le i \le n; 1 \le j \le n\}$.

By the definition of Corona graph, each vertex of P_n is adjacent to every vertex of a copy of K_n i.e., every vertex $v_i \in V(P_n)$ is adjacent to every vertex from the set $\{u_{ij} : 1 \le i \le n; 1 \le j \le n\}$. Thus the Corona of two graphs Path with Complete graphs is

$$V(P_n \circ K_n) = \{v_i : 1 \le i \le n\} \cup \{u_{ij} : 1 \le i \le n; 1 \le j \le n\}.$$

Assign the following $2n$ coloring for $P_n \circ K_n$ as achromatic:

Let $K_n^{(1)}, K_n^{(2)}, \ldots, K_n^{(n)}$ be the n-copies of the Complete graph K_n.

Consider the color class $C = \{c_1, c_2, \ldots, c_{2n}\}$.

- For $1 \le i \le n$, assign the color c_i to v_i.
- For $1 \le i \le n-1$, assign the color c_{i+2} to u_{ij}.
- For $K_n^{(i)}$, where $1 \le i \le n-1$, $2 \le j \le n-1$, assign the color c_{n+j}.
- Assign the color c_i to all the vertices of $V(K_n^{(n)})$ for $i = n$ and $1 \le j \le n-1$.
- Assign the color c_{n+1} to the missing vertex in $V(K_n^{(i)})$ for $i = n$ and $j = n$.

Thus any pair in the color class is adjacent by atleast one edge and by the very construction this coloring accommodates maximum number of pairs in the color class.

Therefore, $\psi(P_n \circ K_n) = 2n$, for $n \ge 2$. □

Theorem 3.4. *For $n \ge 3$, the achromatic coloring of Corona of P_n with $K_{1,n}$ is $2n$. i.e., $\psi(P_n \circ K_{1,n}) = 2n$.*

Proof. Let $v_1, v_2, \ldots, v_n$ be the n-vertices of P_n and $u_1, u_2, \ldots, u_n$ be the n-vertices of $K_{1,n}$. i.e., $V(P_n) = \{v_1, v_2, \ldots, v_n\}$ and $V(K_{1,n}) = \{u_i, u_{ij} : 1 \le i \le n; 1 \le j \le n\}$.

By the definition of corona graph, each vertex of P_n is adjacent to every copy of $K_{1,n}$ i.e., every vertex $v_i \in V(P_n)$ is adjacent to every vertex from the set $\{u_{i,} u_{ij} : 1 \le i \le n; 1 \le j \le n\}$. Thus the Corona of two graphs is,

$$V(P_n \circ K_{1,n}) = \{v_i : 1 \le i \le n\} \cup \{u_i : 1 \le i \le n\} \cup \{u_{ij} : 1 \le i \le n; 1 \le j \le n\}.$$

Let $K_{1,n}^{(1)}, K_{1,n}^{(2)}, \ldots, K_{1,n}^{(n)}$ be the n-copies of the Star graph $K_{1,n}$.

Now assign a proper vertex coloring as follows:

Consider the color class $C = \{c_1, c_2, \ldots, c_{2n}\}$.

- Assign the color c_1 to v_1 and for $1 \le i \le n$, assign the color c_{1+2i} to v_{i+1}.
- For $1 \le i \le n$, color the vertices u_i with color c_{2i}.

To make the coloring as achromatic one, color the remaining vertices as follows:

- Color the vertices u_{ij} with color c_{ui+j} up to c_{2n}, $n \ge 3$, for $1 \le i \le n$ and $1 \le j \le n$.

Now this coloring will accommodates maximum number of pairs of the color class and an easy check shows that the above said coloring is achromatic.

Hence $\psi(P_n \circ K_{1,n}) = 2n$, for $n \ge 3$. □

Theorem 3.5. *For $n \ge 3$, the achromatic coloring of Corona of P_n with W_n is $2n$. i.e., $\psi(P_n \circ W_n) = 2n$.*

Proof. Let $v_1, v_2, \dots, v_n$ be the vertices of Path graph P_n and $u_1, u_2, \dots, u_n$ be the vertices of Wheel graph W_n with u_n as the hub.
i.e., $V(P_n) = \{v_1, v_2, \dots, v_n\}$ and $V(W_n) = \{u_1, u_2, \dots, u_n\}$.

By the definition of corona graph, each vertex of P_n is adjacent to every vertex of a cop of W_n i.e., every vertex of $V(P_n)$ is adjacent to every vertex from the set $V(W_n)$. Thus the Corona of two graphs Path with Wheel is,

$$V(P_n \circ W_n) = \{v_i : 1 \le i \le n\} \cup \{u_{ij} : 1 \le i \le n; 1 \le j \le n\}.$$

Let $G_1 = P_n$ be a Path graph with n-vertices and $G_2 = W_n$ be a Wheel graph with n-vertices. Now the corona $G = P_n \circ W_n$ is obtained by taking one copy of P_n of order n and n-copies of W_n and then joining the ith vertex of P_n to every vertex on the ith copy of W_n.

Let $W_n^{(1)}, W_n^{(2)}, \dots, W_n^{(n)}$ be the n-copies of the Wheel graph W_n.

Now we consider the following two cases:

Case (i): when n is odd
Consider the color class $C = \{c_1, c_2, \dots, c_{2n}\}$.

- Assign the color c_1 to v_1 and assign the color c_{1+2i} to the vertices v_{i+1} for $1 \le i < n$.
- Assign the color c_{2i} to u_{i1} for $1 \le i \le n$.
- Assign the color c_{ij} to u_{ij} and assign $c_{ui1+j-1}$ to u_{ij} up to c_{2n}, $n \ge 3$ for $1 \le i \le n$, $2 \le j \le n$.
    - Color the remaining vertices of u_{ij} with the color c_i for $1 \le i < n$.

This makes the above said coloring is achromatic one, since we have assigned colors to satisfy the definition of achromatic coloring.

Case (ii): when n is even

- Assign the color c_1 to v_1 and assign the color c_{1+2i} to the vertices v_{i+1} for $1 \le i < n$.
- Assign the color c_{2i} to u_{i1} for $1 \le i \le n$.
- For $1 \le i \le n/2$, $2 \le j \le n$, assign the color $c_{ui1+j-1}$ to u_{ij} up to c_{2n}, $n \ge 3$ and for the remaining u_{ij}'s, assign the color c_i for $1 \le i \le n$.

This coloring will accommodate all the missing pairs and an easy check shows that the above said coloring is achromatic. Therefore $\psi(P_n \circ K_{1,n}) = 2n$, for $n \ge 3$. □

Theorem 3.6. *For $n \ge 6$, the achromatic coloring of corona of L_n with P_n is $2n+2$.*
i.e., $\psi(L_n \circ P_{,n}) = 2n+2$.

Proof. Let $v_1, v_2, \dots, v_n$, $u_1, u_2, \dots, u_n$ be the vertices of the Ladder graph L_n and $p_{1,} p_2, \dots, p_n$ be the vertices of the Path graph P_n.
i.e., $V(L_n) = \{v_1, v_2, \dots, v_n, u_1, u_2, \dots, u_n\}$ and $(P_n) = \{p_{1,} p_{2,\dots,} p_n\}$.

Let $P_n^{(1)}, P_n^{(2)}, \dots, P_n^{(n)}$ be the n-copies of the Path graph P_n and p_{ij} be the corresponding vertices of each $P_n^{(i)}$ where $i = 1, 2, \dots, n$ and $j = 1, 2, \dots, n$.

Now by the definition of corona graph, each vertex of L_n is adjacent to every vertex in each copy of P_n i.e., every vertex $v_i, u_i \in V(L_n)$, $i = 1,2,\dots,n$ is adjacent to every vertex in each copy of $P_n^{(i)}$, $i = 1,2,\dots,n$. i.e.,

$$\begin{aligned} V(L_n \circ P_n) &= v_i \cup u_i \cup p_{ij} \\ &= \{v_{i:} 1 \le i \le n\} \cup \{u_{i:} 1 \le i \le n\} \cup \{p_{ij} : 1 \le i \le n \text{ and } 1 \le j \le n\}. \end{aligned}$$

Assign the following $2n+2$ coloring for $L_n \circ P_n$ as achromatic:

Consider the color class $C = \{c_1, c_2, \dots, c_{2n+2}\}$.

- For $1 \le i \le n$, assign the color c_i to v_i and for $1 \le i \le n$, assign the color c_{n+i} to $u_n, u_{n-1}, \dots, u_{n-(n-1)}$.
- Assign the color c_{i+2} to p_{i1}, where $1 \le i \le n$ for all v_i and $u_n, u_{n-1}, \dots, u_{n-(n-1)}$.
- For $P_n^{(i)}$, $1 \le i \le n/2$, $j = 2,3,\dots,n-2$, assign the color c_{i+j+2} and for $p_{i,n-1}$ and $p_{i,n}$, where $i = 1,2,\dots,2n-1$, assign the colors c_{2n+1} and c_{2n+2}.
- For all p_{ij}, assign the color c_{i+j+1}, where $n/2 < i \le n-1$ and $2 \le j \le n-2$. But for v_{n-i}, $i = 1,2,\dots,n-1$.
- For $n < i \le 2n-2$, $2 \le j \le 2n-i-1$, assign the color c_{i+j+1} for some p_{ij}, for $i > n$ and for the remaining vertices of p_{ij}, assign the color c_i where $1 \le i \le n$.
- For $i = 2n-1$,
  - (i) Assign the colors $c_1, c_3, c_5, \dots, c_{n-2}$, for $j = 3,5,7,\dots,n$.
  - (ii) Assign the colors $c_4, c_6, c_8, \dots, c_{n-1}$, for $j \equiv 0 \bmod 2$ but $j \ne 2$.
- For $i = 2n$,
  - (i) Assign the color $c_{2n+2} for j = 1$.
  - (ii) Assign the color c_j for $j \equiv 0 \bmod 2$.
  - (iii) Assign the color c_{j+2}, where $j = 3,5,7,\dots,n+2$.

If we add one more color we will miss some more pairs. This will contradict the non adjacency condition. Hence $\psi(L_n \circ P_{,n}) \le 2n+2$. This proves that this coloring is maximal and achromatic one. Therefore, $\psi(L_n \circ P_{,n}) = 2n+2$, for $n \ge 6$. □

4. Conclusion

In this paper, we have presented an achromatic coloring for corona graphs on several graphs which are Cycle with Path, Path and Cycle, Path with Complete, Path with Wheel and Ladder with Path on the same order and n th order Path graph with $n+1$ th order Star graph.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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