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Research Article

Some Results on Anti-Invariant Submanifolds of $(LCS)_N$-Manifold

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Abstract. The object of the present paper is to study anti-invariant submanifolds M of $(LCS)_n$-manifold $\bar{M}$. The basic equations are decomposed into horizontal and vertical homomorphisms and geometric properties of anti-invariant submanifolds are studied.

Keywords. Anti-invariant submanifold; $(\mathrm{LCS})_n$-manifold; Horizontal and vertical projections; Totally umbilical; Totally geodesic

MSC. 53C15; 53C20; 53C50

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1. Introduction

In 2003, the author [14] introduced the notion of Lorentzian concircular structure manifolds (briefly $(\mathrm{LCS})_n$-manifolds) with an example, which generalize the notion of LP-Sasakian manifolds introduced by Matsumoto [8]. Furthermore, $(LCS)_n$-manifolds have been studied by several authors (see for examples, [1, 5, 6, 12, 15, 16, 25]).

The research work on the geometry of invariant submanifolds of contact and complex manifolds is carried out by Kon [7] in 1973, C.S. Bagewadi [2] in 1982, Yano and Kano [29] in 1984, and others [9, 17–22, 30] etc. Also the study of geometry of anti-invariant submanifolds is carried out by [3, 10, 13, 28] in various contact manifolds. Motivated by the studies of the above authors, we study anti-invariant submanifolds of $(LCS)_n$-manifolds.

The paper is organized as follows: Section 2 consists of preliminaries of $(LCS)_n$-manifolds and in section 3, decomposition of basic equations of $(LCS)_n$-manifolds is carried out in horizontal and vertical projections and further results pertaining to geometric properties of the anti-invariant submanifolds are obtained.

2. Preliminaries

An n-dimensional Lorentzian manifold $\bar{M}$ is a smooth connected paracompact Hausdorff manifold with a Lorentzian metric g, that is $\bar{M}$ admits a smooth symmetric tensor field g of type $(0,2)$ such that for each point the tensor $g_p : T_p\bar{M} \times T_p\bar{M} \to R$ is a non-degenerate inner product of signature $(-,+,\dots,+)$, where $T_p\bar{M}$ denotes the tangent vector space of M at p and R is the real number space.

Definition 2.1. In a Lorentzian manifold $(\bar{M}, g)$ a vector field P defined by $g(X,P) = A(X)$ for any $X \in \Gamma(T\bar{M})$, is said to be a concircular vector field if

$$(\bar{\nabla}_X A)(Y) = \alpha[g(X,Y) + \omega(X)A(Y)],$$

where α is a non-zero scalar and ω is a closed 1-form and $\bar{\nabla}$ denotes the operator of covariant differentiation of $\bar{M}$ with respect to the Lorentzian metric g.

Let $\bar{M}$ admit a unit timelike concircular vector field ξ, called the characteristic vector field of the manifold, then we have $g(\xi,\xi) = -1$, since ξ is a unit concircular vector field, it follows that there exists a non-zero 1-form η such that $g(X,\xi) = \eta(X)$.

The equation of the following form holds

$$(\bar{\nabla}_X \eta)(Y) = \alpha[g(X,Y)\xi + \eta(X)\eta(Y)], \quad \alpha \neq 0,$$

$$\bar{\nabla}_X \alpha = X\alpha = d\alpha(X) = \rho\eta(X),$$

for all vector fields X,Y and α is a non-zero scalar function related to ρ by $\rho = -(\xi\alpha)$. Let us take $\phi X = \frac{1}{\alpha}\bar{\nabla}_X\xi$ from which it follows that ϕ is symmetric $(1,1)$ tensor and called the structure tensor manifold. Thus the Lorentzian manifold $\bar{M}$ together with unit time like concircular vector field ξ, its associated 1-form η and a $(1,1)$ tensor field ϕ is said to be a Lorentzian concircular structure manifold (briefly, $(LCS)_n$-manifold) [14]. Especially, if we take $\alpha = 1$ then we can obtain the LP-Sasakian structure of Matsumoto [8]. In $(LCS)_n$-manifold $(n > 2)$ the following relations hold.

$$\phi^2 = I + \eta \otimes \xi, \quad \eta(\xi) = -1, \tag{2.1}$$

where I denotes the identity transformation of the tangent space TM. Also in a $(LCS)_n$-manifold the following relations are satisfied

$$\phi\xi = 0, \quad \eta \cdot \phi = 0, \quad g(X,\phi Y) = g(\phi X, Y), \tag{2.2}$$

$$g(\phi X, \phi Y) = g(X,Y) + \eta(X)\eta(Y), \quad g(X,\xi) = \eta(X), \tag{2.3}$$

$$\bar{R}(X,Y)\xi = (\alpha^2 - \rho)[\eta(Y)X - \eta(X)Y], \tag{2.4}$$

$$\bar{R}(\xi,X)\xi = (\alpha^2-\rho)[\eta(X)\xi + X], \tag{2.5}$$

for $X,Y \in T(\bar{M})$.

Also $\bar{M}(\phi,\xi,\eta,g)$ an almost contact metric structure is a $(LCS)_n$-manifold if

$$(\bar{\nabla}_X\bar{\phi})Y = \alpha[g(X,Y)\xi + 2\eta(X)\eta(Y)\xi + \eta(Y)X] \tag{2.6}$$

$$\bar{R}(X,Y)Z = \phi R(X,Y)Z + (\alpha^2-\rho)\{g(Y,Z)\eta(X) - g(X,Z)\eta(Y)\} \tag{2.7}$$

$$\bar{\nabla}_X\xi = \alpha\phi X. \tag{2.8}$$

Let M be a submanifold of $\bar{M}$. Let $T_x(M)$ and $T_x^{\perp}(M)$ denote the tangent and normal space of M at $x \in M$ respectively. Then, the Gauss and Weingarten formulas are given by

$$\bar{\nabla}_X Y = \nabla_X Y + \sigma(X,Y) \tag{2.9}$$

$$\bar{\nabla}_X N = -A_N X + \nabla_X^{\perp} N \tag{2.10}$$

for any vector fields X,Y tangent to M and any vector field N normal to M, where $\bar{\nabla}$ and ∇ are the operators of covariant differentiation on $\bar{M}$ and M, $\nabla^{\perp}$ is the linear connection induced in the normal space $T_x^{\perp}(M)$. Both A_N and σ are called the shape operator and the second fundamental form and they are related as

$$\bar{g}(\sigma(X,Y),N) = g(A_N X,Y) \tag{2.11}$$

for any $X,Y \in TM$ and $N \in T^{\perp}M$.

A submanifold M of $(LCS)_n$-manifold $\bar{M}$ is said to be invariant if the structure vector field ξ of $\bar{M}$ is tangent to M and $\phi(T_x(M) \subset T_x(M)$, where $T_x(M)$ is the tangent space for all $x \in M$ and if $\phi(T_x(M) \subset T_x^{\perp}(M)$ where $T_x^{\perp}(M)$ is the normal space at $x \in M$ then M is said to be anti-invariant in $\bar{M}$. The submanifold M is called totally umbilical if $\sigma(X,Y) = g(X,Y)H$, where H is the mean curvature and if $\sigma(X,Y) = 0$ then M is said to be totally geodesic.

If M is an anti-invariant submanifolds of $(LCS)_n$-manifold $\bar{M}$. Then for every vector $\bar{Z}$ of $\bar{M}$ at a point of M, we put

$$\bar{Z} = \bar{Z}_t + \bar{Z}_n \tag{2.12}$$

where $\bar{Z}_t$ and $\bar{Z}_n$ are tangential and normal vectors to M, respectively. Define homomorphisms P and Q of the normal bundle of M respectively by

$$PN = (\phi N)_t, \quad QN = (\phi N)_n \tag{2.13}$$

for every normal vector field N of M.

If X is a vector field on an anti-invariant submanifold M, then ϕX is a vector field in the normal bundle of M.

Now, pre-multiplying ϕX, ϕN and ξ and comparing tangential and normal components, we get the following:

$$X + \eta(X)\xi_t = P\phi(X), \quad \eta(X)\xi_n = Q\phi X, \tag{2.14}$$

$$\eta(N)\xi_t = PQN, \quad N + \eta(N)\xi_n = \phi PN + Q^2N, \tag{2.15}$$

$$P\xi_n = 0, \quad P\xi_t + Q\xi_n = 0, \tag{2.16}$$

for any $X \in TM$ and $N \in T^{\perp}M$.

We study the case when characteristic vector field ξ of $\bar{M}$ is tangent and normal to M.

3. The Case in which ξ is Tangent to M

In this section we assume that ξ is tangent to M so $\xi_n = 0$ thus equation (2.14) gives

$$X + \eta(X)\xi = P\phi X, \quad Q\phi X = 0, \tag{3.1}$$

$$PQN = 0, \qquad N = \phi PN + Q^2 N, \tag{3.2}$$

for any $X \in TM$ and $N \in T^{\perp}M$.

From (3.1), we find that $Q^3 + Q = 0$ and hence Q defines f-structure in the normal bundle [26].

Lemma 3.1. *Let M be an anti-invariant submanifold of a $(LCS)_n$-manifold $\bar{M}$ such that ξ is tangent to M. Then*

$$-A_{\phi X}Y - P\sigma(X,Y) = \alpha[g(Y,X)\xi + 2\eta(Y)\eta(X)\xi + \eta(X)Y] \tag{3.3}$$

$$\nabla^{\perp}_Y \phi X - \phi(\nabla_Y X) - Q\sigma(X,Y) = 0 \tag{3.4}$$

Proof. From (2.6) for $X, Y \in TM$, we have

$$(\bar{\nabla}_Y \phi)X = \alpha\{g(Y,X)\xi + 2\eta(Y)\eta(X)\xi + \eta(X)Y\},$$

i.e.,

$$\bar{\nabla}_Y \phi X - \phi(\bar{\nabla}_Y X) = \alpha\{g(Y,X)\xi + 2\eta(Y)\eta(X)\xi + \eta(X)Y\}.$$

Since $\phi X \in T^{\perp}_x M$ for $X \in T_x M$, we have by (2.9) and (2.10) in L.H.S. of the above

$$-A_{\phi X}Y + \nabla^{\perp}_Y \phi X - \phi(\nabla_Y X) - \phi\sigma(X,Y) = \alpha\{g(Y,X)\xi + 2\eta(Y)\eta(X)\xi + \eta(X)Y\}. \tag{3.5}$$

Again using (2.12) in the above, we have

$$-A_{\phi X}Y + \nabla^{\perp}_Y \phi X - \phi(\nabla_Y X) - P\sigma(X,Y) - Q\sigma(X,Y) = \alpha\{g(Y,X)\xi + 2\eta(Y)\eta(X)\xi + \eta(X)Y\}.$$

Comparing tangential and normal components, we get (3.3) and (3.4), respectively. □

Lemma 3.2. *Let M be an anti-invariant submanifold of $(LCS)_n$-manifold $\bar{M}$ such that ξ is tangent to M. Then*

$$\nabla_X PN = P\nabla^{\perp}_X N + A_{QN}X, \tag{3.6}$$

$$Q\nabla^{\perp}_X N = \sigma(X, PN) + \phi(A_N X) + \nabla^{\perp}_X QN. \tag{3.7}$$

Proof. From (2.6) and for $X \in TM$ and $N \in T^{\perp}M$, i.e., $X, N \in T\bar{M}$, we have

$$(\bar{\nabla}_X \phi)N = \alpha\{g(X,N)\xi + 2\eta(X)\eta(N)\xi + \eta(N)X\},$$

$$\bar{\nabla}_X \phi N - \phi(\bar{\nabla}_X N) = 0.$$

Using (2.10) and (2.12) in the above, we have

$$\bar{\nabla}_X(PN) + \bar{\nabla}_X(QN) - \phi(-A_N X + \nabla^{\perp}_X N) = 0.$$

Again using (2.9) and (2.12), we have

$$\nabla_X PN + \sigma(X,PN) + -A_{QN}X + \nabla^{\perp}_X QN + \phi(A_N X) - P\nabla^{\perp}_X N - Q\nabla^{\perp}_X N = 0.$$

Equating tangential and normal components of the above we get (3.6) and (3.7), respectively. □

Lemma 3.3. *Let M be an anti-invariant submanifold of $(LCS)_n$-manifold $\bar{M}$ such that ξ is tangent to M. Then*

$$\nabla_X \xi = \alpha\phi X, \tag{3.8}$$

$$\sigma(X,\xi) = 0. \tag{3.9}$$

Further, if M is totally umbilical then M is totally geodesic.

Proof. From Gauss formula and (2.8), we have

$$\alpha\phi X = \bar{\nabla}_X \xi = \nabla_X \xi + \sigma(X,\xi) \tag{3.10}$$

Equating the tangential and normal components we get (3.8) and (3.9), respectively.

Let M be totally umbilical then $\sigma(X,Y) = g(X,Y)H$, where H is the mean curvature. By (3.8), we have $\sigma(X,\xi) = g(X,\xi)H = 0$, This implies $g(\xi,\xi)H = 0$ or $H = 0$, hence $\sigma(X,Y) = 0$. So by definition M is totally geodesic. □

Proposition 3.1. *Let M be an anti-invariant submanifold of $(LCS)_n$-manifold $\bar{M}$ such that ξ is tangent to M. Then*

(a) *P and Q are parallel along ξ.*

(b) *The directional derivative of ξ is normal to M and $\sigma(\xi,\xi)$ vanishes in the direction of ξ.*

Proof. (a) Taking $X = \xi$ in (3.6), we have

$$\nabla_\xi PN - P\nabla^{\perp}_\xi N = -A_{QN}\xi. \tag{3.11}$$

Let $X \in TM$ and taking inner product of the above equation with X, we have

$$g(\nabla_\xi PN - P\nabla^{\perp}_\xi N, X) = -g(A_{QN}\xi, X). \tag{3.12}$$

Using (2.11) and (3.9) in R.H.S. of the above, we have

$$g(A_{QN}\xi, X) = g(\sigma(X,\xi), QN) = \sigma(0,QN) = 0.$$

But

$$(\bar{\nabla}_\xi P)N = \bar{\nabla}_\xi PN - P(\bar{\nabla}_\xi N),$$

$$(\bar{\nabla}_\xi P)N = \nabla_\xi PN + \sigma(\xi,PN) - P(-A_N\xi + \nabla^{\perp}_\xi N). \tag{3.13}$$

Using (3.11) and (3.13)

$$(\bar{\nabla}_\xi P)N = \sigma(\xi,PN) + PA_N\xi + A_{QN}\xi. \tag{3.14}$$

Thus by virtue of (3.12), we have

$$g((\bar{\nabla}_\xi P)N, X) = 0.$$

This is true for all vector fields X. Hence

$$(\bar{\nabla}_\xi P)N = 0.$$

Similarly (3.7) gives

$$g(\bar{\nabla}_\xi Q)N, X) = 0,$$

for all vector fields X tangent to M.

Hence $(\bar{\nabla}_\xi Q)N = 0$, by the above. Therefore P and Q are parallel along ξ.

(b) Follows from (3.8) and (3.9). □

Proposition 3.2. *Let M be an anti-invariant submanifold of $(LCS)_n$-manifold $\bar{M}$ with ξ tangent to M. Then we have*

$$(\bar{\nabla}_X \Phi)(X,\xi) = -\alpha[\|X\|^2 + \eta^2(X)], \tag{3.15}$$

$$\bar{\nabla}_X \eta = 0, \tag{3.16}$$

where Φ is the fundamental 2-form given by $\Phi(X,Y) = g(\phi X, Y)$.

Proof. By definition of covariant derivative for $X,Y,Z \in TM$ we have

$$\begin{aligned}(\bar{\nabla}_X \Phi)(Y,Z) &= \bar{\nabla}_X \Phi(Y,Z) - \Phi(\bar{\nabla}_X Y, Z) - \Phi(Y, \bar{\nabla}_X Z)\\ &= \bar{\nabla}_X g(\phi Y, Z) - g(\phi(\bar{\nabla}_X Y), Z) - g(Y, \phi(\bar{\nabla}_X Z)\\ &= g(\bar{\nabla}_X \phi Y, Z) + g(\phi Y, \bar{\nabla}_X Z) - g(\phi(\bar{\nabla}_X Y), Z) - g(Y, \phi(\bar{\nabla}_X Z))\\ &= g((\bar{\nabla}_X \phi)Y, Z).\end{aligned}$$

Using (2.6) in the above

$$\begin{aligned}(\bar{\nabla}_X \Phi)(Y,Z) &= g(\alpha\{g(X,Y)\xi + 2\eta(X)\eta(Y)\xi + \eta(Y)X\}, Z)\\ &= \alpha[g(X,Y)\eta(Z) + 2\eta(X)\eta(Y)\eta(Z) + \eta(Y)g(X,Z)].\end{aligned}$$

Take $X = Y = X$, and $Z = \xi$ in the above and by virtue of (2.1), we have

$$\begin{aligned}(\bar{\nabla}_X \Phi)(X,\xi) &= \alpha[-g(X,X) - 2\eta^2(X) + \eta^2(X)]\\ &= -\alpha[\|X\|^2 + \eta^2(X)]\end{aligned}$$

and

$$\begin{aligned}(\bar{\nabla}_X \eta)(Y) &= (\bar{\nabla}_X \eta Y) - \eta(\bar{\nabla}_X Y)\\ &= \bar{\nabla}_X g(Y,\xi) - g(\bar{\nabla}_X Y, \xi)\\ &= g(\bar{\nabla}_X Y, \xi) + g(Y, \bar{\nabla}_X \xi) - g(\bar{\nabla}_X Y, \xi)\\ &= g(Y, \alpha\phi X) = \alpha g(Y, \phi X) = 0\end{aligned}$$

by virtue of (3.8), this is true for all vector fields Y and so $\bar{\nabla}_X \eta = 0$. □

We have the following geometric meaning from the Proposition 3.2.

Remark 3.1. (1) The volume $[\|X\|^2 + \eta^2(X)]$ of an anti-invariant submanifold M of $(LCS)_n$-manifold formed by the tangent vectors X and ξ is the derivative of the of the second fundamental form of ϕ on these vectors.

(2) The derivative of 1-form η dual to the characteristic vector ξ always vanishes in all directions of the anti-invariant submanifold M of $(LCS)_n$-manifold.

Proposition 3.3. *Let M be an anti-invariant submanifold of $(LCS)_n$-manifold $\bar{M}$ with ξ is tangent to M. If $A_N X = 0$ for any $N \in T_x^\perp M$ then $\phi T_x M$ is parallel with respect to the normal connection.*

Proof. Using Gauss and Weingarten formulas and equation (2.6), we have

$$\begin{aligned}\nabla_X^\perp \phi Y &= \bar{\nabla}_X \phi Y + A_{\phi Y} X \\ &= (\bar{\nabla}_X \phi)Y + \phi(\bar{\nabla}_X Y) + A_{\phi Y} X \\ &= \alpha[g(X,Y)\xi + 2\eta(X)\eta(Y)\xi + \eta(Y)X] + \phi\nabla_X Y + A_{\phi Y} X + \phi(\bar{\nabla}_X, Y)\end{aligned}$$

Since $A_N = 0$ for any $N \in T^\perp M$, in order to show that $\phi(T_x M)$ is parallel with respect to the normal connection, we have to show that for every local section ϕY in $\phi(T_x M)$, $\nabla_X \phi(Y)$ is also a local section in $\phi(T_x M)$, i.e., we have to show that

$$g(\nabla_X^\perp \phi Y, N) = 0.$$

Taking inner product of the above equation with N, we have

$$g(\nabla_X^\perp \phi Y, N) = g(\phi\nabla_X Y, N) + g(\phi(\sigma(X,Y)), N).$$

Using (2.3) in the above

$$g(\nabla_X^\perp \phi Y, N) = g(\nabla_X Y, \phi N) + g(\sigma(X,Y), \phi N) = g(\nabla_X Y, \phi N) + g(A_{\phi N} X, Y).$$

Also $\phi N \in T_X^\perp M$ for $N \in T_X^\perp M$.
Hence $g(\nabla_X^\perp \phi Y, N) = 0$. □

4. The Case in which ξ is Normal

In this section, we assume that ξ is normal to M so ξ_t=0 and (2.14) gives

$$\begin{aligned} &X = P\phi X, && Q\phi X = 0 \\ &PQN = 0, && N + \eta(N)\xi = \phi PN + Q^2 N \end{aligned}$$

for any $X \in TM$, $N \in T^\perp M$.

Lemma 4.1. *Let M be an anti-invariant submanifold of $(LCS)_n$-manifold $\bar{M}$ such that ξ is normal to M. Then*

$$-A_{\phi Y} X = Ph(X,Y), \tag{4.1}$$

$$\nabla_X^\perp \phi Y = \alpha g(X,Y)\xi + Q\sigma(X,Y) + \phi(\nabla_X Y). \tag{4.2}$$

Proof. Since ξ is normal to M, by virtue of (2.6) for $X, Y \in TM$, we have

$$(\bar{\nabla}_X \phi)Y = \alpha g(X,Y)\xi.$$

Simplifying (2.9) and (2.10), we have

$$\bar{\nabla}_X \phi Y - \phi(\bar{\nabla}_X Y) = \alpha g(X,Y)\xi$$
$$-A_{\phi Y}X + \nabla^{\perp}_X \phi Y - \phi(\nabla_X Y) - \phi\sigma(X,Y) = \alpha g(X,Y)\xi$$
$$-A_{\phi Y}X + \nabla^{\perp}_X \phi Y - \phi(\nabla_X Y - P\sigma(X,Y) - Q\sigma(X,Y) = \alpha g(X,Y)\xi$$

Comparing tangential and normal components we get (4.1) and (4.2), respectively. □

Lemma 4.2. *Let M be an anti-invariant submanifold of a $(LCS)_n$-manifold $\bar{M}$ such that ξ is normal to M. Then*

$$PA_N X + \nabla_X(PN) - A_{QN}X - P\nabla^{\perp}_X N - \alpha\eta(N)X = 0 \tag{4.3}$$
$$h(X,PN) - Q\nabla^{\perp}_X N + QA_N X + \nabla^{\perp}_X(QN) = 0 \tag{4.4}$$

for any $X \in TM$, $N \in T^{\perp}M$.

Proof. Using (2.6) for $X \in TM, N \in T^{\perp}M$, we have

$$(\bar{\nabla}_X \phi)N = \alpha\eta(N)X$$

Simplifying and using (2.9), (2.10) and (2.12), we have

$$\bar{\nabla}_X \phi N - \phi(\bar{\nabla}_X N) = \alpha\eta(N)X$$

i.e.,

$$\nabla_X(PN) + h(X,PN) + (-A_{QN}X + \nabla^{\perp}_X QN) + PA_N X$$
$$-P\nabla^{\perp}_X N + QA_N X - Q\nabla^{\perp}_X NX = \alpha\eta(N)X$$

Comparing tangential and normal components we get (4.3) and (4.4), respectively. □

Lemma 4.3. *Let M be an anti-invariant submanifold of a $(LCS)_n$-manifold $\bar{M}$ such that ξ is normal to M. Then*

$$A_\xi X = 0, \tag{4.5}$$
$$\nabla^{\perp}_X \xi = \alpha\phi X. \tag{4.6}$$

Further, M is totally geodesic.

Proof. From Weingarten formula

$$\bar{\nabla}_X \xi = -A_\xi X + \nabla^{\perp}_X \xi.$$

From (2.8), we have

$$-A_\xi X + \nabla^{\perp}_X \xi = \alpha\phi X.$$

Equating the tangential and normal components we have (4.5) and (4.6) for any $X \in TM$ and $\xi \in T^{\perp}_x M$.

By (4.5), we have

$$g(A_\xi X, Y) = 0$$
$$\Rightarrow \quad g(\sigma(X,Y),\xi) = 0$$

$$\Rightarrow \quad \sigma(X,Y) = 0.$$

Therefore M is totally geodesic. □

Lemma 4.4. *If M is an anti-invariant submanifold of a $(LCS)_n$-manifold $\bar{M}$ such that ξ is normal to M. Then the curvature tensor of the normal bundle annihilates ξ.*

Proof. Now

$$R^{\perp}(X,Y)\xi = \nabla^{\perp}_X \nabla^{\perp}_Y \xi) - \nabla^{\perp}_Y \nabla^{\perp}_X \xi - \nabla^{\perp}_{[X,Y]}\xi.$$

Using (4.6), we have

$$R^{\perp}(X,Y)\xi = \nabla^{\perp}_X(\alpha\phi Y) - \nabla^{\perp}_Y(\alpha\phi X) - \alpha\phi([X,Y]).$$

Now using (4.2) in the above

$$\begin{aligned} R^{\perp}(X,Y)\xi = {} & \alpha g(X,\alpha Y)\xi + Qh(X,\alpha Y) + \phi(\nabla_X \alpha Y) - \alpha g(Y,\alpha X)\xi \\ & - Qh(Y,\alpha X) - \phi(\nabla_Y \alpha X) - \alpha\phi([X,Y]). \end{aligned}$$

Simplifying the above

$$\begin{aligned} R^{\perp}(X,Y)\xi &= \phi(\nabla_X \alpha Y - \nabla_Y \alpha X - \alpha([X,Y]) \\ &= \phi((X\alpha)Y - (Y\alpha)X) \\ &= (X\alpha)\phi Y - (Y\alpha)\phi X \\ &= \rho[\eta(X)\phi Y - \eta(Y)\phi X] = 0. \end{aligned}$$

By definition of ρ Since ξ is normal to M for $X,Y \in TM$ $R^{\perp}(X,Y)\xi = 0$. □

Lemma 4.5. *Let M be an anti-invariant submanifold of $(LCS)_n$-manifold $\bar{M}$ such that ξ is normal to M. Then*

$$A_{\phi Y}X = P\sigma(X,Y), \tag{4.7}$$

$$\nabla^{\perp}_X(\phi Y) = \alpha g(X,Y)\xi - Q\sigma(X,Y) - \phi\nabla_X Y, \tag{4.8}$$

$$(A_{\phi N}X) + P(\nabla^{\perp}_X N) = -\alpha\eta(N)X, \tag{4.9}$$

$$\nabla^{\perp}_X(\phi N) + \phi(A_N X) = \phi(\nabla^{\perp}_X N), \tag{4.10}$$

for $X,Y \in TM$.

Proof. Using (2.10), we have

$$\begin{aligned} & \bar{\nabla}_X \phi Y = -A_{\phi Y}X + \nabla^{\perp}_X \phi Y \\ \Rightarrow \quad & (\bar{\nabla}_X \phi)Y - \phi(\bar{\nabla}_X Y) = -A_{\phi Y}X + \nabla^{\perp}_X \phi Y. \end{aligned} \tag{4.11}$$

Using (2.9), (2.6) and hypothesis in the above

$$\alpha[g(X,Y)\xi] - \phi(\nabla_X Y) - \phi(\sigma(X,Y)) = -A_{\phi Y}X + \nabla^{\perp}_X \phi Y.$$

Further using (2.12) in the above

$$\alpha[g(X,Y)\xi] - \phi(\nabla_X Y) - P\sigma(X,Y) - Q\sigma(X,Y) = A_{\phi Y}X + \nabla^{\perp}_X \phi Y.$$

Equating the tangential and normal components we obtain (4.7). Again using (2.10), we have

$$(\bar{\nabla}_X\phi N)-\phi(\bar{\nabla}_X N)=\alpha\eta(N)X.$$

Using (2.9), we have

$$-A_{\phi N}X+\nabla^{\perp}_X\phi N-\phi(-A_N X+\nabla^{\perp}_X N)=\alpha\eta(N)X$$

$$-A_{\phi N}X+\nabla^{\perp}_X\phi N+\phi(A_N X)-P(\nabla^{\perp}_X N)-Q(\nabla^{\perp}_X N)=\alpha\eta(N)X$$

Equating the tangential and normal components, we have the (4.9) and (4.10) □

Proposition 4.1. *Let M be an anti-invariant submanifold of $(LCS)_n$-manifold $\bar{M}$ such that ξ is normal to M. Then M is flat in the normal direction if and only if $\bar{M}$ is a space of curvature $-(\alpha^2-\rho)$.*

Proof. Using (2.4), (2.7), (3.8), (4.6), (4.8) and simplifying we have

$$\phi\bar{R}(X,Y)Z+(\alpha^2-\rho)[g(Y,Z)\phi(X)-g(X,Z)\phi(Y)]=R^{\perp}(X,Y)\phi Z$$

for any $X,Y,Z\in TM$. From (4.11), if M is flat in the normal direction then, $R^{\perp}=0$. Thus $\bar{M}$ is a space of curvature-$(\alpha^2-\rho)$.

Conversely, if $\bar{M}$ is a space of curvature $-(\alpha^2-\rho)$, then from (4.11), we have $R^{\perp}(X,Y)\phi Z=0$. Thus M is flat in the normal direction. □

Corollary 4.1. *If α= constant then $\rho=0$, it is seen that M is flat in the normal direction if and only if $\bar{M}$ is a space of constant curvature-α^2.*

Remark 4.1. If $\alpha=1$, $(LCS)_n$-manifold reduces to LP-Sasakian manifold and the results proved are also true for LP-Sasakian manifold.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

References

[1] S.R. Ashoka, C.S. Bagewadi and G. Ingahhalli, A Geometry on Ricci solitons in $(LCS)_n$ manifolds, *Differential Geometry-Dynamical Systems* **16** (2014), 50 – 62, DOI: http://vectron.mathem.pub.ro/dgds/v16/D16-as-768.pdf.

[2] C.S. Bagewadi, On totally real submanifolds of a Kahlerian manifold addmitting semisymmetric metric connection, *Indian J. Pure Appl. Math.* **13**(5) (1982), 528 – 536, DOI: https://www.insa.nic.in/writereaddata/UpLoadedFiles/IJPAM/20005a7d_528.pdf.

[3] A. Brasil, G.A. Lobos and M. Marlano, C-Totally real submanifolds with parallel mean curvature in λ-Sasakian space forms, *Matematica Contemporacnca Sociedade Brasileia de Matematicano*, **34**(c) (2008), `http://mc.sbm.org.br/docs/mc/pdf/34/a5.pdf`.

[4] B.Y. Chen, *Geometry of Submanifolds and its Applications*, Science University of Tokyo, Tokyo (1981).

[5] S.K. Hui and D. Chakraborty, Some types of Ricci Solitons on $(LCS)_n$ manifolds, *Journal of Mathematical Science Advances and Applications* **37** (2016), 1 – 17.

[6] S.K. Hui, M. Ateken and S. Nandy, Contact CR-Wrapped product $(LCS)_n$ manifolds, *Act. Math. Univ.* **1** (xxxv)(1.1) (2017), 101 – 109.

[7] M. Kon, Invariant submanifolds of normal contact metric manifolds, *Kodai Math. Sem. Rep.* **27** (1973), 330 – 336.

[8] K. Matsumoto, On Lorentzian almost paracontact manifolds with a structure of concircular type, *Bull. Yamagata Uni. Natur. Sci.* **12** (1989), 151 – 156.

[9] C. Ozgur and C. Murathan, On invariant submanifolds of Lorenzian para-Sasakian manifolds, *The Arabian J. Sci. Engg.* **34**(24) (2009), 177 – 185.

[10] H.B. Pandey and Anilkumar, Anti-invariant Submanifolds of almost para contact manifolds, *Indian J. Pure and Appl. Math.* **16**(6) (1985), 586 – 590, `https://www.insa.nic.in/writereaddata/UpLoadedFiles/IJPAM/20005a6a_586.pdf`.

[11] G.P. Pokhariyal and R.S. Mishra, Curvature tensors and their relativistics significance, *Yokohama Mathematical Journal* **18** (1970), 105 – 108, `http://hdl.handle.net/11295/38452`.

[12] D.G. Prakasha, On Ricci η-recurrent $(LCS)_n$-manifolds, *Acta Universitatis Apulensis* **24** (2010), 449 – 461, `https://www.emis.de/journals/AUA/acta24/Paper-11-Acta24.2010.pdf`.

[13] M.H. Shahid, Some results on anti-invariant submanifolds of a trans-Sasakian manifold, *Bull. Malays. Math. Sci.* **27**(2) (2004), 117 – 127.

[14] A.A. Shaikh, On Lorentzian almost paracontact manifolds with a structure of concircular type, *Kyungpook Math. J.* **43**(2) (2003), 305 – 314, `https://www.emis.de/journals/BMMSS/pdf/v27n2/v27n2p3.pdf`.

[15] A.A. Shaikh, Some results on $(\text{LCS})_n$-manifolds, *J. Korean Math. Soc.* **46**(3) (2009), 449 – 461, DOI: 10.4134/JKMS.2009.46.3.449.

[16] A.A. Shaikh, T. Basu and S. Eyasmin, On the existence of ϕ-Recurrent $(LCS)_n$-manifolds, *Extracta Mathematicae* **23**(1) (2008), 71 – 83, `https://www.eweb.unex.es/eweb/extracta/Vol-23-1/23B1Shai.pdf`.

[17] A.A. Shaikh, Y. Matsuyama and S.K. Hui, On invariant submanifolds of $(\text{LCS})_n$-manifolds, *Journal of the Egyptian Mathematical Society* **24**(2) (2016), 263 – 269, DOI: 10.1016/j.joems.2015.05.008.

[18] M.S. Siddesha and C.S. Bagewadi, Submanifold of a (k,μ)-Contact manifold, *CUBO A Mathematical Journal* **18**(01) (2016), 59 – 68, DOI: https://scielo.conicyt.cl/pdf/cubo/v18n1/art05.pdf.

[19] M.S. Siddesha and C.S. Bagewadi, On some classes of invariant submanifolds (k,μ)-contact manifold, *Journal of Informatics and Mathematical Sciences* **9**(1) (2017), 13 – 26, DOI: 10.26713/jims.v9i1.451.

[20] M.S. Siddesha, D. Nirmala and C.S. Bagewadi, On invariant submanifolds of Lorentzian β-Kenmotsu manifold, *Asian Journal of Mathematics and Computer Research* **19**(4) (2017), 203 – 213.

[21] S. Sular and C. Ozgur, On some submanifolds of Kenmotsu manifolds, *Chaos, Solitons and Fractals* **42** (2009), 1990 – 1995, DOI: 10.1016/j.chaos.2009.03.185.

[22] M.M. Tripathi, T. Sasahara and J.S. Kim, On invariant submanifolds of contact metric manifolds, *Tsukuba J. Math.* **29**(2) (2005), 495 – 510, https://www.jstor.org/stable/43686379?seq=1#page_scan_tab_contents.

[23] Venkatesha and C.S. Bagewadi, On concircular ϕ-recurrent LP-Sasakian manifolds, *Differ. Geom. Dyn. Syst.* **10** (2008), 312 – 319, URL: http://www.mathem.pub.ro/dgds/v10/D10-VE.pdf.

[24] Venkatesha, C.S. Bagewadi and K.T. Pradeep Kumar, Some results on Lorentzian para-Sasakian manifolds, *ISRN Geometry* **2011** (2011), Article ID 161523, 9 pages, DOI: 10.5402/2011/161523.

[25] K.R. Vidyavathi and C.S. Bagewadi, A study on $(LCS)_n$ manifolds admitting η-Ricci soliton, *JMI International Journal of Mathematical Science* **5** (2017), 47 – 52.

[26] K. Yano, On structure f-satisfying $f^3 + f = 0$, *Technical Report*, University of Washington, 12 (1961).

[27] K. Yano, On structure defined by a tensor field of type $(1,1)$ satisfying $f^3 + f = 0$, *Tensor* **14** (1963), 99 – 109.

[28] K. Yano and M. Kon, Anti-invariant submanifolds of a Sasakian space forms-II, *J. Korean Math. Soc.* **13** (1976), 1 – 14.

[29] K. Yano and M. Kon, *Structures on Manifolds*, World Scientific Publishing (1984).

[30] A. Yildz and C. Murathan, Invariant submanifolds of Sasakian space forms, *J. Geom.* **95** (2009), 135 – 150, DOI: 10.1007/s00022-009-0011-9.