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The Coefficients of the Polynomial Interpolation in Terms of Finite Differences and Numerical Differentiations

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Abstract. In this note, the polynomial interpolation of degree n passing through $n+1$ distinct points is considered. The coefficients of the polynomial interpolation are investigated in terms of finite differences and numerical differentiations. The coefficients are formulated by the use of divided differences and correlated with forward, backward differences and numerical differentiations. It is seen that the coefficients of the polynomial interpolation can be found and computed by using finite differences, numerical differentiations and generating special formulae for equidistant points or not.

1. Introduction

The polynomial interpolation plays an important role both in mathematics and applied sciences [1, 2]. The problem of interpolating is fairly common in many engineering and scientific applications [1, 2, 3, 4]. The polynomial interpolation is investigated by various methods and algorithms [3, 4, 5, 6].

The simplest and best known way to construct an nth-degree polynomial approximation $p_n(x)$ to a continuous function $y=f(x)$ in the interval $[a,b]\subset\Re$ is by interpolation. Let $\{x_0,x_1,\dots,x_n\}$ be $n+1$ distinct points in $[a,b]$, let $\{y_0,y_1,\dots,y_n\}$ be any set of $n+1$ real numbers. Then there exists a unique polynomial of degree at most n

$$p_n(x)=a_0+a_1x+a_2x^2+\cdots+a_nx^n \tag{1}$$

such that $p_n(x_i)=f(x_i)$ for (x_i,y_i), $0\le i\le n$ [1, 6]. There are many ways to represent this interpolating polynomial $p_n(x)=\sum_{i=0}^{n}a_ix^i$. Some representations are more useful for computation than others to find $p_n(x)$. The most commonly used polynomial interpolations are the Lagrange and Newton's forms. They are

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expressed as

$$p_n(x) = \sum_{i=0}^{n} y_i L_i(x), \tag{2}$$

$$p_n(x) = y_0 + \sum_{i=1}^{n} f[x_0, x_1, \cdots, x_i] \prod_{j=0}^{i-1}(x - x_j) \tag{3}$$

respectively, where $L_i(x)$'s are Lagrange polynomials and $f[x_0, x_1, \cdots x_i]$'s are divided differences [1, 2, 3, 4, 5, 6]. The most convenient form of these operations is usually $p_n(x)$ defined in (1) when given $n+1$ distinct points are large, but to find the values a_i's we must go through the effort of solving the system of coefficients of the polynomial interpolation satisfying $n+1$ distinct points or collecting coefficients of like powers of x from another form of $p_n(x)$ such as Lagrange and Newton's form [1, 2].

In this note, the coefficients $a_0, a_1, \cdots, a_n$ of the polynomial (1) are directly formulated by constructing relationships between coefficients and factorizations of Newton's interpolation formula (3). The formulations of coefficients are written in an open form by the use of divided differences. At equally spaced points, the coefficients of the interpolation in terms of forward, backward differences and numerical differentiations are given by using relations to divided differences. It is seen that the coefficients of polynomial interpolation can be found by the use of finite differences, numerical differentiations and computed by generating special formulae for $n + 1$ distinct equidistant points or not.

2. **The coefficients of the polynomial interpolation**

In this section, the polynomial (1) and Newton's form of the interpolating polynomial (3) of degree at most n passing through $n + 1$ distinct points are considered. Since there is a unique polynomial of degree n passing through $n + 1$ distinct points, the polynomials (1) and (3) are equal and the coefficients of them are the same.

The following result is obtained for the coefficients of polynomial $p_n(x)$ passing through $n + 1$ distinct points in the interval $[a, b]$.

Theorem 1. *Let a_i for $i = 0, 1, 2, \ldots, n$ be the coefficients of the polynomial* (1) *satisfying $n + 1$ distinct points. Then the coefficients of the interpolation of degree at most n are formulated as*

$$a_0 = y_0 - x_0 f[x_0, x_1] + x_0 x_1 f[x_0, x_1, x_2] - \cdots$$
$$+ (-1)^n x_0 x_1 \cdots x_{n-1} f[x_0, x_1, x_2, \cdots, x_n]$$

$$a_1 = f[x_0, x_1] - \left(\sum_{i_0=0}^{1} x_{i_0}\right) f[x_0, x_1, x_2] + \left(\sum_{i_0=0}^{1}\sum_{i_1=1}^{2} x_{i_0} x_{i_1}\right) f[x_0, x_1, x_2, x_3] + \cdots$$

$$+(-1)^{n-1}\left(\sum_{i_0=0}^{1}\sum_{i_1=1}^{2}\cdots\sum_{i_{n-2}=n-2}^{n-1} x_{i_0}x_{i_1}\cdots x_{i_{n-2}}\right)f[x_0,x_1,x_2,\cdots,x_n]$$

$$a_2 = f[x_0,x_1,x_2] - \sum_{i_0=0}^{2} x_{i_0} f[x_0,x_1,x_2,x_3] + \cdots$$

$$+(-1)^{n-2}\left(\sum_{i_0=0}^{2}\sum_{i_1=1}^{3}\cdots\sum_{i_{n-3}=n-3}^{n-1} x_{i_0}x_{i_1}\cdots x_{i_{n-3}}\right)f[x_0,x_1,x_2,\cdots,x_n]$$

$$\vdots$$

$$a_{n-2} = f[x_0,x_1,x_2,\cdots,x_{n-2}] - \sum_{i_0=0}^{n-2} x_{i_0} f[x_0,x_1,x_2,\cdots,x_{n-1}]$$

$$+\sum_{i_0=0}^{n-2}\sum_{i_1=1}^{n-1} x_{i_0}x_{i_1} f[x_0,x_1,x_2,\cdots,x_n]$$

$$a_{n-1} = f[x_0,x_1,x_2,\cdots,x_{n-1}] - \left(\sum_{i_0=0}^{n-1} x_{i_0}\right) f[x_0,x_1,x_2,\cdots,x_n]$$

$$a_n = f[x_0,x_1,x_2,\cdots,x_n] \tag{4}$$

where $0 \le i_0 < i_1 < \ldots < i_n \le n$ *and* $[i_r = i_{r-1}+1,\ r = 1,2,\ldots,n.$

Proof. The terms $\prod_{i=0}^{k-1}(x-x_i)$ for $k = 1,2,\ldots,n$ of Newton's form of interpolation (3) are written by the relationships with coefficients and variables x_i using fundamental theorem of algebra as follows:

$$(x-x_0)(x-x_1)\cdots(x-x_{k-1})$$

$$= x^k - \left(\sum_{i_0=0}^{k-1} x_{i_1}\right)x^{k-1} + \left(\sum_{i_0=0}^{k-2}\sum_{i_1=1}^{k-1} x_{i_0}x_{i_1}\right)x^{k-2} + \cdots + (-1)^k\prod_{i_0=0}^{k-1} x_{i_0}. \tag{5}$$

If the equation (5) for $k = 1,2,\ldots,n$ is replaced at polynomial (3), the polynomial interpolation is obtained as

$$P_n(x)$$

$$= y_0 + (x-x_0)f[x_0,x_1] + \{x^2 - (x_0+x_1)x + x_0x_1\}f[x_0,x_1,x_2]$$

$$+\{x^3 - (x_0+x_1+x_2)x^2 + (x_0x_1+x_0x_2+x_1x_2)x - x_0x_1x_2\}f[x_0,x_1,x_2,x_3] + \cdots$$

$$+\left\{x^n - \left(\sum_{i_0=0}^{n-1} x_{i_0}\right)x^{n-1} + \left(\sum_{i_0=0}^{n-2}\sum_{i_1=1}^{n-1} x_{i_0}x_{i_1}\right)x^{n-2} - \cdots + (-1)^n\prod_{i_0=0}^{n-1} x_{i_0}\right\}$$

$$\times f[x_0,x_1,x_2,\cdots x_n]$$

From rearranging this equation in terms of all power of x, the coefficients of the polynomial $p_n(x)$ are obtained as the equalities (4) and the proof is completed. □

When given $n+1$ distinct points are not equally spaced, it is seen that the coefficients of the polynomial interpolation can be found and computed by generating formulae (4).

The following example is solved by using these formulae.

Example 1. Consider $(-1,0),(0,1),(2,-3),(3,-20),(4,-55)$ five distinct points which are not equally spaced. The coefficients of polynomial interpolation degree at most four passing through five distinct points are obtained by writing $n=4$ in (4) as

$$\begin{aligned}
a_0 &= y_0 - x_0 f[x_0,x_1] + x_0x_1 f[x_0,x_1,x_2] - x_0x_1x_2 f[x_0,x_1,x_2,x_3] \\
&\quad + x_0x_1x_2x_4 f[x_0,x_1,x_2,x_3,x_4] = 1 \\
a_1 &= f[x_0,x_1] - \left(\sum_{i_0=0}^{1} x_{i_0}\right) f[x_0,x_1,x_2] \\
&\quad + \left(\sum_{i_0=0}^{1}\sum_{i_1=1}^{2} x_{i_0}x_{i_1}\right) f[x_0,x_1,x_2,x_3] \\
&\quad + \left(\sum_{i_0=0}^{1}\sum_{i_1=1}^{2}\sum_{i_2=2}^{3} x_{i_0}x_{i_1}x_{i_2}\right) f[x_0,x_1,x_2,x_3,x_4] = 2 \\
a_2 &= f[x_0,x_1,x_2] - \left(\sum_{i_0=0}^{2} x_{i_0}\right) f[x_0,x_1,x_2,x_3] \\
&\quad + \left(\sum_{i_0=0}^{2}\sum_{i_1=1}^{3} x_{i_0}x_{i_1}\right) f[x_0,x_1,x_2,x_4] = 0 \\
a_3 &= f[x_0,x_1,x_2,x_3] - \left(\sum_{i_0=0}^{3} x_{i_0}\right) f[x_0,x_1,x_2,x_3,x_4] = -1 \\
a_4 &= f[x_0,x_1,x_2,x_3,x_4] = 0.
\end{aligned}$$

Note that the value of the last divided difference is zero. This means that the degree of the polynomial interpolation is three less than it could have been based on the number of data points. The polynomial interpolation satisfying five distinct points is obtained as $P(x) = 1 + 2x - x^3$ from the formulations of coefficients.

The coefficients of the interpolation which are formulated in terms of divided differences can be defined by forward and backward differences, when the distinct points are equally spaced. The following results are concerned with these formulations of the coefficients.

Corollary 1. *If the $n+1$ distinct points $(x_0, y_0), (x_1, y_1), \cdots, (x_n, y_n)$ are equally spaced, then the coefficients in terms of forward differences of the interpolation of degree n are formulated as*

$$
\begin{aligned}
a_0 &= y_0 - \frac{1}{h} x_0 \Delta y_0 + \frac{1}{2!h^2} x_0 x_1 \Delta^2 y_0 - \frac{1}{3!h^3} x_0 x_1 x_2 \Delta^3 y_0 + \cdots \\
&\quad + (-1)^n \frac{1}{n!h^n} x_0 x_1 \cdots x_{n-1} \Delta^n y_0 \\
a_1 &= \frac{1}{h} \Delta y_0 - \frac{1}{2!h^2} \left(\sum_{i_0=0}^{1} x_{i_0} \right) \Delta^2 y_0 + \frac{1}{3!h^3} \left(\sum_{i_0=0}^{1} \sum_{i_1=1}^{2} x_{i_0} x_{i_1} \right) \Delta^3 y_0 + \cdots \\
&\quad + (-1)^{n-1} \frac{1}{n!h^n} \left(\sum_{i_0=0}^{1} \sum_{i_1=1}^{2} \cdots \sum_{i_{n-2}=n-2}^{n-1} x_{i_0} x_{i_1} \cdots x_{i_{n-2}} \right) \Delta^n y_0 \\
&\vdots \\
a_{n-2} &= \frac{1}{(n-2)!h^{n-2}} \Delta^{n-2} y_0 - \frac{1}{(n-1)!h^{n-1}} \left(\sum_{i_0=0}^{n-2} x_{i_0} \right) \Delta^{n-1} y_0 \\
&\quad + \frac{1}{n!h^n} \left(\sum_{i_0=0}^{n-2} \sum_{i_1=1}^{n-1} x_{i_0} x_{i_1} \right) \Delta^n y_0 \\
a_{n-1} &= \frac{1}{(n-1)!h^{n-1}} \Delta^{n-1} y_0 - \frac{1}{n!h^n} \left(\sum_{i_0=0}^{n-1} x_{i_0} \right) \Delta^n y_0 \\
a_n &= \frac{1}{n!h^n} \Delta^n y_0 ,
\end{aligned} \tag{6}
$$

where $x_{j+1} - x_j = h$ for $j = 0, 1, 2, \ldots, n-1$, $x_n = x_0 + nh$ and Δ forward difference operator.

Proof. Using the connection between forward differences and divided differences,

$$\Delta^k y_i = k!h^k f[x_i, x_{i+1}, x_{i+2}, \cdots, x_{i+k}],$$

the result is easily proved. □

Corollary 2. *Let $n+1$ distinct points $(x_0, y_0), (x_1, y_1), \cdots, (x_n, y_n)$ be equally spaced. Then the coefficients in terms of backward differences of the interpolation of degree n are obtained as follows:*

$$
\begin{aligned}
a_0 &= y_n - \frac{1}{h} x_n \nabla y_n + \frac{1}{2!h^2} x_{n-1} x_n \nabla^2 y_n - \frac{1}{3!h^3} x_{n-2} x_{n-1} x_n \nabla^3 y_n + \cdots \\
&\quad + (-1)^n \frac{1}{n!h^n} x_1 \cdots x_{n-1} x_n \nabla^n y_n
\end{aligned}
$$

$$a_1 = \frac{1}{h}\nabla y_n - \frac{1}{2!h^2}\left(\sum_{i_{n-1}=n-1}^{n} x_{i_{n-1}}\right)\nabla^2 y_n$$
$$+\frac{1}{3!h^3}\left(\sum_{i_{n-2}=n-2}^{n-1}\sum_{i_{n-1}=n-1}^{n} x_{i_{n-2}}x_{i_{n-1}}\right)\nabla^3 y_n+\cdots$$
$$+(-1)^{n-1}\frac{1}{n!h^n}\left(\sum_{i_1=1}^{2}\sum_{i_2=2}^{3}\cdots\sum_{i_{n-1}=n-1}^{n} x_{i_1}x_{i_2}\cdots x_{i_{n-1}}\right)\nabla^{n-1}y_n$$
$$\vdots$$
$$a_{n-2} = \frac{1}{(n-2)!h^{n-2}}\nabla^{n-2}y_n - \frac{1}{(n-1)!h^{n-1}}\left(\sum_{i_2=2}^{n} x_{i_2}\right)\nabla^{n-1}y_n$$
$$+\frac{1}{n!h^n}\left(\sum_{i_1=1}^{n-1}\sum_{i_2=2}^{n} x_{i_1}x_{i_2}\right)\nabla^n y_n$$
$$a_{n-1} = \frac{1}{(n-1)!h^{n-1}}\nabla^{n-1}y_n - \frac{1}{n!h^n}\left(\sum_{i_1=1}^{n} x_{i_1}\right)\nabla^n y_n$$
$$a_n = \frac{1}{n!h^n}\nabla^n y_n\,, \tag{7}$$

where $x_{j+1} - x_j = h$ for $j = 1, 2, \ldots, n-1$, $x_0 = x_n - nh$ and ∇ backward difference operator.

Proof. Newton's backward formula as interpolating polynomial for equidistant points is formulated as

$$p_n(x) = y_n + \frac{(x-x_n)}{h}\nabla y_n + \frac{(x-x_n)(x-x_{n-1})}{2!h^2}\nabla^2 y_n + \cdots$$
$$+\frac{(x-x_n)(x-x_{n-1})\cdots(x-x_1)}{n!h^n}\nabla^n y_n\,,$$

where $x = x_n - nh$. Using $\nabla^k y_i = k!h^k f[x_{i-k}, x_{i-k+1}, \cdots, x_{i-1}, x_i]$, and Corollary 1, the proof is completed. □

The coefficients of the polynomial interpolation can be investigated by the use of numerical differentiation. Using forward and backward formulae of numerical differentiation for equidistant points,

$$y_i^{(k)} \cong \frac{1}{h^k}\Delta^k y_i, \quad k = 1, 2, \cdots, n$$

and

$$y_i^{(k)} \cong \frac{1}{h^k}\nabla^k y_i, \quad k = 1, 2, \cdots, n$$

the coefficients of the polynomial (1) are computed by forward and backward numerical differentiations. If the numerical differentiations are replaced by the equations (6) and (7), the following results are easily obtained.

Corollary 3. *If the distinct points are equally spaced, then the coefficients in terms of forward formula of numerical differentiation of the polynomial* (1) *are formulated as*

$$
\begin{aligned}
a_0 &= y_0 - x_0 y_0' + \frac{1}{2!} x_0 x_1 y_0'' - \frac{1}{3!} x_0 x_1 x_2 y_0''' + \cdots \\
&\quad + (-1)^n \frac{1}{n!} x_0 x_1 \cdots x_{n-1} y_0^{(n)} \\
a_1 &= y_0' - \frac{1}{2!}\left(\sum_{i_0=0}^{1} x_{i_0}\right) y_0'' + \frac{1}{3!}\left(\sum_{i_0=0}^{1}\sum_{i_1=1}^{2} x_{i_0} x_{i_1}\right) y_0''' + \cdots \\
&\quad + (-1)^{n-1} \frac{1}{n!}\left(\sum_{i_0=0}^{1}\sum_{i_1=1}^{2}\cdots\sum_{i_{n-2}=n-2}^{n-1} x_{i_0} x_{i_1} \cdots x_{i_{n-2}}\right) y_0^{(n)} \\
&\vdots \\
a_{n-2} &= \frac{1}{(n-2)!} y_0^{(n-2)} - \frac{1}{(n-1)!}\left(\sum_{i_0=0}^{n-2} x_{i_0}\right) y_0^{(n-1)} + \frac{1}{n!}\left(\sum_{i_0=0}^{n-2}\sum_{i_1=1}^{n-1} x_{i_0} x_{i_1}\right) y_0^{(n)} \\
a_{n-1} &= \frac{1}{(n-1)!} y_0^{(n-1)} - \frac{1}{n!}\left(\sum_{i_0=0}^{n-1} x_{i_0}\right) y_0^{(n)} \\
a_n &= \frac{1}{n!} y_0^{(n)},
\end{aligned}
\tag{8}
$$

where $x_{j+1} - x_j = h$ *for* $j = 1, 2, \ldots, n-1$.

Corollary 4. *Let distinct points* $(x_0, y_0), (x_1, y_1), \cdots, (x_n, y_n)$ *be equally spaced. Then the coefficients in terms of backward formula of numerical differentiation of the polynomial* (1) *are formulated as follows:*

$$
\begin{aligned}
a_0 &= y_n - x_n y_n' + \frac{1}{2!} x_{n-1} x_n y_n'' - \frac{1}{3!} x_{n-2} x_{n-1} x_n y_n''' + \cdots \\
&\quad + (-1)^n \frac{1}{n!} x_1 \cdots x_{n-1} x_n y_n^{(n)} \\
a_1 &= y_n' - \frac{1}{2!}\left(\sum_{i_{n-1}=n-1}^{n} x_{i_{n-1}}\right) y_n'' + \frac{1}{3!}\left(\sum_{i_{n-2}=n-2}^{n-1}\sum_{i_{n-1}=n-1}^{n} x_{i_{n-2}} x_{i_{n-1}}\right) y_n''' + \cdots \\
&\quad + (-1)^{n-1} \frac{1}{n!}\left(\sum_{i_1=1}^{2}\sum_{i_2=2}^{3}\cdots\sum_{i_{n-1}=n-1}^{n} x_{i_1} x_{i_2} \cdots x_{i_{n-1}}\right) y_n^{(n)} \\
&\vdots
\end{aligned}
$$

$$a_{n-2} = \frac{1}{(n-2)!} y_n^{(n-2)} - \frac{1}{(n-1)!h^{n-1}} \left(\sum_{i_2=2}^{n} x_{i_2} \right) y_n^{(n-1)} + \frac{1}{n!h^n} \left(\sum_{i_1=1}^{n-1} \sum_{i_2=2}^{n} x_{i_1} x_{i_2} \right) y_n^{(n)}$$

$$a_{n-1} = \frac{1}{(n-1)!} y_n^{(n-1)} - \frac{1}{n!h^n} \left(\sum_{i_1=1}^{n} x_{i_1} \right) y_n^{(n)}$$

$$a_n = \frac{1}{n!} y_n^{(n)} \tag{9}$$

where $x_{j+1} - x_j = h$ *for* $j = 1, 2, \ldots, n-1$.

The coefficients $a_0, a_1, \cdots, a_n$ of the polynomial (1) are directly formulated by constructing relationships between coefficients and factorizations of Newton's interpolation formula (3). The formulations of coefficients are written in an open form by using divided differences. At equally spaced points, the coefficients of the interpolation in terms of forward, backward differences and numerical differentiations are given by using relations to divided differences. It is seen that the coefficients of polynomial interpolation can be found by using finite differences and numerical differentiations and computed by generating special formulae for $n+1$ distinct equidistant points or not.

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