Journal of Informatics and Mathematical Sciences
Vol. 10, Nos. 1 & 2, pp. 227–236, 2018
ISSN 0975-5748 (online); 0974-875X (print)
Published by RGN Publications
http://dx.doi.org/10.26713/jims.v10i1-2.777

http://www.rgnpublications.com

Research Article

Binet Forms Involving Golden Ratio and Two Variables: Convolution Identities

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Abstract. The irrational number $\Phi=\frac{1+\sqrt{5}}{2}$ or $\phi=\frac{-1+\sqrt{5}}{2}$ is well known as golden ratio. The binet forms $L_n=\Phi^n+(-\phi)^n$ and $F_n=\frac{\Phi^n-(-\phi)^n}{\sqrt{5}}$ define the well known Lucas and Fibonacci numbers. In the present paper, we generalize the binet forms $\Phi_n(x,y)=\frac{1}{y\cdot\sqrt{5}}[(x+y\Phi)^n-(x-y\phi)^n]$ and $\phi_n(x,y)=[(x+y\Phi)^n+(x-y\phi)^n]$. As a result we obtain a pair of two variable polynomial which are new combinatorial entities. Many convolution identities of L_n and F_n are getting added to the recent literature. A generalized convolution identities will be a worthy enrichment of such combinatorial identities to the current literature.

Keywords. Golden ratio; Binet forms; Combinatorial identities

MSC. 11B33; 30B70; 05A19

Received: January 11, 2018 **Accepted:** June 3, 2018

1. Introduction

In combinatorial Number Theory, Binet forms express naturally the following pair of two variable polynomials $(P_n(u,v),Q_n(u,v))$:

$$P_n(u,v)=u^n+v^n,\quad Q_n(u,v)=\frac{u^n-v^n}{u-v}.$$

They produce many particular pairs of numbers or polynomials such as Fibonacci and Lucas numbers [2], Tchebyshev polynomials of first and second kind [6] and so on.

They satisfy the following pair of difference equations which is also quite interesting from the point of combinatorial number theory:

$$P_{n+1}(u,v) = (u+v)P_n(u,v) - u\cdot vP_{n-1}(u,v), \tag{1.1}$$
$$P_0(u,v) = 2,\ P_1(u,v) = u+v, \quad n = 1,2,3,\cdots$$

and

$$Q_{n+1}(u,v) = (u+v)Q_n(u,v) - u\cdot vQ_{n-1}(u,v) \tag{1.2}$$
$$Q_0(u,v) = 0,\ Q_1(u,v) = 1, \quad n = 1,2,3,\cdots.$$

As an important special case, one can take $u = \Phi = \frac{1+\sqrt{5}}{2}$ and $v = \phi = \frac{-1+\sqrt{5}}{2}$. Then $u+v = 1$, $-uv = 1$ and $u-v = \sqrt{5}$.

$$P_n(\Phi,-\phi) = L_n = \Phi^n + (-\phi)^n, \quad Q_n(\Phi,-\phi) = F_n = \frac{\Phi^n - (-\phi)^n}{\Phi - (\phi)}$$

are the well known Lucas and Fibonacci numbers given by the following pair of beautiful difference equations:

$$L_{n+1} = L_n + L_{n-1}, L_0 = 2, L_1 = 1, \quad n = 1,2,\cdots, \tag{1.3}$$
$$F_{n+1} = F_n + F_{n-1}, F_0 = 2, F_1 = 1, \quad n = 1,2,\cdots. \tag{1.4}$$

Recently, a two variable generalization of L_n and F_n, called two variable Hybrid Lucas and Fibonacci polynomials are studied in [7, 8]. They are also a special case with $u = \frac{x+\sqrt{x^2+4y}}{2}$ and $v = \frac{x-\sqrt{x^2+4y}}{2}$, when $x = y = 1$, $u = \Phi$ and $v = -\phi$. So that we get back Lucas and Fibonacci numbers. Two variable hybrid Fibonacci and Lucas polynomials are given by

$$l_{n+1}^{(H)}(x,y) = x\ l_n^{(H)}(x,y) + y\ l_{n-1}^{(H)}(x,y), \tag{1.5}$$
$$sl_0^{(H)}(x,y) = 2,\ l_1^{(H)}(x,y) = x, \quad n = 1,2,\cdots$$
$$f_{n+1}^{(H)}(x,y) = x\ f_n^{(H)}(x,y) + y\ f_{n-1}^{(H)}(x,y), \tag{1.6}$$
$$f_0^{(H)}(x,y) = 0,\ f_1^{(H)}(x,y) = 1, n = 1,2,\cdots$$

More recently, the authors with Rangaswamy have stated and proved convolution identities of $f_n^{(H)}(x,y)$ and $l_n^{(H)}(x,y)$ ([9, 10]). The general case $P_n(u,v)$ and $Q_n(u,v)$ has motivated us to consider one more simple and interesting special case by choosing $u = x+y(\Phi)$ and $v = x+y(-\phi)$ where x, y are two real variables, $\Phi = \frac{1+\sqrt{5}}{2}$ and $\phi = \frac{-1+\sqrt{5}}{2}$.

In the present paper, we introduce the following Binet form:

$$P_n(u,v) = P_n(x+y(\Phi), x+y(-\phi)) = \psi_n(x,y) = (x+y(\Phi))^n + (x+y(-\phi))^n \tag{1.7}$$

and

$$Q_n(u,v) = Q_n(x+y(\Phi), x+y(-\phi)) = \phi_n(x,y) = \frac{(x+y(\Phi))^n + (x+y(-\phi))^n}{y\cdot\sqrt{5}}. \tag{1.8}$$

Let us note that

$$\psi_n(x,y) = \xi^n + \eta^n, \tag{1.9}$$

$$\phi_n(x,y) = \frac{\xi^n + \eta^n}{\xi - \eta}, \tag{1.10}$$

$$\xi + \eta = (x + y(\Phi)) + (x + y(-\phi) = 2x + y, \tag{1.11}$$

$$\xi - \eta = (x + y(\Phi)) - (x + y(-\phi) = y \cdot \sqrt{5}, \tag{1.12}$$

$$\xi \cdot \eta = (x + y(\Phi)) \cdot (x + y(-\phi) = x^2 + xy - y^2. \tag{1.13}$$

Hence $(\phi_n(x,y), \psi_n(x,y))$ are solutions of following difference equation.

$$\phi_{n+1}(x,y) = (2x+y)\ \phi_n(x,y) - (x^2 + xy - y^2)\phi_{n-1}(x,y), \tag{1.14}$$

$$\phi_0(x,y) = 0,\ \phi_1(x,y) = 1 \quad \text{for } n = 1,2,3,\cdots$$

$$\psi_{n+1}(x,y) = (2x+y)\ \psi_n(x,y) - (x^2 + xy - y^2)\psi_{n-1}(x,y), \tag{1.15}$$

$$\psi_0(x,y) = 2,\ \psi_1(x,y) = (2x+y) \quad \text{for } n = 1,2,3,\cdots$$

When $x = 0$ and $y = 1$, we get back $\phi_n(0,1) = F_n$ and $\psi_n(0,1) = L_n$. The generalized polynomials $\phi_n(x,y)$ and $\psi_n(x,y)$ does not include $f_n^{(H)}(x,y)$ and $l_n^{(H)}(x,y)$ as special case and also vice-versa. Like $f_n^{(H)}(x,y)$ and $l_n^{(H)}(x,y)$, $\phi_n(x,y)$ and $\psi_n(x,y)$ also exhibit many combinatorial identities ([11, 12]).

In the present paper we state and prove certain convolution identities of $\phi_n(x,y)$ and $\psi_n(x,y)$ which are of recent interest in the literature [4, 5, 7, 8]. In the second section, convolution identities of $\phi_n(x,y)$ and $\psi_n(x,y)$ with a fixed power m of the summing variable, $m = 0,1,2,3$ are stated and proved. In the last section, Binomial convolution identities of $\phi_n(x,y)$ and $\psi_n(x,y)$ with a fixed power m of summing variable, $m = 0,1,2,3$ are stated and proved.

2. Convolution Identities with a Fixed Power of Expanding Variable

One of the remarkable identities is the following well known Bernoulli's identity ([1, 9]):
If $S_n(m) = \sum_{k=1}^{n} k^m$.

In this section, convolution identities of the following polynomials in two variables with a fixed power m of the summing variable, $m = 0,1$, are stated and proved and $m = 2,3$, are stated without proving.

Theorem 1. *The convolution identities at the level $m = 0$ are*

(1a) $\displaystyle\sum_{k=1}^{n} \psi_k(x,y)\psi_{n-k}(x,y) = n\psi_n(x,y) + (2x+y)\ \phi_n(x,y)$,

(1b) $\displaystyle\sum_{k=1}^{n} \phi_k(x,y)\phi_{n-k}(x,y) = \frac{n\psi_n(x,y) - (2x+y)\ \phi_n(x,y)}{5 \cdot y^2}$,

(1c) $\displaystyle\sum_{k=1}^{n} \psi_k(x,y)\phi_{n-k}(x,y) = (n-1)\phi_n(x,y)$, (1d) $\sum_{k=1}^{n} \phi_k(x,y)\ \psi_{n-k}(x,y) = (n+1)\phi_n(x,y)$.

Proof.

$$\text{(1a)}\quad \sum_{k=1}^{n} \psi_k(x,y)\psi_{n-k}(x,y) = \sum_{k=1}^{n}\left(\xi^k+\eta^k\right)\left(\xi^{n-k}+\eta^{n-k}\right) = \left[n(\xi^n+\eta^n)) + \left(\frac{\xi^n-\eta^n}{\xi-\eta}\right)(\xi+\eta)\right]$$
$$= n\psi_n(x,y) + (2x+y)\phi_n(x,y),$$

$$\text{(1b)}\quad \sum_{k=1}^{n} \phi_k(x,y)\phi_{n-k}(x,y) = \sum_{k=1}^{n}\left(\frac{\xi^k-\eta^k}{\xi-\eta}\right)\left(\frac{\xi^{n-k}-\eta^{n-k}}{\xi-\eta}\right) = \frac{1}{(\xi-\eta)^2}\left[n(\xi^n+\eta^n) - \left(\frac{\xi^n-\eta^n}{\xi-\eta}\right)(\xi+\eta)\right]$$
$$= \frac{n\ \psi_n(x,y) - (2x+y)\phi_n(x,y)}{5\cdot y^2}.$$

The proofs of (1c) and (1d) are similar to that of (1a) and (1b) except in the final stages where we need to apply a suitable recurrence relations (1.12) or (1.13) and apply (1.7) or (1.8) according to the situation. □

Theorem 2. *The convolution identities at the level $m = 1$ are*

$$\text{(2a)}\quad \sum_{k=1}^{n} k\ \psi_k(x,y)\ \psi_{n-k}(x,y)$$
$$= \frac{n(n+1)\ \psi_n(x,y)}{2} + \frac{1}{5\cdot y^2}[n\ (2x+y)\ \psi_{n+1}(x,y)\ - 2n(x^2+xy-y^2)\psi_n(x,y)],$$

$$\text{(2b)}\quad \sum_{k=1}^{n} k\ \phi_k(x,y)\ \psi n-k(x,y)$$
$$= \frac{n(n+1)\ \psi_n(x,y)}{10\cdot y^2} - \frac{1}{25\cdot y^4}[n\ (2x+y)\ \psi_{n+1}(x,y)\ - 2n(x^2+xy-y^2)\psi_n(x,y)],$$

$$\text{(2c)}\quad \sum_{k=1}^{n} k\ \psi_k(x,y;t)\ \phi_{n-k}(x,y;t)$$
$$= \frac{n(n+1)\phi_n(x,y)}{2} - \frac{1}{5\cdot y^2}\left[n(2x+y)\ \phi_{n+1}(x,y) - [(2n+2)(x^2+xy-y^2)]\phi_n(x,y)\right],$$

$$\text{(2d)}\quad \sum_{k=1}^{n} k\ \phi_k(x,y;t)\ \psi_{n-k}(x,y;t)$$
$$= \frac{n(n+1)\phi_n(x,y;t)}{2} + \frac{1}{5\cdot y^2}\left[n(2x+y)\ \phi_{n+1}(x,y) - [(2n+2)(x^2+xy-y^2)]\phi_n(x,y)\right].$$

Proof.

$$\text{(2a)}\quad \sum_{k=1}^{n} k\ \psi_k(x,y)\ \psi_{n-k}(x,y)$$
$$= \sum_{k=1}^{n} k\ (\xi^k+\eta^k)(\xi^{n-k}+\eta^{n-k})$$
$$= \left[\frac{n(n+1)}{2}(\xi^n+\eta^n) + \frac{1}{(\xi-\eta)^2}\left(n(\xi^{n+2}+\eta^{n+2}) - (n+1)(\xi\cdot\eta)(\xi^n+\eta^n) + (\xi\cdot\eta)(\xi^n+\eta^n)\right)\right]$$

$$= \frac{n(n+1)\,\psi_n(x,y)}{2} + \frac{1}{5\cdot y^2}[n\ (2x+y)\ \psi_{n+1}(x,y) - 2\ n\ (x^2+xy-y^2)\psi_n(x,y)]$$

(by repeated deductions using (1.12) and (1.13)).

(2b) $\sum_{k=1}^{n} k\ \phi_k(x,y)\ \phi_{n-k}(x,y)$

$$= \sum_{k=1}^{n} k\left(\frac{\xi^k-\eta^k}{\xi-\eta}\right)\left(\frac{\xi^{n-k}-\eta^{n-k}}{\xi-\eta}\right)$$

$$= \frac{1}{(\xi-\eta)^2}\left[\frac{n(n+1)}{2}\psi_n(x,y) - \frac{\xi^{n+2}}{(\xi-\eta)^2}\left(n\frac{\eta^{n+2}}{\xi^{n+2}} - (n+1)\frac{\eta^{n+1}}{\xi^{n+1}} + \frac{\eta}{\xi}\right)\right.$$
$$\left. - \frac{\eta^{n+2}}{(\xi-\eta)^2}\left(n\frac{\xi^{n+2}}{\eta^{n+2}} - (n+1)\frac{\xi^{n+1}}{\eta^{n+1}} + \frac{\xi}{\eta}\right)\right]$$

$$= \frac{1}{(\xi-\eta)^2}\left[\frac{n(n+1)}{2}(\xi^n+\eta^n) - \frac{1}{(\xi-\eta)^2}\left(n(\xi^{n+2}+\eta^{n+2}) - (n+1)(\xi\cdot\eta)(\xi^n+\eta^n)\right.\right.$$
$$\left.\left. +(\xi\cdot\eta)(\xi^n+\eta^n)\right)\right]$$

$$= \frac{n(n+1)\,\psi_n(x,y)}{10.y^2} - \frac{1}{25\cdot y^4}[n\ (2x+y)\ \psi_{n+1}(x,y)\ - 2n(x^2+xy-y^2)\psi_n(x,y)]$$

(by repeated deductions using (1.12) and (1.13)).

The proofs of (2c) and (2d) are similar to that of (2a) and (2b) except in the final stages where we need to apply a suitable recurrence relations (1.12) or (1.13) and apply (1.7) or (1.8) according to the situation. □

Theorem 3. *The convolution identities at the level $m=2$ are*

(3a) $\sum_{k=1}^{n} k^2\ \psi_k(x,y)\ \psi_{n-k}(x,y)$

$$= \frac{n(n+1)(2n+1)\psi_n(x,y)}{6} + \frac{1}{5\cdot y^2}\big[[5n^2y^2 - 4n(x^2+xy-y^2)]\phi_{n+1}(x,y)$$
$$+[(2n+2)(2x^3+3x^2y-xy^2-y^3)]\phi_n(x,y)\big],$$

(3b) $\sum_{k=1}^{n} k^2\ \phi_k(x,y)\ \phi_{n-k}(x,y)$

$$= \frac{n(n+1)(2n+1)\psi_n(x,y)}{30\cdot y^2} - \frac{1}{25\cdot y^4}\big[[5n^2y^2 - 4n(x^2+xy-y^2)]\phi_{n+1}(x,y)$$
$$+[(2n+2)(2x^3+3x^2y-xy^2-y^3)]\phi_n(x,y)\big],$$

(3c) $\sum_{k=1}^{n} k^2\ \psi_k(x,y)\ \phi_{n-k}(x,y)$

$$= \frac{n(n+1)(2n+1)\phi_n(x,y)}{6} - \frac{1}{25\cdot y^4}\big[[5n^2y^2 - 4n(x^2+xy-y^2)]\psi_{n+1}(x,y)$$
$$+[2n(2x^3+3x^2y-xy^2-y^3)]\psi_n(x,y)\big],$$

(3d) $\sum_{k=1}^{n} k^2\ \phi_k(x,y)\ \psi_{n-k}(x,y)$

$$= \frac{n(n+1)(2n+1)\phi_n(x,y)}{6} + \frac{1}{25\cdot y^4}\big[[5n^2y^2 - 4n(x^2+xy-y^2)]\psi_{n+1}(x,y) \\ +[2n(2x^3+3x^2y-xy^2-y^3)]\psi_n(x,y)\big].$$

Theorem 4. *The convolution identities at the level $m = 3$ are*

(4a) $\displaystyle\sum_{k=1}^{n} k^3\ \psi_k(x,y)\psi_{n-k}(x,y)$

$$= \frac{n^2(n+1)^2\psi_n(x,y)}{4} + \frac{1}{25\cdot y^4}\big[[n^3b_{31} + (-2n^3+6n)b_{32}]\psi_{n+1}(x,y) \\ +[-n^3b_{33} + (7n^3+12n^2)b_{34} + (-n^3-3n^2-3n)b_{36}]\psi_n(x,y)\big],$$

(4b) $\displaystyle\sum_{k=1}^{n} k^3\ \phi_k(x,y)\phi_{n-k}(x,y)$

$$= \frac{n^2(n+1)^2\psi_n(x,y)}{20.y^2} - \frac{1}{125\cdot y^6}\big[[n^3b_{31} + (-2n^3+6n)b_{32}]\psi_{n+1}(x,y) \\ +[-n^3b_{33} + (7n^3+12n^2)b_{34} + (-n^3-3n^2-3n)b_{36}]\psi_n(x,y)\big],$$

(4c) $\displaystyle\sum_{k=1}^{n} k^3\ \psi_k(x,y)\phi_{n-k}(x,y)$

$$= \frac{n^2(n+1)^2\phi_n(x,y)}{4} - \frac{1}{25\cdot y^4}\big[[n^3b_{31} + (-2n^3+6n)b_{32}]\phi_{n+1}(x,y) \\ +[-n^3b_{33} + (7n^3+12n^2-4)b_{35} + (-n^3-3n^2-3n-2)b_{36}]\phi_n(x,y)\big],$$

(4d) $\displaystyle\sum_{k=1}^{n} k^3\ \phi_k(x,y)\psi_{n-k}(x,y)$

$$= \frac{n^2(n+1)^2\phi_n(x,y)}{4} + \frac{1}{25\cdot y^4}\big[[n^3b_{31} + (-2n^3+6n)b_{32}]\phi_{n+1}(x,y) \\ +[-n^3b_{33} + (7n^3+12n^2-4)b_{35} + (-n^3-3n^2-3n-2)(b_{36}]\phi_n(x,y)\big].$$

Here $b_{31} = (4x^3+6x^2y+8xy^2+3y^3)$, $b_{32} = (2x+y)(x^2+xy-y^2)$, $b_{33} = (3x^4+6x^3y+2x^2y^2-xy^3-2y^4)$, $b_{34} = (x^2+xy-y^2)^2$, $b_{35} = (2x+y)^2(x^2+xy-y^2)^2$ *and* $b_{36} = (2x+y)^2(x^2+xy-y^2)$.

Proof. The proofs of Theorems 3 and 4 are similar to that of Theorems 1 and 2 except in the final stages where we need to apply a suitable recurrence relations and apply according to the situation. The same procedure of employing generalized Bernoulli identity can be applied to compute convolution identities at any level. □

3. Binomial Convolution Identities

The following Bernoulli type identity for

$$B_n(m,x) = \sum_{k=0}^{n} \binom{n}{k} k^m\ x^k$$

([1, 10]) will be applied:

In this section, Binomial convolution identities of the following polynomials in two variables with a fixed power m of the summing variable, $m = 0, 1$, are stated and proved and $m = 2, 3$, are stated without proving.

Theorem 5. *The Binomial convolution identities at the level $m = 0$ are*

(5a) $\sum_{k=0}^{n} \binom{n}{k} \psi_k(x,y)\, \psi_{n-k}(x,y) = 2^n \psi_n(x,y) + 2(2x+y)^n$,

(5b) $\sum_{k=0}^{n} \binom{n}{k} \phi_k(x,y)\, \phi_{n-k}(x,y) = \dfrac{2^n\ \psi_n(x,y) - 2(2x+y)^n}{5.y^2}$,

(5c) $\sum_{k=0}^{n} \binom{n}{k} \psi_k(x,y)\, \phi_{n-k}(x,y) = 2^n \phi_n(x,y)$,

(5d) $\sum_{k=0}^{n} \binom{n}{k} \phi_k(x,y)\, \psi_{n-k}(x,y) = 2^n \phi_n(x,y)$,

Proof. By using equations (1.7) to (1.11) will take us through the derivation step by step for all four identities.

(5a)
$$\begin{aligned}\sum_{k=0}^{n} \binom{n}{k} \psi_k(x,y)\, \psi_{n-k}(x,y) &= \sum_{k=0}^{n} \binom{n}{k} \left(\xi^k + \eta^k\right)\left(\xi^{n-k} + \eta^{n-k}\right) \\ &= \left[2^n\ (\xi^n + \eta^n) + \eta^n \left(1 + \frac{\xi}{\eta}\right)^n + \xi^n \left(1 + \frac{\eta}{\xi}\right)^n\right] \\ &= 2^n\ \psi_n(x,y) + 2(2x+y)^n,\end{aligned}$$

(5b)
$$\begin{aligned}\sum_{k=0}^{n} \binom{n}{k} \phi_k(x,y)\phi_{n-k}(x,y) &= \sum_{k=0}^{n} \binom{n}{k} \left(\frac{\xi^k - \eta^k}{\xi - \eta}\right)\left(\frac{\xi^{n-k} - \eta^{n-k}}{\xi - \eta}\right) \\ &= \frac{1}{(\xi-\eta)^2}\left[2^n\ (\xi^n + \eta^n) + \eta^n \left(1 + \frac{\xi}{\eta}\right)^n + + \xi^n \left(1 + \frac{\eta}{\xi}\right)^n\right] \\ &= \frac{2^n\ \psi_n(x,y) - 2(2x+y)^n}{5 \cdot y^2}.\end{aligned}$$

The proofs of (5c) and (5d) are similar to that of (5a) and (5b) except in the final steps where we need to apply (1.7) or (1.8) according to the situation. □

Theorem 6. *The Binomial convolution identities at the level $m = 1$ are*

(6a) $\sum_{k=0}^{n} \binom{n}{k} k\ \psi_k(x,y)\, \psi_{n-k}(x,y) = n\ 2^{n-1} \psi_n(x,y) + n(2x+y)^n$,

(6b) $\sum_{k=0}^{n} \binom{n}{k} k\ \phi_k(x,y)\, \phi_{n-k}(x,y) = \dfrac{n\ 2^{n-1}\ \psi_n(x,y) - n\ (2x+y)^n}{5 \cdot y^2}$,

(6c) $\sum_{k=0}^{n} \binom{n}{k} k\ \psi_k(x,y)\, \phi_{n-k}(x,y) = n\ 2^{n-1}\ \phi_n(x,y) - n(2x+y)^{n-1}$,

(6d) $\sum_{k=0}^{n} \binom{n}{k} k\ \phi_k(x,y)\, \psi_{n-k}(x,y) = n\ 2^{n-1}\ \phi_n(x,y) + n(2x+y)^{n-1}$.

Proof. By using equations (1.7) to (1.11) will take us through the derivation step by step for all four identities.

(6a) $$\sum_{k=0}^{n}\binom{n}{k}k\ \psi_k(x,y)\ \psi_{n-k}(x,y)=\sum_{k=0}^{n}\binom{n}{k}\ k\ \left(\xi^k+\eta^k\right)\left(\xi^{n-k}+\eta^{n-k}\right)$$
$$=\left[n\ 2^{n-1}(\xi^n+\eta^n)+n(\xi+\eta)^{n-1}(\xi+\eta)\right]$$
$$=n\ 2^{n-1}\ \psi_n(x,y)+n(2x+y)^n\,,$$

(6b) $$\sum_{k=0}^{n}\binom{n}{k}k\ \phi_k(x,y)\phi_{n-k}(x,y)\ =\sum_{k=0}^{n}\binom{n}{k}k\left(\frac{\xi^k-\eta^k}{\xi-\eta}\right)\left(\frac{\xi^{n-k}-\eta^{n-k}}{\xi-\eta}\right)$$
$$=\frac{1}{(\xi-\eta)^2}\left[2^{n-1}n\ (\xi^n+\eta^n)+n.\xi(\xi+\eta)^{n-1}+n\cdot\eta(\xi+\eta)^{n-1}\right]$$
$$=\frac{n\ 2^{n-1}\ \psi_n(x,y)-n\ (2x+y)^n}{5\cdot y^2}\,.$$

The proofs of (6c) and (6d) are similar to that of (6a) and (6b) except in the final stages where we need to apply a suitable recurrence relations (1.12) or (1.13) and apply (1.7) or (1.8) according to the situation. □

Theorem 7. *The Binomial convolution identities at the level $m=2$ are*

(7a) $$\sum_{k=0}^{n}\binom{n}{k}k^2\ \psi_k(x,y)\ \psi_{n-k}(x,y)$$
$$=2^{n-2}n(3n-1)\psi_n(x,y)+n(n-1)(2x+y)^{n-2}(2x^2+3y^2+2xy)+n\ (2x+y)^n\,,$$

(7b) $$\sum_{k=0}^{n}\binom{n}{k}k^2\ \phi_k(x,y)\ \phi_{n-k}(x,y)$$
$$=\frac{1}{5.y^2}\left[2^{n-2}n(3n-1)\psi_n(x,y)-[n(n-1)(2x+y)^{n-2}(2x^2+3y^2+2xy)+n\ (2x+y)^n]\right],$$

(7c) $$\sum_{k=0}^{n}\binom{n}{k}k^2\ \psi_k(x,y)\ \phi_{n-k}(x,y)=2^{n-2}n(3n-1)\phi_n(x,y)-n^2\ (2x+y)^{n-1}\,,$$

(7d) $$\sum_{k=0}^{n}\binom{n}{k}k^2\ \phi_k(x,y)\ \psi_{n-k}(x,y)=2^{n-2}n(3n-1)\phi_n(x,y)+n^2\ (2x+y)^{n-1}\,.$$

Theorem 8. *The Binomial convolution identities at the level $m=3$ are*

(8a) $$\sum_{k=0}^{n}\binom{n}{k}k^3\ \psi_k(x,y)\ \psi_{n-k}(x,y)$$
$$=\left[2^{n-3}n^2(n+3)\ \psi_n(x,y)+[(n^3-3n^2+2n)(2x+y)^{n-3}(2x^3+3x^2y+9xy^2+4y^3)\right.$$
$$\left.+3(n^2-n)(2x+y)^{n-2}(2x^2+2xy+3y^2)+n(2x+y)^n]\right],$$

(8b) $$\sum_{k=0}^{n}\binom{n}{k}k^3\ \phi_k(x,y)\ \phi_{n-k}(x,y)$$
$$=\frac{1}{5.\ y^2}\left[2^{n-3}n^2(n+3)\ \psi_n(x,y)+[(n^3-3n^2+2n)(2x+y)^{n-3}(2x^3+3x^2y+9xy^2+4y^3)\right.$$
$$\left.+3(n^2-n)(2x+y)^{n-2}(2x^2+2xy+3y^2)+n(2x+y)^n]\right],$$

(8c) $$\sum_{k=0}^{n}\binom{n}{k}k^3\ \psi_k(x,y)\ \phi_{n-k}(x,y)$$
$$=2^{n-3}n^2(n+3)\phi_n(x,y)-[(n^3-3n^2+2n)(2x+y)^{n-3}(x^3+xy+4y^2)+(3n^2-2n)(2x+y)^{n-1}],$$

(8d) $$\sum_{k=0}^{n}\binom{n}{k}k^3\ \phi_k(x,y)\ \psi_{n-k}(x,y)$$
$$=2^{n-3}n^2(n+3)\phi_n(x,y)+[(n^3-3n^2+2n)(2x+y)^{n-3}(x^3+xy+4y^2)+(3n^2-2n)(2x+y)^{n-1}].$$

Proof. The proofs of Theorems 7 and 8 are similar to that of Theorems 5 and 6 except in the final stages where we need to apply a suitable recurrence relations applied according to the situation. The same procedure of employing generalized binomial summation can be applied to compute convolution identities at any level. □

4. Conclusion

The convolution identities with the power of expanding variable and binomial convolution identities with the power of expanding variable are two important types of convolution identities. They are very useful to analyze discrete dynamical systems. Computing such convolution identities with the higher power of expanding variable by applying the Bernoulli's identity at any level is a challenging task for computer engineers.

Acknowledgement

The second author would like to thank both UGC-SWRO, F. No. FIP/12th Plan/KADA018 TF02 and Government of Karnataka (DCE).

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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