Journal of Informatics and Mathematical Sciences
Volume 4 (2012), Number 2, pp. 157–165

http://www.rgnpublications.com

On $\widetilde{g}_\alpha$-Grill in Topological Spaces

M. Lellis Thivagar and Nirmala Rebecca Paul

Abstract. The aim of this paper is to introduce a new type of topology in terms of grills and $\widetilde{g}_\alpha$-sets and study the related topological properties. We define new types of sets to study the relationship between the new sets and existing sets and to derive a new decomposition of continuity.

1. Introduction

The idea of grill was introduced by Choquet [2]. The power set of X will be denoted by $P(X)$. A collection G of nonempty subsets of a space X is called a grill on X if (i) $A \in G$ and $A \subseteq B \subseteq X \Rightarrow B \in G$ (ii) $A, B \subseteq X$ and $A \cup B \in G \Rightarrow A \in G$ or $B \in G$. The grill concept has been found useful in proximity spaces, closure spaces and in the theory of compactification. Hence the grill concept is a powerful tool similar to nets and filters in dealing with many topological situations. In this paper we have introduced a new operator which satisfies Kuratowski's closure axioms which induces a new topology which is finer than the given topology. This new topology is associated with the $\widetilde{g}_\alpha$-sets [4] and some of its basic properties have been highlighted. The class of $\widetilde{g}_\alpha$-closed sets is one among the few classes of closed sets which forms a topology. New sets have been defined in studying the relationship between the sets and for deriving the decomposition of continuity.

2. Preliminaries

We list some definitions which are useful in the following sections. The interior and the closure of a subset A of (X, τ) are denoted by $Int(A)$ and $Cl(A)$, respectively. Throughout the present paper (X, τ) and (Y, σ) (or X and Y) represent non-empty topological spaces on which no separation axiom is defined, unless otherwise mentioned.

2010 *Mathematics Subject Classification.* 54C08, 54C10, 54A10.
Key words and phrases. $^{\#}gs$-open sets; $\widetilde{g}_\alpha$-closed; $\widetilde{g}_\alpha$-open sets.

Definition 2.1. A subset A of a topological space (X,τ) is called

(i) an ω-closed set [9] $(=\widehat{g}-closed)$ if $Cl(A)\subseteq U$ whenever $A\subseteq U$ and U is semi-open in (X,τ),

(ii) a ${}^{*}g$-closed set [10] if $Cl(A)\subseteq U$ whenever $A\subseteq U$ and U is ω-open in (X,τ),

(iii) a $^{\#}$g-semi-closed set(briefly $^{\#}$gs-closed) [11] if $sCl(A)\subseteq U$ whenever $A\subseteq U$ and U is *g-open in (X,τ) and

(iv) a $\widetilde{g}_\alpha$-closed [4] if $\alpha Cl(A)\subseteq U$ whenever $A\subseteq U$ and U is $^{\#}$gs-open in X.

The complement of $\widetilde{g}_\alpha$-closed set is said to be $\widetilde{g}_\alpha$-open.

Remark 2.2. The family of all $\widetilde{g}_\alpha$-open sets of (X,τ) is denoted by $\widetilde{G}_\alpha O(X)$ and the family of all $\widetilde{g}_\alpha$-closed sets of (X,τ) is denoted by $\widetilde{G}_\alpha C(X)$.

We set $\widetilde{G}_\alpha O(X,x)=\{V\in\widetilde{G}_\alpha O(X)/x\in V\}$ for $x\in X$.

Definition 2.3. Let (X,τ) be a topological space and E be subset of X. We define $\widetilde{g}_\alpha$-Interior of E denoted by $\widetilde{g}_\alpha$-$Int(E)$ to be the union of all $\widetilde{g}_\alpha$-open sets contained in E [4]. We define $\widetilde{g}_\alpha$-closure of E denoted by $\widetilde{g}_\alpha$-$Cl(E)$ to be the intersection of all $\widetilde{g}_\alpha$-closed sets containing E [4].

3. Topology induced by a $\widetilde{g}_\alpha$-Grill

Let (X,τ) be a topological space and G be a grill on X. We define a mapping $\eta: P(X)\to P(X)$ denoted by $\eta(A)$ as $\eta_G(A)=\eta(A)=\{x\in X: A\cap U\in G,\ \forall\, U\in\widetilde{G}_\alpha O(X,x)\}$.

Remark 3.1. (i) If G is a grill on X then η is an increasing function. If $A\subseteq B\subseteq X$ then $\eta(A)\subseteq\eta(B)$ and if G_1 and G_2 are two grills on X with $G_1\subseteq G_2$ then $\eta_{G_1}(A)=\{x\in X: A\cap U\in G_1,\ \forall\, U\in\widetilde{G}_\alpha O(X,x)\}\subseteq\{x\in X: B\cap U\in G_2,\ \forall\, U\in\widetilde{G}_\alpha O(X,x)\}=\eta_{G_2}(A)$.

(ii) For any grill G on X and $A\subseteq X$ if $A\notin G$ then $\eta_G(A)=\phi$.

Proposition 3.2. *Let (X,τ) be a topological space and G a grill on X. Then for all $A,B\subseteq X$*

(i) $\eta(A\cup B)=\eta(A)\cup\eta(B)$

(ii) $\eta(\eta(A))\subseteq\eta(A)=\widetilde{g}_\alpha-\mathrm{Cl}(\eta(A))\subseteq\widetilde{g}_\alpha-\mathrm{Cl}(A)$

Proof. (i) Since η is an increasing function $\eta(A)\cup\eta(B)\subseteq\eta(A\cup B)$. To show that $\eta(A\cup B)\subseteq\eta(A)\cup\eta(B)$ Let $x\notin\eta(A)\cup\eta(B)$. There are $U,V\in\widetilde{G}_\alpha O(X,x)$ such that $A\cap U, B\cap V\notin G$ and hence $(A\cap U)\cup(B\cap V)\notin G$. Also $U\cap V\in\widetilde{G}_\alpha O(X,x)$ and $(A\cup B)\cap(U\cap V)\subseteq(A\cap U)\cup(B\cap V)\notin G$, proving that $x\notin\eta(A\cup B)$. Hence $\eta(A\cup B)=\eta(A)\cup\eta(B)$.

(ii) Let $x\notin\widetilde{g}_\alpha-Cl(A)\Rightarrow\exists U\in\widetilde{G}_\alpha O(X,x)$ such that $U\cap A=\phi\notin G\Rightarrow x\notin\eta(A)$. Hence $\eta(A)\subseteq\widetilde{g}_\alpha-Cl(A)$. We have to prove that $\widetilde{g}_\alpha-Cl(\eta(A))\subseteq\eta(A)$. Let $x\in\widetilde{g}_\alpha-Cl(\eta(A))$ and $U\in\widetilde{G}_\alpha O(X,x)\Rightarrow U\cap\eta(A)\neq\phi$. Let $y\in U\cap\eta(A)$. i.e. $y\in U$ and $y\in\eta(A)$. Then $U\cap A\in G$ and so $x\in\eta(A)$. Thus $\widetilde{g}_\alpha-Cl(\eta(A))=\eta(A)$. Also $\eta(\eta(A))\subseteq\widetilde{g}_\alpha-Cl(\eta(A))=\eta(A)\subseteq\widetilde{g}_\alpha-Cl(A)$. □

Definition 3.3. Let G be a grill on a space X. We define a map $\lambda : P(X) \to P(X)$ as $\lambda(A) = A \cup \eta(A)$, for all $A \in P(X)$.

Theorem 3.4. *The function λ satisfies Kuratowski's closure axioms.*

Proof. By Remark 3.1 we have $\lambda(\phi) = \phi$ and $A \subseteq \lambda(A)$, $\forall\, A \subseteq X$.

$$\lambda(A \cup B) = (A \cup B) \cup \eta(A \cup B) = A \cup B \cup \eta(A) \cup \eta(B) = \lambda(A) \cup \lambda(B).$$

For any $A \subseteq X$,

$$\begin{aligned}\lambda(\lambda(A)) &= \lambda(A \cup \eta(A)) = A \cup \eta(A) \cup \eta(A \cup \eta(A)) = A \cup \eta(A) \cup \eta(\eta(A)) \\ &= A \cup \eta(A) = \lambda(A).\end{aligned}$$

4. New topology

Definition 4.1. Corresponding to a grill G on the space (X, τ) we define a topology τ_G on X as $\tau_G = \{U \subseteq X : \lambda(X - U) = X - U\}$ where $A \subseteq X, \lambda(A) = A \cup \eta(A) = \tau_G - \widetilde{g}_\alpha Cl(A)$.

(i) $\phi \subseteq X, \lambda(X - \phi) = \lambda(X) = X, X \subseteq X, \lambda(X - X) = \lambda(\phi) = \phi$. Hence $\phi, X \in \tau_G$.

(ii) Let $\{U_i\}_{i \in I} \in \tau_G$ then $\lambda(X - U_i) = X - U_i\ \forall\ i$. i.e. $(X - U_i) \cup \eta(X - U_i) = X - U_i\ \forall\ i$. Therefore $\eta(X - U_i) \subseteq X - U_i\ \forall\ i$. Claim $\lambda(X - \cup U_i) = X - \cup Ui = \cap(X - U_i)$. $\eta((\cap(X - U_i)) \subseteq \cap(X - U_i)\ \forall\ i$. $\lambda(\cap(X - U_i)) = \cap(X - U_i) \cup \eta(\cap(X - U_i) \Rightarrow \lambda(\cap(X - U_i) \supseteq \cap(X - U_i)$. Thus $\cap(X - U_i) \cup \eta(\cap(X - Ui)) \subseteq \cap(X - U_i)$ i.e. $\lambda(X - \cup U_i) = X - \cup U_i$. Hence $\cup U_{i \in I} \in \tau_G$.

(iii) Let $U_1, U_2 \in \tau_G$ then $\lambda(X - U_1) = X - U_1, \lambda(X - U_2) = X - U_2$. $\lambda((X - (U_1 \cap U_2)) = \lambda(\cup_{i=1}^{i=2}(X - Ui) = \lambda(X - U_1) \cup \lambda(X - U_2) = X - U_1 \cup X - U_2$. Hence τ_G is a topology.

Theorem 4.2. (i) *If G_1 and G_2 are two grills on a space X with $G_1 \subseteq G_2$ then $\tau_{G_1} \subseteq \tau_{G_2}$.*

(ii) *If G is a grill on a space X and $B \notin G$, then B is $\widetilde{g}_\alpha$-closed in (X, τ_G).*

(iii) *For any subset A of a space X and any grill G on X, $\eta(A)$ is τ_G-closed.*

Proof. (i) Let $U \in \tau_{G_2} \Rightarrow \tau_{G_2} - \widetilde{g}_\alpha Cl(X - U) = \lambda(X - U) \Rightarrow X - U = (X - U) \cup \eta_{G_2}(X - U) \Rightarrow \eta_{G_2}(X - U) \subseteq (X - U) \Rightarrow \eta_{G_1}(X - U) \subseteq (X - U) \Rightarrow X - U = \tau_{G_1} - \widetilde{g}_\alpha Cl(X - U) \Rightarrow U \in \tau_{G_1}$.

(ii) If $B \notin G \Rightarrow \eta(B) = \phi$ then $\tau_G - \widetilde{g}_\alpha Cl(B) = \lambda(B) = B \cup \eta(B) = B$ and hence B is $\tau_G - \widetilde{g}_\alpha$-Closed.

(iii) $\lambda(\eta(A)) = \eta(A) \cup \eta(\eta(A)) = \eta(A), \Rightarrow \eta(A)$ is $\tau_G - \widetilde{g}_\alpha$-closed. □

Theorem 4.3. *Let G be a grill on a topological space (X, τ). If $U \in \widetilde{G}_\alpha O(X)$ then $U \cap \eta(A) = U \cap \eta(U \cap A)$, for any $A \subseteq X$.*

Proof. We have $U \cap \eta(A) \supseteq U \cap \eta(U \cap A)$. Let $x \in U \cap \eta(A)$ and $V \in \widetilde{G}_\alpha O(X, x)$. Then $U \cap V \in \widetilde{G}_\alpha O(X, x)$ and $x \in \eta(A) \Rightarrow (U \cap V) \cap A \in G$. $(U \cap A) \cap V \in G \Rightarrow x \in \eta(U \cap A) \Rightarrow x \in U \cap \eta(U \cap A)$. Thus $U \cap \eta(A) = U \cap \eta(U \cap A)$. □

Theorem 4.4. *If G is a grill on a space (X,τ) with $\widetilde{G}_\alpha O(X)-\phi \subseteq G$, then for all $U \in \widetilde{G}_\alpha O(X), U \subseteq \eta(U)$.*

Proof. If $U=\phi$ then $\eta(U)=\phi=U$. If $\widetilde{G}_\alpha O(X)-\phi \subseteq G$, then $\eta(X)=X$. If $x \notin \eta(X) \Rightarrow \exists V \in \widetilde{G}_\alpha O(X,x)$ such that $V \cap X \notin G \Rightarrow V \notin G$, a contradiction. By using the Theorem 4.3 we have for any $U \in \widetilde{G}_\alpha O(X)-\phi, U \cap \eta(X)=U \cap \eta(U \cap X)$ and hence $U=U \cap X=U \cap \eta(U)$. Thus $\eta(U) \supseteq U$. □

5. New class of sets via $\widetilde{g}_\alpha$-Grills

Definition 5.1. Let (X,τ) be a topological space and G be a grill on X. A subset A in X is said to be

(i) η-open if $A \subseteq \widetilde{g}_\alpha - Int(\eta(A))$.
(ii) h-set if $\widetilde{g}_\alpha - Int(\lambda(A)) = \widetilde{g}_\alpha - Int(A)$.
(iii) hη-set if $\widetilde{g}_\alpha - Int(\eta(A)) = \widetilde{g}_\alpha - Int(A)$.

Proposition 5.2. *Every hη-set is a h-set.*

Proof. Let A be a hη-set then $\widetilde{g}_\alpha - Int(\lambda(A) = \widetilde{g}_\alpha - Int(A \cup \eta(A)) \subseteq \widetilde{g}_\alpha - Int(\eta(A)) \cup \widetilde{g}_\alpha - Int(A) \subseteq \widetilde{g}_\alpha - Int(A)$.
$\widetilde{g}_\alpha - Int(A) \subseteq \widetilde{g}_\alpha - Int(\eta(A) \cup (A)) = \widetilde{g}_\alpha - Int(\lambda(A))$. Hence $\widetilde{g}_\alpha - Int(\lambda(A)) = \widetilde{g}_\alpha - Int(A)$. Hence A is a h-set. □

Remark 5.3. The converse of the Proposition 5.2 need not be true.

Example 5.4. Let $X=\{a,b,c\}$, $\tau=\{\phi,\{a\},\{b,c\},X\}$ and
$\widetilde{G}_\alpha O(X)=\{\phi,X,\{a\},\{b,c\}\}$
$G=\{\{a\},\{b\},\{a,b\}\}$
h sets are $\{\phi,X,\{a\},\{c\},\{a,c\},\{b,c\}\}$ and hη sets are $\{\phi,X,\{a\},\{c\},\{a,c\}\}$. The set $\{b,c\}$ is a h-set but not a hη set.

Remark 5.5. Any open $\widetilde{g}_\alpha$-open (open) and η-open set are independent of each other.

Example 5.6. Let $X=\{a,b,c\}$, $\tau=\{\phi,\{a\},X\}$ and
$\widetilde{G}_\alpha O(X)=\{\phi,X,\{a\},\{a,b\},\{a,c\}\}$
$G=\{\{a\},\{b\},\{a,b\}\}$.
The set $\{a,b\}$ is η-open but not open. The set $\{a,c\}$ is $\widetilde{g}_\alpha$-open but not η-open.

Example 5.7. Let $X=\{a,b,c\}$, $\tau=\{\phi,\{a\},\{b,c\},X\}$ and $G=\{\{a\},\{b,c\}\}$.
η-open sets are $\{X,\{a\},\{b\},\{a,b\}\}$. The set $\{a,b\}$ is η open but not open and $\widetilde{g}_\alpha$-open.

Proposition 5.8. *A τ_G-closed set is equivalent to a h-set.*

Proof. Let A be a subset in (X,τ,G) which is a topological space with the grill G. Then $\eta(A)$ is τ_G-closed by Theorem 4.2 $\widetilde{g}_\alpha - int(\lambda(\eta(A)) = \widetilde{g}_\alpha - int(\eta(A) \cup \eta(\eta(A)) = \widetilde{g}_\alpha - int(\eta(A))$. Therefore $\eta(A)$ is a h-set. □

Definition 5.9. A subset A of (X,τ,G) is said to be G-regular if $\widetilde{g}_\alpha - int(\lambda(A)) = A$.

Proposition 5.10. *Every G-regular set is a h-set.*

Proof. Let A be a G-regular set. Then $\widetilde{g}_\alpha - int(\lambda(A)) = A \Rightarrow \widetilde{g}_\alpha - int(\lambda(A)) = \widetilde{g}_\alpha - int(A)$. ☐

Remark 5.11. The converse of Proposition 5.10 need not be true.

Example 5.12. Let $X=\{a,b,c\}$, $\tau=\{\phi,\{a\},\{b,c\},X\}$ and $G=\{a\}$.
The set $\{a,c\}$ is a h-set but not a G-regular set.

Definition 5.13. A set A is said to be a k-set if $\widetilde{g}_\alpha - int(\widetilde{g}_\alpha - Cl(A)) = \widetilde{g}_\alpha - int(A)$.

Proposition 5.14. *Every closed and $\widetilde{g}_\alpha$-closed set is a k-set.*

Proof. Let A be a $\widetilde{g}_\alpha$-closed set. Then $\widetilde{g}_\alpha - Cl(A) = A$, $\widetilde{g}_\alpha - int(\widetilde{g}_\alpha - Cl(A)) = \widetilde{g}_\alpha - int(A)$

Remark 5.15. The converse of Proposition 5. 14 need not be true.

Example 5.16. Let $X=\{a,b,c\}$, $\tau=\{\phi,\{a,b\},X\}, G=\{\{a\},\{b\},\{a,b\}\}$ and k-sets are $\{\phi,X,\{a\},\{b\},\{c\},\{a,c\},\{b,c\}\}$
$\widetilde{G}_\alpha C(X)=\{\phi,X,\{c\},\{a,c\},\{b,c\}\}$. The set $\{a\}$ is a k-set but not a $\widetilde{g}_\alpha$-closed set.

Proposition 5.17. *If A and B are k-sets then $A\cap B$ is a k-set.*

Proof. $\widetilde{g}_\alpha - int(\widetilde{g}_\alpha - Cl(A\cap B)) \subseteq \widetilde{g}_\alpha - int(\widetilde{g}_\alpha - Cl(A) \cap \widetilde{g}_\alpha - Cl(B)) = \widetilde{g}_\alpha - int(\widetilde{g}_\alpha - Cl(A) \cap \widetilde{g}_\alpha - int(\widetilde{g}_\alpha - Cl(B)) = \widetilde{g}_\alpha - int(A\cap B)$. Also $\widetilde{g}_\alpha - int(A\cap B) \subseteq \widetilde{g}_\alpha - int(\widetilde{g}_\alpha - Cl(A\cap B))$. Hence $A\cap B$ is a k-set. ☐

Proposition 5.18. *If A is a k-set and $B\subseteq X$ with $A\subseteq B\subseteq \widetilde{g}_\alpha - \mathrm{Cl}(A)$ then B is a k-set.*

Proof. We have $\widetilde{g}_\alpha - Cl(B) \subseteq \widetilde{g}_\alpha - Cl(A)$. Then $\widetilde{g}_\alpha - Int(B) \subseteq \widetilde{g}_\alpha - Int(\widetilde{g}_\alpha - Cl(B)) \subseteq \widetilde{g}_\alpha - Int(\widetilde{g}_\alpha - Cl(A)) = \widetilde{g}_\alpha - Int(A) \subseteq \widetilde{g}_\alpha - Int(B)$. Hence $\widetilde{g}_\alpha - int(\widetilde{g}_\alpha - Cl(B)) = \widetilde{g}_\alpha - int(B)$. ☐

Proposition 5.19. *Every k-set is a h-set.*

Proof. $A\subseteq A\cup\eta(A), \widetilde{g}_\alpha - int(A) \subseteq \widetilde{g}_\alpha - int(\lambda(A)) = \widetilde{g}_\alpha - int(A\cup\eta(A)) = \widetilde{g}_\alpha - int(A\cup \widetilde{g}_\alpha - Cl(A)) = \widetilde{g}_\alpha - int(\widetilde{g}_\alpha - Cl(A)) = \widetilde{g}_\alpha - int(A)$. Therefore A is a h-set. ☐

Example 5.20. Let $X=\{a,b,c\}$, $\tau=\{\phi,\{a\},\{b,c\},X\}, G=\{a\}$. The set $\{a,c\}$ is a h-set but not a k-set. Hence the converse of the Proposition 5.19 need not be true.

Proposition 5.21. *If A and B are h-sets then $A\cap B$ is a h-set.*

Proof. $\widetilde{g}_\alpha - int(A\cap B) \subseteq \widetilde{g}_\alpha - int(\lambda(A\cap B)) = \widetilde{g}_\alpha - int(\lambda(A\cap B)\cap\lambda(A\cap B)) = \widetilde{g}_\alpha - int(\lambda(A\cap B) \cap \widetilde{g}_\alpha - int(\lambda(A\cap B) \subseteq \widetilde{g}_\alpha - int(\lambda(A)) \cap \widetilde{g}_\alpha - int(\lambda(B)) = \widetilde{g}_\alpha - int(A) \cap \widetilde{g}_\alpha - int(B) = \widetilde{g}_\alpha - int(A\cap B)$. Hence $A\cap B$ is a h-set. ☐

Definition 5.22. Let (X,τ) be a topological space and G be a grill on X. A subset A of X is said to be G-preopen if $A \subseteq \widetilde{g}_\alpha - int(\lambda(A))$.

Proposition 5.23. *If A is a G-preopen set then* $\widetilde{g}_\alpha - Cl(\widetilde{g}_\alpha - \mathrm{int}(\lambda(A)) = \widetilde{g}_\alpha - \mathrm{Cl}(A)$

Proof. $\widetilde{g}_\alpha - Cl(A) \subseteq \widetilde{g}_\alpha - Cl(\widetilde{g}_\alpha - int(\lambda(A)) \subseteq \widetilde{g}_\alpha - Cl(\lambda(A)) = \widetilde{g}_\alpha - Cl(A \cup \eta(A)) = \widetilde{g}_\alpha - Cl(A) \cup \widetilde{g}_\alpha - Cl(\eta(A)) = \widetilde{g}_\alpha - Cl((A) \cup \eta(A)) \subseteq \widetilde{g}_\alpha - Cl(A)$. □

Proposition 5.24. *Every η-open set is G-preopen.*

Proof. Let A be a η-open set then $A \subseteq \widetilde{g}_\alpha - Int(\eta((A)) \subseteq \widetilde{g}_\alpha - Int(A \cup \eta(A)) = \widetilde{g}_\alpha - Int(\lambda(A))$. □

Remark 5.25. The converse of the Proposition 5.24 need not be true. In Example 5.20 the set $\{b,c\}$ is G-preopen but not a η set.

Proposition 5.26. *Let (X,τ,G) be a grill topological space with arbitrary index set I. Then*

(i) *If $\{A_i : i \in I\}$ are G-preopen sets then $\bigcup_{i\in I} A_i$ is a G-preopen set.*
(ii) *If A is a G-preopen set and U is a $\widetilde{g}_\alpha$-open set $A \cap U$ is a G-preopen set.*

Proof. (i) Let $\{A_i : i \in I\}$ are G-preopen sets then $A_i \subseteq \widetilde{g}_\alpha - Int(\lambda(A_i))$ for each $i \in I$. Thus $\bigcup_{i\in I} A_i \subseteq\subseteq \widetilde{g}_\alpha - Int(\bigcup \lambda(A_i)) = \widetilde{g}_\alpha - Int(\bigcup(A_i \cup \eta(A_i))) = \widetilde{g}_\alpha - Int(\bigcup(A_i) \cup (\bigcup \eta(A_i)) = \widetilde{g}_\alpha - Int(\bigcup A_i \cup \eta(\bigcup A_i)) = \widetilde{g}_\alpha - Int(\lambda(\bigcup(A_i))$.
(ii) Let A be a G-preopen set and U is a $\widetilde{g}_\alpha$-open set then by Theorem 4.3 $U \cap A \subseteq U \cap \widetilde{g}_\alpha - Int(\lambda(A)) = U \cap \widetilde{g}_\alpha - Int(A \cup \eta(A)) = \widetilde{g}_\alpha - Int(U \cap (A \cup (\eta(A)) = \widetilde{g}_\alpha - Int((U \cap A) \cup (U \cap (\eta(A))) = \widetilde{g}_\alpha - Int((U \cap A) \cup (U \cap (\eta(U \cap A))) \subseteq \widetilde{g}_\alpha - Int(U \cap A) \cup \eta(U \cap A) = \widetilde{g}_\alpha - Int(\lambda(U \cap A))$. □

Definition 5.27. Let (X,τ) be a topological space. Let G be a grill on X. A subset H in X is said to be a Gh-set if there is a $\widetilde{g}_\alpha$-open set and a h-set A in X such that $H = U \cap A$.

Definition 5.28. Let (X,τ) be a topological space. Let G be a grill on X. A subset H in X is said to be a Gη set if there is a $\widetilde{g}_\alpha$-open set and a hη-set A in X such that $H = U \cap A$.

Proposition 5.29. (i) *A h-set is a Gh-set.*
(ii) *A hη-set is a Gη set.*

Proof. Since every h-set and hη-set A can be written as $X \cap A$. Hence the proof is completed. □

Proposition 5.30. *Any open ($\widetilde{g}_\alpha$-open) is a Gh-set and Gη-set.*

Proof. Since $U = U \cap X$ and every open set is $\widetilde{g}_\alpha$-open and $\widetilde{g}_\alpha - Int(\lambda(X)) = \widetilde{g}_\alpha - Int(X)$. □

Remark 5.31. The converse of the Proposition 5.30 need not be true.

Example 5.32. In Example 5.20 $\{c\}$ is Gh-set and Gη-set but neither open nor $\widetilde{g}_\alpha$-open.

Proposition 5.33. *A τ_G-closed set is a Gh-set but not conversely.*

Proof. It follows from Proposition 5.8 and Proposition 5.29.

Remark 5.34. The converse of the Proposition 5.33 need not be true.

Example 5.35. In Example 5.16 the set $\{a, b\}$ is a Gh-set but not τ_G-closed.

Definition 5.36. A set A is said to be a H-set if $A = U \cap B$ where U is the $\widetilde{g}_\alpha$-open set and B is a k-set.

Proposition 5.37. *Any k-set is a H-set.*

Proof. Let S be a k-set then $S = X \cap S$ where X is $\widetilde{g}_\alpha$-open set and S is k-set. □

Remark 5.38. The converse of the Proposition 5.37 need not be true.

Example 5.39. In Example 5.6 $\{a, b\}$ is a H-set but not a k-set.

Proposition 5.40. *Any $\widetilde{g}_\alpha$-closed set is a H-set.*

Proof. Since any $\widetilde{g}_\alpha$-closed set is a k set and any k-set is a H-set. □

Remark 5.41. In Example 5.6 the set $\{a, b\}$ is a H-set but not a $\widetilde{g}_\alpha$-closed.

Proposition 5.42. *Any $\widetilde{g}_\alpha$-open set is G-preopen.*

Proof. $\widetilde{g}_\alpha - Int(A) = A \subseteq \widetilde{g}_\alpha - Int(\lambda(A))$. □

Proposition 5.43. (i) *A H-set is a Gh-set.*
(ii) *A Gη set is a Gh-set.*

Proof. (i) Let C be a H-set then $C = U \cap A$ where U is $\widetilde{g}_\alpha$-open and A is a k-set. $C = U \cap \widetilde{g}_\alpha - Int(A) = U \cap \widetilde{g}_\alpha - Int(\widetilde{g}_\alpha - Cl(A)) = U \cap \widetilde{g}_\alpha - Int(A \cup \widetilde{g}_\alpha - Cl(A)) \supseteq U \cap \widetilde{g}_\alpha - Int(A \cup (\eta(A)) = U \cap \widetilde{g}_\alpha - Int(\lambda(A)) \supseteq U \cap \widetilde{g}_\alpha - Int(A) = C$.
(ii) Since any hη set is a h-set the result follows. □

Example 5.44. In the Example 5.20 the set $\{a, c\}$ is a Gh-set but not a H-set. In Example 5.16 the set $\{b, c\}$ is a Gh-set but not a Gη-set.

Proposition 5.45. *A subset A in a grill space (X, τ, G) is $\widetilde{g}_\alpha$-open if and only if it is a G-preopen and a Gh-set.*

Proof. *Necessity*. Let A be a $\widetilde{g}_\alpha$-open set. Then $A = \widetilde{g}_\alpha - Int(A) \subseteq \widetilde{g}_\alpha - Int(A \cup \eta(A)) = \widetilde{g}_\alpha - Int(\lambda(A))$. Hence it is G-preopen and by Proposition 5.30 it is a Gh-set.
Sufficiency. Since S is a Gh-set $S = U \cap A$ where U is $\widetilde{g}_\alpha$-open and $\widetilde{g}_\alpha - Int(\lambda(A)) = \widetilde{g}_\alpha - Int(A)$. Since S is G-preopen $S \subseteq \widetilde{g}_\alpha - Int(\lambda(S)) = \widetilde{g}_\alpha - Int(\lambda(U \cap A)) = \widetilde{g}_\alpha - Int(\lambda(U \cap A) \cap (\lambda(U \cap A))) \subseteq \widetilde{g}_\alpha - Int(\lambda(U) \cap \lambda(A)) = \widetilde{g}_\alpha - Int(\lambda(U)) \cap \widetilde{g}_\alpha - Int(\lambda(A)) =$

$\widetilde{g}_\alpha - Int(U \cup \eta(U)) \cap \widetilde{g}_\alpha - Int(\lambda(A)) \subseteq \widetilde{g}_\alpha - Int(\widetilde{g}_\alpha - Cl(U)) \cap \widetilde{g}_\alpha - Int(\lambda(A)) = \widetilde{g}_\alpha - Int(\widetilde{g}_\alpha - Cl(U)) \cap \widetilde{g}_\alpha - Int(A)$. Hence $S = U \cap A = (U \cap A) \cap U \subseteq \widetilde{g}_\alpha - Int(\widetilde{g}_\alpha - Cl(U)) \cap \widetilde{g}_\alpha - Int(A) \cap U = \widetilde{g}_\alpha - Int(\widetilde{g}_\alpha - Cl(U)) \cap U) \cap \widetilde{g}_\alpha - Int(A) = U \cap \widetilde{g}_\alpha - Int(A)$. Therefore $S = U \cap A \supseteq U \cap \widetilde{g}_\alpha - Int(A)$ and $S = U \cap \widetilde{g}_\alpha - Int(A)$. Thus S is a $\widetilde{g}_\alpha$-open set. □

Corollary 5.46. If S is both Gη-set and η-open set in (X, τ, G), then S is $\widetilde{g}_\alpha$-open.

Proof. Let S be a Gη-set and η set then $S = U \cap A$ where U is $\widetilde{g}_\alpha$-open and $\widetilde{g}_\alpha - Int(A) = \widetilde{g}_\alpha - Int(\eta(A))$

$S \subseteq \widetilde{g}_\alpha - Int(\eta(S)) = \widetilde{g}_\alpha - Int(\eta(U \cap A)) \subseteq \widetilde{g}_\alpha - Int(\eta(U) \cap \eta(A)) = \widetilde{g}_\alpha - Int(\eta(U) \cap \widetilde{g}_\alpha - Int(\eta(A)) \subseteq \widetilde{g}_\alpha - Int(\widetilde{g}_\alpha - Cl(U)) \cap \widetilde{g}_\alpha - Int(\eta(A)) = \widetilde{g}_\alpha - Int(\widetilde{g}_\alpha - Cl(U)) \cap \widetilde{g}_\alpha - Int(A)$

Hence $S = U \cap A = (U \cap A) \cap U \subseteq \widetilde{g}_\alpha - Int(\widetilde{g}_\alpha - Cl(U)) \cap \widetilde{g}_\alpha - Int(A) \cap U = \widetilde{g}_\alpha - Int(\widetilde{g}_\alpha - Cl(U)) \cap U) \cap \widetilde{g}_\alpha - Int(A) = U \cap \widetilde{g}_\alpha - Int(A)$.

Therefore $S = U \cap A \supseteq U\widetilde{g}_\alpha - Int(A)$ and $S = U \cap \widetilde{g}_\alpha - Int(A)$. Thus S is a $\widetilde{g}_\alpha$-open set.

Definition 5.47. Let (X, τ, G) be a grill space and A is a subset of X. Then A is said to be a G-dense set if $\lambda(A) = X$.

Proposition 5.48. *A subset A of a grill in a space (X, τ, G) is G-dense if and only if for every $\widetilde{g}_\alpha$-open set U containing $x \in X$, $A \cap U \in G$.*

Proof. *Necessity*. Let A be a G-dense set. Then, for every $\widetilde{g}_\alpha$-open set U containing $x \in X, x \in \lambda(A) = A \cup \eta(A)$. Hence if $x \in A$ then $A \cap U \in G$ and if $x \in \eta(A)$, Then $A \cap U \in G$.

Sufficiency. Let every $x \in X$ and every $\widetilde{g}_\alpha$-open set U containing $x \in X$ such that $A \cap U \in G$. Then if $x \in A$ or $x \in \eta(A)$. We have $A \cap U \in G$. Then $x \in \lambda(A)$ and thus $X \subseteq \lambda(A)$. Therefore $\lambda(A) = X$. □

Proposition 5.49. *If U is a $\widetilde{g}_\alpha$-open set U and A is a G-dense set in (X, τ, G), then $\lambda(U) = \lambda(U \cap A)$.*

Proof. Since $A \cap U \subseteq U$, we have $\lambda(U \cap A) \subseteq (U)$. Conversely, if $x \in \lambda(U), x \in U$ and $x \in \eta(U)$. Then for every $\widetilde{g}_\alpha$-open set V containing x, $U \cap V \in G$. Let $W = U \cap V \in \widetilde{G}_\alpha O(X, x)$. Since $\lambda(A) = X, W \cap A \in G$. i.e. $W = (U \cap A) \cap V \in G$. Therefore $x \in \lambda(U \cap A)$ and $\lambda(U) = \lambda(U \cap A)$. □

References

[1] A. Ahmed and S. Mohd, On decomposition of continuity via b-open sets, *Bol. Soc. Paran. Mat.* **26**(2008), 53–64.

[2] G. Choquet, Sur les notions de filtere et gille, *Comptes Rendus Acad. Sci. Paris* **224**(1974), 171–173.

[3] Y. Erguang and Y. Pongfei, On Decomposition of A-continuity, *Acta Math. Hungar.* **110**(4) (2006), 309–313.

[4] S. Jafari, M.L. Thivagar and Nirmala Rebecca Paul, Remarks on $\widetilde{g}_\alpha$-closed sets, *International Mathematical Forum* **5**(2010), 1167–1178.

[5] T. Noiri and O.R. Sayed, On decomposition of continiuty, *Acta Math. Hungar.* **111**(4) (2006), 1–8.

[6] M.L. Thivagar and Nirmala Rebecca Paul, Remarks on $\widetilde{g}_\alpha$-irresolute functions, *Bol. Soc. Paran. Mat.* **29**(1) (2011), 55–66.

[7] M.L. Thivagar and Nirmala Rebecca Paul, New types of topological mappings, *International Journal of Mathematical Sciences and Engineering Applications* **5** (IV) (2011), 43–61.

[8] W.J. Thron, Proximity structure and grills, *Math. Ann.* **206** (1973), 35–62.

[9] P. Sundaram, On ω-closed sets in topology, *Acta Ciencia Indica* **4** (2000), 389–392.

[10] M.K.R.S. Veera Kumar, Between g^* and g-closed sets, *Mem. Fac. Sci. Kochi Univ. Ser. App. Math.* **21** (2000), 1–19.

[11] M.K.R.S. Veera Kumar, $^{\#}$g-semi-closed sets in topological spaces, *Antarctica J. Math.* **2** (2005), 201–222.

M. Lellis Thivagar, *Department of Mathematics, Madurai Kamaraj University, Madurai 625021, Tamilnadu, India*.
E-mail: mlthivagar@yahoo.co.in

Nirmala Rebecca Paul, *Department of Mathematics, Lady Doak College, Madurai 625002, Tamilnadu, India*.
E-mail: nimmi_rebecca@yahoo.com

Received April 14, 2011
Accepted October 21, 2011