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Group Theoretical Study of Certain Generating Functions for Modified Jacobi Polynomials

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Abstract. The object of the preset paper is to derive some generating functions with five parameters Lie-group for the modified Jacobi polynomial $P_n^{(\alpha-n,\beta)}(x)$ by interpreting n, $\alpha-n$, β simultaneously by using the Weisner's group–theoretic method.

1. Introduction

The modified Jacobi polynomials $P_n^{(\alpha-n,\beta)}(x)$, defined by

$$P_n^{(\alpha-n,\beta)}(x)=\frac{(1+\alpha-n)_n}{n!}2F_1\left[\begin{matrix}-n, & 1+\alpha+\beta\,; \\ & 1+\alpha-n\,;\end{matrix}\ \frac{1-x}{2}\right] \tag{1.1}$$

is the solution of following ordinary differential equation:

$$(1-x^2)\frac{d^2}{dx^2}P_n^{(\alpha-n,\beta)}(x)+[\beta-\alpha+n-(2+\alpha-\beta-n)x] \times\frac{d}{dx}P_n^{(\alpha-n,\beta)}(x)+n(1+\alpha+\beta)P_n^{(\alpha-n,\beta)}(x)=0\,. \tag{1.2}$$

W. Miller's (Jr.) Lie theoretic method is utilized of modified Jacobi polynomials $P_n^{(\alpha-n,\beta)}(x)$ by making suitable interpretation to the index n in order to obtain new generating functions.

In 1986 Ghosh obtained some generating functions for $P_n^{(\alpha-n,\beta)}(x)$ with the help of Weisner's method by given suitable interpretation to the index n.

The object of the present paper investigation to apply Miller's method to obtain some generating functions for modified Jacobi polynomial $P_n^{(\alpha-n,\beta)}(x)$ by interpreting the index n, with the help of Weisner's group theoretic method (Mcbride 1971).

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Here the following generating functions are derived for $P_n^{(\alpha-n,\beta)}(x)$ by finding a set of infinitesimal operators $A_i(i=1,2,3,4,5)$ constituting a Lie-algebra:

$$P_n^{(\alpha-n,\beta)}\left[\frac{yx-t_1}{y-t_1}\right]=\sum_{P=0}^{n}\frac{1}{P!}\frac{1}{2^P}(\alpha+\beta+1)_P\,P_{n-P}^{(\alpha-n+p,\beta+P)}(x)(t_1)^P, \tag{1.3}$$

$$\begin{aligned}&[1-t_2y(1+x)]^{1+\alpha}[1+t_2y(1-x)]^{\beta}P_n^{(\alpha-n,\beta)}[x+t_2y(1-x^2)]\\&\quad=\sum_{k=0}^{\infty}\frac{(-2)^k}{k!}(n+1)_kP_{n+k}^{(\alpha-n-k,\beta-k)}(x)(t_2)^k,\end{aligned} \tag{1.4}$$

$$\begin{aligned}&(1-t_3(1+x))^{1+\alpha}(1+t_3(1-x)^{\alpha}\left(1+\frac{1}{yt_3w_1}(1+yt_3(1-x))\right)^n\\&\quad\times P_n^{(\alpha-n,\beta)}\left[(yt_3(1-yt_3(1+x)))(x+yt_3(1-x^2)+\frac{1}{w_1}(1+yt_3(1-x))\right]\\&\quad=\sum_{k=0}^{\infty}\sum_{p=0}^{n-k}\frac{(-1)^k}{k!}\frac{(-1/w_1)^p}{p!}2^{k-p}(\alpha+\beta+1)_p(n-p+1)_k\\&\qquad\times P_n^{(\alpha-n+p-k,\beta+p-k)}(x)(t_3)^{k-p}.\end{aligned} \tag{1.5}$$

2. **Group theoretic method**

Replacing d/dx by $\partial/\partial x$, α by $y\frac{\partial}{\partial y}$, β by $z\frac{\partial}{\partial z}$, n by $t\frac{\partial}{\partial t}$ and $P_n^{(\alpha-n,\beta)}(x)$ by $u(x,y,z,t)$ we get the following partial differential equation

$$\begin{aligned}&(1-x^2)\frac{\partial^2u}{\partial x^2}+z(1-x)\frac{\partial^2u}{\partial z\partial x}-y(1+x)\frac{\partial^2u}{\partial y\partial x}-tx\frac{\partial^2u}{\partial t\partial x}+ty\frac{\partial^2u}{\partial t\partial y}\\&\quad+tz\frac{\partial^2u}{\partial t\partial z}+2x\frac{\partial u}{\partial x}+t\frac{\partial u}{\partial t}=0.\end{aligned} \tag{2.1}$$

Thus $u(x,y,z,t)=P_n^{(\alpha-n,\beta)}(x)y^{\alpha-n}z^{\beta}t^n$ is a solution of the differential equation (2.1).

Lets us defined the infinitesimal operators A_i $(i=1,2,3,4,5)$

$$A_i=A_i^{(1)}\partial/\partial x+A_i^{(2)}\partial/\partial y+A_i^{(3)}\partial/\partial z+A_i^{(4)}\partial/\partial t+A_i^{0}. \tag{2.2}$$

As follows

$$\left.\begin{aligned}&A_1=y\partial/\partial y\\&A_2=z\partial/\partial z\\&A_3=t\partial/\partial t\\&A_4=(x-1)zt^{-1}\partial/\partial x-z\partial/\partial t\\&A_5=(1-x^2)z^{-1}t\partial/\partial x-(x+1)z^{-1}ty\partial/\partial y-(x-1)t\partial/\partial z\\&\qquad-(x+1)z^{-1}t^2\partial/\partial t-(x+1)z^{-1}t\end{aligned}\right\} \tag{2.3}$$

which satisfy the following rules:

$$\left.\begin{array}{l} A_1[P_n^{(\alpha-n,\beta)}(x)y^{\alpha-n}z^{\beta}t^n] = (\alpha-n)P_n^{(\alpha-n,\beta)}(x)y^{\alpha-n}z^{\beta}t^n \\ A_2[P_n^{(\alpha-n,\beta)}(x)y^{\alpha-n}z^{\beta}t^n] = \beta P_n^{(\alpha-n,\beta)}(x)y^{\alpha-n}z^{\beta}t^n \\ A_3[P_n^{(\alpha-n,\beta)}(x)y^{\alpha-n}z^{\beta}t^n] = nP_n^{(\alpha-n,\beta)}(x)y^{\alpha-n}z^{\beta}t^n \\ A_4[P_n^{(\alpha-n,\beta)}(x)y^{\alpha-n}z^{\beta}t^n] = -(\alpha)P_{n-1}^{(\alpha-n,\beta+1)}(x)y^{\alpha-n}z^{\beta+1}t^{n-1} \\ A_5[P_n^{(\alpha-n,\beta)}(x)\ y^{\alpha-n}z^{\beta}t^n] = -(n+1)P_{n+1}^{(\alpha-n,\beta-1)}(x)\ y^{\alpha-n}z^{\beta-1}t^{n+1} \end{array}\right\} \quad (2.4)$$

3. **Lie Algebra**

Now we shall find the commutator relations by using commutator notation with

$$[A,B]u = (AB - BA)u$$

$$\begin{array}{llll} [A_1,A_2] = 0; & [A_2,A_3] = 0; & [A_3,A_4] = -A_4; & [A_4,A_5] = 2(A_1 + A_2) \\ [A_1,A_3] = 0; & [A_2,A_4] = 0; & [A_3,A_5] = A_5 & \\ [A_1,A_4] = 0; & [A_2,A_5] = -A_5; & & \\ [A_1,A_5] = 0; & & & \end{array}$$

So we see from the above commutator relations that set of operators $\{1, A_i,\ i = 1,2,3,4,5\}$ generating a lie Algebra.

Now the partial differential operator L, given by:

$$\begin{aligned} L &= (1-x^2)\frac{\partial^2 u}{\partial x^2} + z(1-x)\frac{\partial^2 u}{\partial z \partial x} - y(1+x)\frac{\partial^2 u}{\partial y \partial x} - tx\frac{\partial^2 u}{\partial t \partial x} \\ &\quad + ty\frac{\partial^2 u}{\partial t \partial y} + tz\frac{\partial^2 u}{\partial t \partial z} + 2x\frac{\partial u}{\partial x} + t\frac{\partial u}{\partial t} \\ &= 0\,. \end{aligned}$$

Which can be express as:

$$(x-1)L = A_5A_4 - 2A_3(A_1 + A_3)\,. \quad (3.1)$$

It can be easy verified that the operator A_i $(i = 1,2,3,4,5)$ commute with $(x-1)L$, i.e.

$$[(x-1)L, A_i] = 0\,. \quad (3.2)$$

The extended form of the group generated by A_i $(i = 1,2,3,4,5)$ are given by

$$e^{a_1A_1}u(x,y,z,t) = u(x, e^{a_1}y, z, t),$$

$$e^{a_2A_2}u(x,y,z,t) = u(x, y, e^{a_2}z, t),$$

$$e^{a_3A_3}u(x,y,z,t) = u(x, y, z, e^{a_3}t),$$

$$e^{a_4A_4}u(x,y,z,t) = u\left(\frac{tx - a_4z}{t - a_4z}, y, z, t - a_4z\right),$$

$$e^{a_5A_5}u(x,y,z,t)=\left(\frac{z-t(1+x)(a_5)}{z}\right)u\left[\frac{xz+t(1-x^2)a_5}{z},\frac{yz-yt(1+x)a_5}{z},\right.$$
$$\left.z+t(1-x)a_5,\frac{tz-t^2(1+x)a_5}{z}\right],$$

where a_i $(i=1,2,3,4,5)$ are constants.

Thus we have

$$e^{a_5A_5}e^{a_4A_4}e^{a_3A_3}e^{a_2A_2}e^{a_1A_1}u(x,y,z,t)=\left(\frac{z-t(1+x)a_5}{z}\right)u(\xi,\eta\cdot\rho,\theta). \quad (3.3)$$

Where

$$\xi=\frac{(tz-t^2(1+x)a_5)(xz+t(1-x^2)a_5)-(a_4z(z+t(1-x)a_5)}{tz^2-t^2(1+x)a_5-a_4z(z+t(1-x)a_5)},$$
$$\eta=e^{a_1-n}y\left(\frac{z-t(1+x)(a_5-n)}{z}\right),$$
$$\rho=e^{a_2}\left(z+t(1-x)a_5\right),$$
$$\theta=e^{a_5}\left(\frac{z-t(1+x)a_5}{z}\right)\left[t-\frac{a_4z^2(z+t(1-x)a_5}{z-t(1+x)a_5}\right].$$

It may be interest to remark that, by virtue of the commutator relation given above

$$\exp(a_5A_5+a_4A_4+a_3A_3+a_2A_2+a_1A_1)$$
$$\neq\exp(a_5A_5)\exp(a_4A_4)\exp(a_3A_3)\exp(a_2A_2)\exp(a_1A_1). \quad (3.4)$$

The relation (3.3) is obtained by using the operator mentioned in the right side of (3.4). Thus the order of A_i $(i=1,2,3,4,5)$ can be change at case without uttering the effect in the left member of (3.4), while that can not be change in the right member of (3.4). So if we change the order of the operator mentioned in the right side of (3.4) the relation (3.3) will be changed

4. **Generating Functions**

From the (2,1), $u(x,y,z,t)=P_n^{(\alpha-n,\beta)}(x)y^{\alpha-n}z^{\beta}t^n$ is a solution of the systems.

$$\begin{array}{ll} Lu=0 & Lu=0 \\ (A_1-\alpha-n)u=0; & (A_2-\beta)u=0; \end{array}$$

$$\begin{array}{ll} Lu=0 & Lu=0 \\ (A_3-n)u=0; & (A_1+A_2+A_3-\beta-\alpha)u=0; \end{array}$$

From (3.2) we easily get

$$S((x-1)L)P_n^{(\alpha-n,\beta)}(x)y^{\alpha-n}z^{\beta}t^n=((x-1)L)SP_n^{(\alpha-n,\beta)}(x)y^{\alpha-n}z^{\beta}t^n=0,$$

where

$$S=e^{a_5A_5}e^{a_4A_4}e^{a_3A_3}e^{a_2A_2}e^{a_1A_1}.$$

There fore the transformations $S[P_n^{(\alpha-n,\beta)}(x)y^{\alpha-n}z^{\beta}t^n]$ is annulled by L.

By putting $a_1 = a_2 = a_3 = 0$ in (3.3) we get

$$e^{a_5A_5}e^{a_4A_4}[P_n^{(\alpha-n,\beta)}(x)y^{\alpha-n}z^{\beta}t^n]$$
$$= y^{\alpha-n}\left(\frac{z-t(1+x)a_5}{z}\right)^{1+\alpha}(z+t(1-x)a_5)^{\beta}\left[t-\frac{a_4z(z+t(1-x)a_5}{z-t(1+x)a_5}\right]$$
$$\times P_n^{(\alpha-n,\beta)}\frac{(tz-t^2(1+x)a_5)(xz+t(1-x^2)a_5)-a_4z^2(z+t(1-x)a_5)}{tz^2-t^2z(1+x)a_5-a_4z^2(z+t(1-x)a_5)} \tag{4.1}$$

If we change the order of $e^{a_5A_5}e^{a_4A_4}$ we shall get the relation different from (4.1). But

$$e^{a_5A_5}e^{a_4A_4}[P_n^{(\alpha-n,\beta)}(x)y^{\alpha-n}z^{\beta}t^n]$$
$$=\sum_{k=0}^{\infty}\sum_{p=0}^{n+k}\frac{(a_5)^k}{k!}\frac{(a_4)^p}{p!}(\alpha+\beta+1)_p(-1)^k(2)^{(k-p)}(n-p+1)_k$$
$$\times P_{n-p+k}^{(\alpha-n+p-k,\beta+p-k)}(x)y^{\alpha-n+p-k}z^{\beta+p-k}t^{n-p+k}. \tag{4.2}$$

Equating the result (4.1) and (4.2) we get

$$y^{\alpha-n}\left(\frac{z-t(1+x)a_5}{z}\right)^{1+\alpha}(z+t(1-x)a_5)^{\beta}\left[t-\frac{a_4z(z+t(1-x)a_5}{z-t(1+x)a_5}\right]$$
$$\times P_n^{(\alpha-n,\beta)}\frac{(tz-t^2(1+x)a_5)(xz+t(1-x^2)a_5)-a_4z^2(z+t(1-x)a_5)}{tz^2-t^2z(1+x)a_5-a_4z^2(z+t(1-x)a_5)}$$
$$=\sum_{k=0}^{\infty}\sum_{p=0}^{n+k}\frac{(a_5)^k}{k!}\frac{(a_4)^p}{p!}(\alpha+\beta+1)_p(-1)^k(2)^{(k-p)}(n-p+1)_k$$
$$\times P_{n-p+k}^{(\alpha-n+p-k,\beta+p-k)}(x)y^{\alpha-n+p-k}z^{\beta+p-k}t^{n-p+k}. \tag{4.3}$$

Now we shall consider the following cases:

Case 1: Letting $a_5 = 0$, $a_4 = 1$ and writing $t_1 = \frac{z}{t}$ in (4.3) we get

$$P_n^{(\alpha-n,\beta)}\left[\frac{yx-t_1}{y-t_1}\right]=\sum_{P=0}^{n}\frac{1}{P!}\frac{1}{2^P}(\alpha+\beta+1)_P P_{n-P}^{(\alpha-n+p,\beta+P)}(x)(t_1)^P. \tag{4.4}$$

Which is (1.3).

Case 2: Let $a_5 = 1$, $a_4 = 0$and writing $t_2 = \frac{t}{z}$ in (4.3) we get

$$[1-t_2y(1+x)]^{1+\alpha}[1+t_2y(1-x)]^{\beta}P_n^{(\alpha-n,\beta)}[x+t_2y(1-x^2)]$$
$$=\sum_{k=0}^{\infty}\frac{(-2)^k}{k!}(n+1)_k P_{n+k}^{(\alpha-n-k.\beta-k)}(x)(t_2)^k). \tag{4.5}$$

Which is (1.4).

Case 3: Finally substituting $a_5 = 1$, $a_4 = \frac{-1}{w_1}$ and writing $\frac{t}{yz} = t_3$ in (4.3). We get

$$[1 - t_3(1+x)]^{1+\alpha}[1+t_3(1-x)]^{\beta}\left[1 + \frac{1}{yt_3w_1}(1+yt_3(1-x))\right]^n$$
$$\times P_n^{(\alpha-n,\beta)}\left[(yt_3(1-yt_3(1+x)))(x+yt_3(1-x^2)+\frac{1}{w_1}(1+yt_3(1-x))\right]$$
$$= \sum_{k=0}^{\infty}\sum_{p=0}^{n-k}\frac{(-1)^k}{k!}\frac{(-1/w_1)^p}{p!}2^{k-p}(\alpha+\beta+1)_p(n-p+1)_k$$
$$\times\ [P_n^{(\alpha-n+p-k,\beta+p-k)}(x)(t_3)^{k-p}]\,. \tag{4.6}$$

Which is (1.5).

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