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Research Article

An Umbrella Surface in $\mathbb{R}^3$ as An Orbit Surface

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Abstract. In this study, an umbrella surface as an orbit surface of an umbrella motion is defined. Matlab applications and some examples are given.

Keywords. Two parametric motion; Motion; Umbrella Surface

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1. Introduction

An orbit of a point under a displacement or a mechanism is a geometrical object. For example, the orbit of a point under a linear motion is a straight line. Similarly, a point of orbit of a rotation effect is a circle or a spherical curve. Orbits are sometimes referred to as the displacement or mechanism that causes the orbit. Sometimes geometric objects are named using the name of the person who presented it or a similarity. Whitney's umbrella [3], cylindrical helix, etc. Are some examples. Previously we had identified umbrella motion. Since the article is in print, a detailed summary was presented here. In this study, umbrella surface will be introduced using umbrella motion. If $U \subset \mathbb{R}^2$ is an open set and $\varphi : U \to \mathbb{R}^3$ is a mapping, then $\varphi(u)$ is called a surface. If φ is a continuous (differentiable), $\varphi(u)$ is called topological (differentiable) surface.

Definition 1. A subset $S \subset \mathbb{R}^3$ is a regular surface if, for each $p \in S$, there exists a neighborhood V in $\mathbb{R}^3$ and a map $\mathbf{x} : U \to V \cap S$ of an open set $U \subset \mathbb{R}^2$ onto $V \cap S \subset \mathbb{R}^3$ such that

(i) $\mathbf{x}$ is differentiable. This means that if we write

$$\mathbf{x}(u,v) = (x(u,v), y(u,v), z(u,v)), \quad (u,v) \in U,$$

the functions $x(u,v)$, $y(u,v)$, $z(u,v)$ have continuous partial derivatives of all orders in U.

(ii) $\mathbf{x}$ is a homeomorphism. Since $\mathbf{x}$ is continuous by condition (i), this means that $\mathbf{x}$ has an inverse $\mathbf{x}^{-1} : V \cap S \to U$ which is continuous; that is, $\mathbf{x}^{-1}$ is the restriction of a continuous map $F : W \subset \mathbb{R}^3 \to \mathbb{R}^2$ defined on an open set W containing $V \cap S$.

(iii) The regularity condition. For each $q \in U$, the differential $d\mathbf{x}_q : \mathbb{R}^2 \to \mathbb{R}^3$ is one to one [2].

For example, V is an open set in the xy-plane. The set

$$(x,y,z) \in \mathbb{R}^3, \quad z = 0 \text{ and } (x,y) \in V$$

is a regular surface.

Also the unit sphere

$$S^2 = (x,y,z) \in \mathbb{R}^3, \quad x^2 + y^2 + z^2 = 1$$

is a regular surface [2].

Sometimes, we can obtain a surface as a trajectory of a point under a motion with two parameters.

In this paper we define a special two parametric motion called an umbrella motion, and an umbrella surface which is an orbit of a point under an umbrella motion.

The definitions of an umbrella mechanism, an umbrella motion and n-umbrella mechanism are defined by us as follows.

Definition 2. The planar section of a mechanism with planes passing the mechanism axis or a fixed axis are the same in formal, then the mechanism is called an umbrella mechanism.

Definition 3. A motion which defined by an umbrella mechanism is called an umbrella motion.

Definition 4. If the property of an umbrella mechanism is possible at a mechanism which intersects finite planes, the mechanism is called n-umbrella mechanism.

Let M be a planar mechanism in α plane and d be fixed axis at the plane, axis of mechanism.

Let α ve β be two intersecting planes along d.

A mechanism structure can be transported from one of the intersecting planes to another planes without changing its mechanism algorithm.

Suppose that α-plane is xoz-plane. β-plane is occurred, rotating α-plane about y-axis an angle of γ. Transformation matrix from α to β is

$$R_y(\gamma) = \begin{bmatrix} C\gamma & 0 & S\gamma & 0 \\ 0 & 1 & 0 & 0 \\ -S\gamma & 0 & C\gamma & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

where $C\gamma = \cos\gamma$ and $S\gamma = \sin\gamma$. The mechanism at xoz-plane is at rotating plane which is rotated about y-axis an angle of γ and transformation matrix of the mechanism is

$$R_y(\gamma)\,{}^0A_3 = \begin{bmatrix} C\gamma C\theta_{123} & -C\gamma S\theta_{123} & S\gamma & C\gamma(l_1C\theta_1 + l_2C\theta_{12} + l_3C\theta_{123}) \\ S\theta_{123} & C\theta_{123} & 0 & l_1S\theta_1 + l_2S\theta_{12} + l_3S\theta_{123} \\ -S\gamma C\theta_{123} & S\gamma S\theta_{123} & C\gamma & -S\gamma(l_1C\theta_1 + l_2C\theta_{12} + l_3C\theta_{123}) \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

where $\theta_{123} = \theta_1 + \theta_2 + \theta_3$. It is clear that the position of the end-effector belonging to ${}^0B_3$ is the same as the position of the end-effector belonging to $R_y(\gamma)\,{}^0A_3$ [1].

Reduplicating this transfer n-times or infinite times, n umbrella mechanism or umbrella mechanisms can be established. Namely, let M be a planar mechanism at α_1-plane and D be a fixed axis at α_1. Let α_i be plane bundle which intersects with D. Suppose that the angle between the successive planes is

$$\text{angle}(\alpha_i, \alpha_{i-1}) = \theta_i,$$

and the transformation matrix is $R_i(\theta_i)$ from α_i to α_{i+1}.

In the present case, the following two theorems, which we define, can be given for

$$M_i = R_i(\theta_i)M_{i-1}. \tag{1.1}$$

Theorem 1.

$$M_n = \bigcap_{i=1}^{n} M_i$$

is a n-umbrella mechanism.

Proof. If we take $i = n$ in equation (1.1), proof is completed. □

Theorem 2.

$$\lim_{n\to\infty} M_n = M$$

is an umbrella mechanism.

Proof. Because of Definition 3 and equation (1.1), proof is clear. □

2. The Umbrella Surfaces

Let M be umbrella motion in $\mathbb{R}^3$. We know that M can be written as

$$M = R_\theta M_0,$$

where M_0 is a planar mechanism and R_θ is a rotation about the axis of M_0. We define an umbrella surface as follows.

Definition 5. Trajectory surface $M(p)$ is called an umbrella surface, where p is a point in the plane of α_0.

If planar curve $M_0(p)$ in α_0 is a differentiable curve then trajectory surface $M(p)$ is a differentiable surface.

3. Surface of an Umbrella

We pay attention that an umbrella surface and surface of an umbrella are different. We introduce an umbrella surface. Surface of an umbrella is defined as this.

A projection in plane of planar mechanism is called a planar image of a mechanism.

A planar projection of link axes of a planar mechanism is called form of planar curve of mechanism.

If form of a planar curve of a planar mechanism is differentiable, surface of the mechanism is a differentiable surface.

We denote difference between surface of an umbrella and an umbrella surface with an umbrella mechanism. In addition, an umbrella motion M_n will be used for this.

Let we define planar mechanism, M_0 as follows.

The rotating angle of motion is $\theta = \theta(x)$. a, b, c are constant (Figure 1).

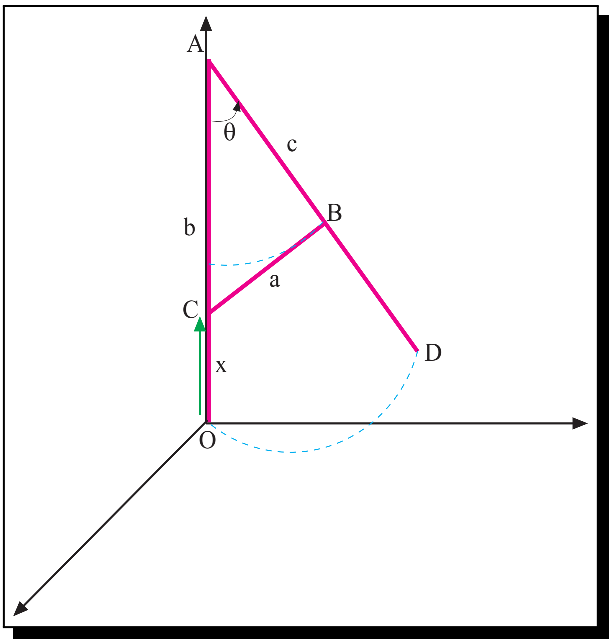

Figure 1. First position of the mechanism

C is translated along $\overrightarrow{OA}$. Let axis of $\overrightarrow{OA}$ be axis of $\overrightarrow{OZ}$.

Point A, $c = AB$ and $a = CD$ are constant, and also

$$x + b = a + c.$$

x is parameter of the motion and $b = b(x)$. θ is the rotating angle of the motion. In accordance with cosinus theorem, for a, b, c and x,

$$a^2 = b^2 + c^2 - 2bc\cos\theta \tag{3.1}$$

can be written.

For $\theta = 0$, $b = a + c$. x is variable and

$$b = a + c - x, \quad 0 < x < a + c.$$

From equation (3.1),

$$\cos\theta = \frac{b^2 + c^2 - a^2}{2bc}$$

is obtained. Depend on $\cos\theta$

$$\sin\theta = \sqrt{1 - \cos^2\theta}$$

also can be written.

Then the first step of the motion is on yoz-plane. The pole point is $A = (0, 0, a + c)$. The rotating axis is x-axis. The rotation matrix is

$$R_0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{(a+c-x)^2+c^2-x^2}{2(a+c-x)c} & -\sqrt{1-\left(\frac{(a+c-x)^2+c^2-x^2}{2(a+c-x)c}\right)^2} & -p_2\sqrt{1-\left(\frac{(a+c-x)^2+c^2-x^2}{2(a+c-x)c}\right)^2} \\ 0 & \sqrt{1-\left(\frac{(a+c-x)^2+c^2-x^2}{2(a+c-x)c}\right)^2} & \frac{(a+c-x)^2+c^2-x^2}{2(a+c-x)c} & p_2 - p_2\frac{(a+c-x)^2+c^2-x^2}{2(a+c-x)c} \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

The motion which is defined on $\alpha_1 = yoz$-plane can be translated α_2-plane which passes from z-axis and θ_{12} with α_1 without changing its mechanism algorithm. z, the common axis, is known with $\cos\theta$, $\sin\theta$, 0. The pole point of the rotation is A as in (Figure 2). The rotating angle at the first mechanism is $\theta = \theta(x)$.

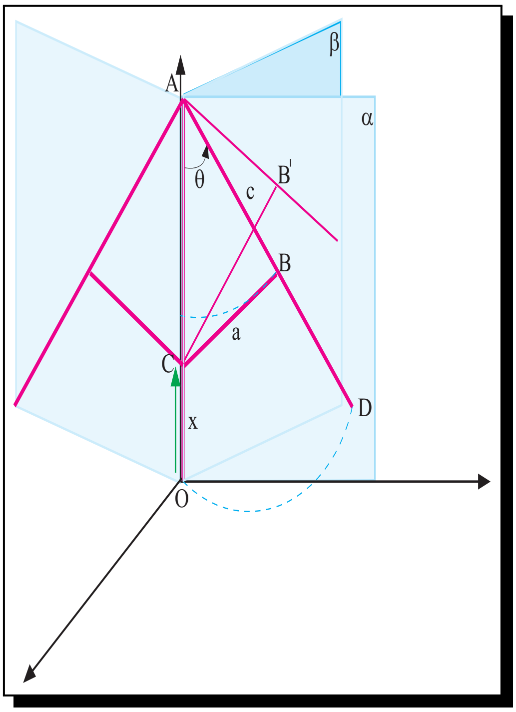

Figure 2. Transitions from α to α_i

$\frac{2\pi}{\theta}$ is the mechanism transfer number. If $\frac{2\pi}{\theta} = n$ is integer, the mechanism returns the first plane and also there are α_i-planes and n-term $R_{i(i+1)}$, the motion transfer matrix, $i = 0, 1, \dots, n$. The transformation matrix from α_i-plane to α_{i+1} can be written

$$R = (\Pi R_{i(i+1)}) R_0\,,$$

where $R_{i(i+1)}$ is the rotating matrix around z-axis and the rotating angle is γ. In addition, we calculate this matrix as

$$R = (R_z(\gamma))^n R_0.$$

If we define a body with this motion, and the points B and D are selected as acted point, the act of the piece of AB_iD_i, can be written under this motion.

The image of under umbrella motion, M_n, of mechanism is known as

$$M_n = \prod_{i=1}^{n} R_i(\theta_i) M_0\,.$$

If θ_i are equal, then $\theta_i = \frac{2\pi}{n}$.

If successive two mechanisms are M_i and M_{i+1}, the triangle $M_i P M_{i+1}$ is the part of classic umbrella surface.

The triangle mesh in space can be defined as a close surface. If A,B and C are not collinear.

$$\varphi(u,v) = \overrightarrow{OA} + \lambda\overrightarrow{AB} + \lambda\beta\overrightarrow{BC}$$

$0 \le u$, $v \le 1$, is a triangle region.

Consequently, classic umbrella surfaces as a combination of triangle mesh can be written as follows

$$\overline{\varphi}(u,\lambda,\beta) = \bigcup_{i=0}^{n} (\overrightarrow{OP} + \lambda M_i M_{i+1} + \lambda\beta M_{i+1} M_{i+2}).$$

For every fixed parameter u, we have a surface of an umbrella while umbrella is open.
Matlab programme, m-file and (Figure 3) are as follows:

```
close; clear all, clc
k=20;
a=9;
c=3;
A=[0;0;a+c]
D=[0;0;0;1]
for x=0:0.5:a/3
axis([-k k -k k -k k]);
e=10;
line([0,0],[-e,e],[0,0])
line([-e,e],[0,0],[0,0])
line([0,0],[0,0],[-e+5,e])
hold on
T=60
hold on
b=a+c-x;
```

```
CC=(b*b+c*c-a*a)/(2*b*c)
S=(1-CC*CC)^(1/2)
p1=0;
p2=a+c;
d1=p1-p1*CC-p2*S;
d2=p2-p1*S-p2*CC;
R=[1 0 0 0 ;0 CC S d1;0 -S CC d2;0 0 0 1]
R1=[cosd(T) -sind(T) 0 0 ;sind(T) cosd(T) 0 0 ;0 0 1 0;0 0 0 1]
P=[0;p1;p2;1];
PP=R*P
B=[0;0;c;1]
C=[0;0;x;1]
B1=R*B
B2=R1*B1
B3=R1*B2
B4=R1*B3
B5=R1*B3
B5=R1*B5
B6=R1*B5
B7=R1*B6

D1=R*D
D2=R1*D1
D3=R1*D2
D4=R1*D3
D5=R1*D4
D6=R1*D5
D7=R1*D6

line([C(1),B1(1)],[C(2),B1(2)],[C(3),B1(3)])
line([C(1),B2(1)],[C(2),B2(2)],[C(3),B2(3)])
line([C(1),B3(1)],[C(2),B3(2)],[C(3),B3(3)])
line([C(1),B5(1)],[C(2),B5(2)],[C(3),B5(3)])
line([C(1),B5(1)],[C(2),B5(2)],[C(3),B5(3)])
line([C(1),B6(1)],[C(2),B6(2)],[C(3),B6(3)])
line([C(1),B7(1)],[C(2),B7(2)],[C(3),B7(3)])

line([A(1),B1(1)],[A(2),B1(2)],[A(3),B1(3)])
line([A(1),B2(1)],[A(2),B2(2)],[A(3),B2(3)])
line([A(1),B3(1)],[A(2),B3(2)],[A(3),B3(3)])
line([A(1),B4(1)],[A(2),B4(2)],[A(3),B4(3)])
line([A(1),B5(1)],[A(2),B5(2)],[A(3),B5(3)])
line([A(1),B6(1)],[A(2),B6(2)],[A(3),B6(3)])
line([A(1),B7(1)],[A(2),B7(2)],[A(3),B7(3)])

line([A(1),D1(1)],[A(2),D1(2)],[A(3),D1(3)])
line([A(1),D2(1)],[A(2),D2(2)],[A(3),D2(3)])
line([A(1),D3(1)],[A(2),D3(2)],[A(3),D3(3)])
line([A(1),D4(1)],[A(2),D4(2)],[A(3),D4(3)])
```

```
line([A(1),D5(1)],[A(2),D5(2)],[A(3),D5(3)])
line([A(1),D6(1)],[A(2),D6(2)],[A(3),D6(3)])
line([A(1),D7(1)],[A(2),D7(2)],[A(3),D7(3)])

line([D1(1),D2(1)],[D1(2),D2(2)],[D1(3),D2(3)])
line([D2(1),D3(1)],[D2(2),D3(2)],[D2(3),D3(3)])
line([D3(1),D4(1)],[D3(2),D4(2)],[D3(3),D4(3)])
line([D4(1),D5(1)],[D4(2),D5(2)],[D4(3),D5(3)])
line([D5(1),D6(1)],[D5(2),D6(2)],[D5(3),D6(3)])
line([D6(1),D7(1)],[D6(2),D7(2)],[D6(3),D7(3)])
pause(3)
cla
end
cla
```

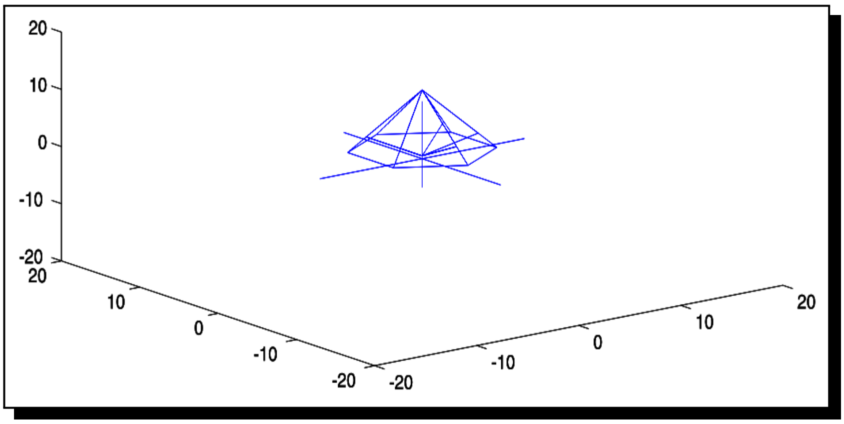

Figure 3. Surface of umbrella

4. Conclusion

A surface is defined in different ways. The essential thing is that a surface can be given as an orbit of a point, a line or a geometric object. In this state, the surface characterizes some geometric objects. In kinematics and geometry of mechanism, sometimes a surface characterizes the motion. For example, a ruled surface is a surface in geometry, but a ruled surface is a set of lines, which every line is an axis of a rotation. In this study it is noted that there is a relation between an umbrella surface and an umbrella mechanism. An umbrella surface and a surface of umbrella should not be confused. The equation of surface of umbrella is given.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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