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Fuzzy Stability of Euler-Lagrange Type Cubic Functional Equation: A Fixed Point Approach

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Abstract. In this paper, the authors investigate the generalized Hyers-Ulam stability of Euler-Lagrange type Cubic functional equation

$$2af(x+ay)+2f(ax-y)=(a^3+a)[f(x+y)+f(x-y)]+2(a^4-1)f(y) \tag{0.1}$$

in fuzzy normed space by direct method and fixed point method, where a is fixed integer with $a \neq 0, \pm 1$.

1. Introduction and Preliminaries

In 1984, Katsaras [16] defined a fuzzy norm on a linear space and at the same year Wu and Fang [34] introduced a notation of fuzzy normed space and gave the generalization of the Kolmogoroff normalized theorem for a fuzzy topological linear space. In [4], Biwas defined and studied fuzzy inner product spaces in a linear space. Since then some mathematicians have defined fuzzy metrics and norms on a linear space from various points of view [11, 18, 31, 33]. In 1994, Cheng and Mordeson introduced a definition of fuzzy norm on a linear space in such a manner that the corresponding induced fuzzy metric is of Kramosil and Michalek type [17]. In 2003, Bag and Samanta [2] modified the definition of Cheng and Mordeson [9] by removing a regular condition. They also established a decomposition theorem of a fuzzy norm into a family of crisp norms and investigated some properties of fuzzy norms (see [3]). Following [2], we give the following notation of a fuzzy norm.

Definition 1.1. A function $N : X \times \mathbb{R} \to [0,1]$ (the so-called fuzzy subset)is said to be a fuzzy norm on X if for all $x, y \in X$ and $s, t \in \mathbb{R}$, $N(x,\cdot)$ is left continuous for every x and satisfies

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(N1) $N(x,c) = 0$ for $c \leq 0$;
(N2) $x = 0$ if and only if $N(x,c) = 1$ for all $c > 0$;
(N3) $N(cx,t) = N\big(x, \frac{t}{|c|}\big)$ if $c \neq 0$;
(N4) $N(x+y,s+t) \leq \min\{N(x,s), N(y,t)\}$;
(N5) $N(x,\cdot)$ is a non-decreasing function on $\mathbb{R}$ and $\lim_{t\to\infty} N(x,t) = 1$.
(N6) For $x \neq 0$, $N(x,\cdot)$ is (upper semi)continuous on $\mathbb{R}$.

A fuzzy normed linear space is a pair (X,N), where X is a real linear space and N is a fuzzy norm on X.

Example 1.2. Let $(X, \|\cdot\|)$ be a normed linear space. Then

$$N(x,t) = \begin{cases} \frac{t}{t+\|x\|} & \text{if } t > 0, x \in X \\ 0 & \text{if } t \leq 0, x \in X \end{cases}$$

is a fuzzy norm on X.

Definition 1.3. Let (X,N) be a fuzzy normed linear space. A sequence $\{x_n\}$ in X is said to be convergent if there exits $x \in X$ such that $\lim_{n\to\infty} N(x_n - x, t) = 1$ $(t > 0)$. In that case, x is called the limit of the sequence $\{x_n\}$ and we write $N - \lim_{n\to\infty} x_n = x$.

Definition 1.4. A sequence $\{x_n\}$ in a fuzzy normed space (X,N) is called Cauchy if for each $\epsilon > 0$ and $\delta > 0$, there exists $n_0 \in N$ such that $N(x_m - x_n, \delta) > 1 - \epsilon$ $(m, n \geq n_0)$. If each Cauchy sequence is convergent, then the fuzzy norm is said to be complete and the fuzzy normed space is called a fuzzy Banach space.

A classical question in the theory of functional equations is the following: "when is it true that a function, which approximately satisfies a functional equation ϵ, must be close to an exact solution of ϵ?". If the problem accepts a solution, we say that the equation ϵ is stable. Such a problem was formulated by Ulam [32] in 1940 and solved in the next year for the Cauchy functional equation by Hyers [14]. It gave rise to the stability theory for functional equations. The result of Hyers was extended by Aoki [1] in 1950 by considering the unbounded Cauchy differences. In 1978, Rassias [27] proved that the additive mapping T, obtained by Hyers or Aoki, is linear if, in addition, for each $x \in E$, the mapping $f(tx)$ is continuous in $t \in \mathbb{R}$. Găvruta [13] generalized the Rassias' result. Following the techniques of the proof of the corollary of Hyers [14], we observed that Hyers introduced (in 1941) the following Hyers continuity condition about the continuity of the mapping for each fixed point and then he proved homogeneity of degree one and, therefore, the famous linearity. This condition has been assumed further till now, through the complete Hyers direct method, in order to prove linearity for generalized Hyers-Ulam stability problem forms (see [19]). Beginning around 1980, the stability problems of several functional equations and approximate homomorphisms have been extensively investigated by a number of authors and there are many interesting results concerning this problem(see [5, 27]).

Rassias [29], following the spirit of the innovative approach of Hyers [14], Aoki [1] and Rassias [27] for the unbounded Cauchy difference, proved a similar stability theorem in which he replaced the factor $\|x\|^p + \|y\|^p$ by $\|x\|^p \cdot \|y\|^q$ for $p, q \in \mathbb{R}$ with $p + q \neq 1$ (see also[30] for a number of other new results).

In 2003, Cadariu and Radu applied the fixed-point method to the investigation of the Jensen functional equation [6]. They could present a short and a simple proof (different of the "direct method" initiated by Hyers in 1941) for the generalized Hyers-Ulam stability of Jensen functional equation [6], for Cauchy functional equation [7], and for quadratic functional equation [8].

The functional equation

$$f(2x+y)+f(2x-y)=2f(x+y)+2f(x-y)+12f(x) \tag{1.1}$$

is called the cubic functional equation, since the function $f(x)=cx^3$ is its solution. Every solution of the cubic functional equation is said to be a cubic mapping. The stability problem for the cubic functional equation was proved by Jun and Kim [15] for mappings $f : X \to Y$, where X is a real normed space and Y is a Banach space. Later a number of mathematicians worked on the stability of some types of the cubic equation [26].

Recently several various fuzzy stability results concerning Cauchy, Jensen, quadratic and cubic functional equations were investigated in [23, 22, 24]. Mohiuddine [25] determined some stability results concerning the Jensen functional equation in intuitionistic fuzzy normed spaces. Mursaleen and Mohiuddine [26] determined some stability results concerning the cubic functional equation (1.1) in intuitionistic fuzzy normed spaces. Mihet and Radu [21] obtained some stability results for the functional equations of Cauchy and Jensen in probabilistic setting using the fixed point method.

We recall fundamental results in fixed-point theory.

Let X be a set. A function $d : X \times X \to [0,\infty]$ is called a generalized metric on X if d satisfies the following:

(1) $d(x,y)=0$ if and only if $x=y$;
(2) $d(x,y)=d(y,x)$ for all $x,y \in X$;
(3) $d(x,z) \le d(x,y)+d(y,z)$ for all $x,y,z \in X$.

Theorem 1.5. *Let (X,d) be a complete metric space and let $J : X \to X$ be strictly contractive, that is,*

$$d(Jx,Jy) \le Ld(x,y), \quad \text{for all } x,y \in X$$

for some $L<1$. Then, for each given $x \in X$, either

$$d(J^n x, J^{n+1}x)=+\infty \quad \text{for all } n \ge 0;$$

or

$$d(J^n x, J^{n+1}x)<\infty \quad \text{for all } n \ge n_0;$$

for some natural number n_0. Moreover, if the second alternative holds then

(i) *The sequence $\{J^n x\}$ is converges to a fixed point y^* of J;*

(ii) *y^* is the unique fixed point of J in the set $Y = \{y \in X : d(J^{n_0}x, y) < +\infty\}$;*

(iii) *$d(y, y^*) \leq \frac{1}{1-L} d(y, J_y)$ for all $y \in Y$.*

Throughout this paper, assume that $X, (Z, N')$ and (Y, N) are linear space, fuzzy normed linear space and fuzzy Banach space respectively.

Now use the following notation for a given mapping $f : X \to Y$

$$Df(x,y)=2af(x+ay)+2f(ax-y)-(a^3+a)[f(x+y)+f(x-y)]-2(a^4-1)f(y)$$

for all $x, y \in X$, where a is fixed integer with $a \neq 0, \pm 1$ in fuzzy normed space.

Let ψ be a function from $X \times X$ to Z. A mapping $f : X \to Y$ is called a ψ-approximately cubic mapping, if

$$N(Df(x,y), t) \geq N'(\alpha\psi(x,y), t) \tag{1.2}$$

for all $x, y \in X$ and all $t \in [0, \infty)$.

2. **Fuzzy Stability for the Functional Equation (0.1)**

In this section, we present fuzzy generalized Hyers-Ulam-stability of Euler-Lagrange type cubic functional equation (0.1).

Theorem 2.1. *Let $\psi : X \times X \to Z$ be a mapping such that, for some $0 < \alpha < a^3$*

$$N(\psi(ax, ay), t) \geq N'(\alpha\psi(x,y), t) \tag{2.1}$$

for all $x, y \in X$ and $t > 0$. Let $f : X \to Y$ be a ψ-approximate cubic mapping with $f(0) = 0$, then there is a unique cubic mapping $C : X \to Y$ such that

$$N[f(x) - C(x), t] \geq N'\left[\frac{\psi(x,0)}{a^3 - \alpha}, t\right] \tag{2.2}$$

for all $x, y \in X$ and $t > 0$.

Proof. Letting $y = 0$ in (1.2), we get

$$N\left(f(ax) - a^3 f(x), \frac{t}{2}\right) \geq N'(\psi(x,0), t) \tag{2.3}$$

for all $x \in X$ and all $t > 0$. Using (2.1) and induction on n, one can verify that

$$N'[\psi(a^n x, a^n y), t] \geq N'[\alpha^n \psi(x,x), t] \tag{2.4}$$

for all $x \in X$ and all $t > 0$. Replacing x by $a^{n-1}x$ in (2.3)

$$N\left(f(a^n x) - a^3 f(a^{n-1}x), \frac{t}{2}\right) \geq N'(\alpha^{n-1}\psi(x,0), t) \tag{2.5}$$

Divided by a^{3n} and replacing t by $\alpha^n t$ in (2.5)

$$N\left(\frac{f(a^n x)}{a^{3n}} - \frac{f(a^{n-1}x)}{a^{3(n-1)}}, \frac{\alpha^n t}{a^{3n}}\right) \geq N'\left(\frac{1}{\alpha}\psi(x,0), 2t\right) \tag{2.6}$$

for all $n \geq 1, x \in X$ and all $t > 0$. So using (2.6)

$$N\left(\frac{f(a^n x)}{a^{3n}} - \frac{f(a^m x)}{a^{3m}}, \sum_{k=m+1}^{n} \frac{\alpha^k t}{a^{3k}}\right) = N\left(\sum_{k=m+1}^{n} \frac{f(a^k x)}{a^{3k}} - \frac{f(a^{k-1} x)}{a^{3(k-1)}}, \sum_{k=m+1}^{n} \frac{\alpha^k t}{a^{3k}}\right)$$

$$\geq N'\left(\frac{1}{\alpha}\psi(x,0), 2t\right)$$

$$N\left(\frac{f(a^n x)}{a^{3n}} - \frac{f(a^m x)}{a^{3m}}, t\right) \geq N'\left(\frac{1}{\alpha}\psi(x,0), \frac{2t}{\sum\limits_{k=m+1}^{n} \frac{\alpha^k}{a^{3k}}}\right) \tag{2.7}$$

for all $n > m \geq 0, x \in X$ and all $t > 0$. Since $\lim\limits_{t\to\infty} N'(\psi(x,0), 2t) = 1$ and that $\sum\limits_{n=0}^{\infty} \left(\frac{\alpha}{a^3}\right)^n$ is convergent, we deduce that $\left\{\frac{f(a^n x)}{a^{3n}}\right\}$ is Cauchy sequence in (Y, N). Since (Y, N) is complete, this sequence converges to some point $C(x) \in Y$. Hence we can define $C : X \to Y$ by

$$C(x) = N - \lim_{n\to\infty} \frac{f(a^n x)}{a^{3n}}.$$

It follows from (1.2) that,

$$\begin{aligned} N(Df(a^n x, a^n y), t) &\geq N'(\psi(a^n x, a^n y), t) \\ &\geq N'(\alpha^n \psi(x,y), t) \\ &\geq N'\left(\psi(x,y), \left(\frac{1}{\alpha}\right)^n t\right) \end{aligned} \tag{2.8}$$

Therefore,

$$N\left(\frac{1}{a^{3n}} Df(a^n x, a^n y), \frac{t}{a^{3n}}\right) \geq N'\left(\psi(x,y), \left(\frac{1}{\alpha}\right)^n t\right)$$

$$N\left[\frac{1}{a^{3n}} Df(a^n x, a^n y), t\right] \geq N'\left(\psi(x,y), \left(\frac{a^3}{\alpha}\right)^n t\right) \tag{2.9}$$

we have

$$\begin{aligned} N(DC(x,y), t) &= N(2aC(x+ay) + 2C(ax-y) \\ &\quad - (a^3+a)[C(x+y) + C(x-y)] - 2(a^4-1)C(y), t) \\ &= N\Bigg\{2aC(x+ay) - 2a\frac{f(a^n(x+ay))}{a^{3n}} \\ &\quad + 2C(ax-y) - 2\frac{f(a^n(ax-y))}{a^{3n}} \\ &\quad - (a^3+a)\left[\left(C(x+y) - \frac{f(a^n(x+y))}{a^{3n}}\right)\right. \\ &\quad \left.+ \left(C(x-y) - \frac{f(a^n(x-y))}{a^{3n}}\right)\right] \end{aligned}$$

$$
\begin{aligned}
&-2(a^4-1)\left(C(y)-\frac{f(a^n y)}{a^{3n}}\right)+\frac{1}{a^{3n}}(Df(a^n x, a^n y), t)\Big\}\\
\geq \min\Big\{&N\left(2aC(x+ay)-2a\frac{f(a^n(x+ay))}{a^{3n}}, \frac{t}{6}\right),\\
&N\left(2C(ax-y)-2\frac{f(a^n(ax-y))}{a^{3n}}, \frac{t}{6}\right),\\
&N\left((a^3+a)\left(C(x+y)-\frac{f(a^n(x+ay))}{a^{3n}}\right), \frac{t}{6}\right),\\
&N\left((a^3+a)\left(C(x-y)-\frac{f(a^n(ax-y))}{a^{3n}}\right), \frac{t}{6}\right),\\
&N\left(2(a^4-1)\left(C(y)-\frac{f(a^n y)}{a^{3n}}\right), \frac{t}{6}\right),\\
&N\left(\frac{(Df(a^n x, a^n y), t)}{a^{3n}}, \frac{t}{6}\right)\Big\}
\end{aligned}
$$

By equations (2.9), (N5) and fact that $\lim_{n\to\infty} N\big(T(z)-\frac{f(a^n z)}{a^{3n}}, r\big)=1$ for all $z \in X$ and $r>0$, each term on the right hand side tends to 1 as $n \to \infty$. Hence

$$N(2aC(x+ay)+2C(ax-y)-(a^3+a)[C(x+y)+C(x-y)]-2(a^4-1)C(y), t)=1.$$

By (N2), it means that C satisfies the Euler Lagrange cubic function and so it is cubic.

Further more, let $x \in X$ and $t>0$ using (2.8) with $m=0$ we obtain

$$
\begin{aligned}
&N(C(x)-f(x), t)\\
&\quad\geq \min\left\{N\left(C(x)-\frac{f(a^n x)}{a^{3n}}, \frac{t}{2}\right), N\left(\frac{f(a^n x)}{a^{3n}}-f(x), \frac{t}{2}\right)\right\}\\
&\quad\geq \min\left\{N\left(C(x)-\frac{f(a^n x)}{a^{3n}}, \frac{t}{2}\right), N'\left(\frac{1}{\alpha}\psi(x,0), \frac{2t}{\sum_{k=1}^{n}\left(\frac{\alpha^k}{a^{3k}}\right)}\right)\right\}
\end{aligned}
$$

Hence

$$
\begin{aligned}
&N(C(x)-f(x), t)\\
&\quad\geq \min\left\{\lim_{n\to\infty} N\left(C(x)-\frac{f(a^n x)}{a^{3n}}, \frac{t}{2}\right), N'\left(\frac{1}{\alpha}\psi(x,0), \lim_{n\to\infty}\frac{t}{\sum_{k=1}^{n}\left(\frac{\alpha^k}{a^{3k}}\right)}\right)\right\}\\
&\quad= N'\left(\frac{\psi(x,0)}{(a^3-\alpha)}, t\right)
\end{aligned}
$$

To prove the uniqueness of the cubic function C, assume that there exists a cubic function $D: X \to Y$ which satisfies (2.2). Fix $x \in X$. Clearly $C(a^n x)=a^{3n}C(x)$

and $D(a^n x) = a^{3n} D(x)$ for all $n \in \mathbb{N}$. It follows from (2.2) that

$$
\begin{aligned}
&N(C(x) - D(x), t) \\
&\quad = N\left(\frac{C(a^n x)}{a^{3n}} - \frac{D(a^n x)}{a^{3n}}, t\right) \\
&\quad \geq \min\left\{N\left(\frac{C(a^n x)}{a^{3n}} - \frac{f(a^n x)}{a^{3n}}, \frac{t}{2}\right), N\left(\frac{f(a^n x)}{a^{3n}} - \frac{D(a^n x)}{a^{3n}}, \frac{t}{2}\right)\right\} \\
&\quad \geq \left(\psi(a^n x, 0), \frac{a^{3n} t(a^3 - \alpha)}{2}\right) \geq N'\left(\psi(x,0), \frac{a^{3n} t(a^3 - \alpha)}{2\alpha^n}\right)
\end{aligned}
$$

Since $\lim_{n\to\infty} \frac{t(a^3-\alpha)a^{3n}}{2\alpha^n} = \infty$, we obtain $N'\big(\psi(x,0), \frac{a^{3n} t(a^3-\alpha)}{2\alpha^n}\big) = 1$.

Therefore, $N(C(x) - D(x), t) = 1$ for all $t > 0$. Hence $C(x) = D(x)$. □

Now we present a result similar to Theorem 2.1 for case where $\alpha > a^3$.

Theorem 2.2. *Let $\psi : X \times X \to Z$ be a function such that for some α with $\alpha > a^3$*

$$
N\left(\psi\left(\frac{x}{a}, 0\right), t\right) \geq N'(\psi(x,0), \alpha t)
$$

and $\lim_{n\to\infty} N'(a^{3n}\psi(a^{-n}x, a^{-n}y, t)) = 1$ for all $x, y \in X$ and $t > 0$. Let $f : X \to Y$ be a ψ-approximately cubic mapping with $f(0) = 0$, then there exists a unique cubic mapping $C : X \to Y$ such that

$$
N(C(x) - f(x), t) \geq N'(\psi(x,0), (\alpha - a^3)t)
$$

for all $x \in X$ and all $t > 0$.

Proof. The techniques are completely similar to that of Theorem 2.1. Hence we present a sketch of proof. Put $y = 0$ in (1.2) to get

$$
N(2f(ax) - 2a^3 f(x), t) \geq N'(\psi(x,0), t) \quad (x \in X, t > 0),
$$

whence

$$
N\left(f(x) - a^3 f\left(\frac{x}{a}\right), t\right) \geq N'(\psi(x,0), 2\alpha t) \quad (x \in X, t > 0).
$$

For each $x \in X, n \geq 0, m \geq 0$ and $t > 0$, one can deduce

$$
N(a^{3n+3m} f(a^{-(n+m)}x) - a^{3m} f(a^{-m}x), t) \geq N'\left(\psi(x,0), \frac{t}{\sum_{k=m+1}^{n+m} \frac{a^{3k}}{2a^3\alpha^k}}\right) \tag{2.10}
$$

from which we conclude that $\{a^{3n} f(a^{-n}x)\}$ is a Cauchy sequence in the fuzzy Banach space (Y, N). Therefore, there is a function $C : X \to Y$ defined by $C(x) = \lim_{n\to\infty} a^{3n} f(a^{-n}x)$. Employing (2.10) with $m = 0$, we obtain

$$
N(C(x) - f(x), t) \geq N'(\psi(x,0), (\alpha - a^3)t)
$$

for all $x \in X$ and $t > 0$. □

3. Fuzzy Stability of Euler-Lagrange Type Cubic Functional Equation (0.1) using Fixed Point

The aim of this section is to give an alternative proof for that result in section 2 based on the fixed point method. And this method provides a better estimation.

Theorem 3.1. *Let $\psi : X \times X \to Z$ be a function such that for some real α with $0 < |\alpha| < a^3$*

$$N'(\psi(ax,0),t) \geq N'(\alpha\psi(x,0),t) \tag{3.1}$$

and $\lim_{n\to\infty} \psi(a^n x, a^n y, a^n t) = 1$ for all $x, y \in X$ and $t > 0$. Let $f : X \to Y$ be a ψ-approximately cubic mapping with $f(0) = 0$, then there exists a unique cubic mapping $C : X \to Y$ such that

$$N(C(x) - f(x), t) \geq N'(\psi(x,0),(a^3 - |\alpha|)2t) \tag{3.2}$$

for all $x \in X$ and all $t > 0$.

Proof. Without loss of generality we may assume that $\alpha > 0$. By setting $y = 0$ in (2.1), we get

$$N\left(\frac{f(ax)}{a^3} - f(x), t\right) \geq N'(\psi(x,0), 2a^3 t). \tag{3.3}$$

Let $G(x,t) = N'(\psi(x,0), 2a^3 t)$.

Consider the $E = \{g : X \to Y, g(0) = 0\}$ and the mapping d_G defined on $E \times E$ by

$$d_G(g,h) = \inf\{a \in R_+ : N(g(x) - h(x), at) \geq G(x,t), \text{ for all } x \in X,\ t > 0\}.$$

By lemma 2.1 of [21], d_G is a complete generalized metric on E. we now define a function $J : E \to E$ by

$$J_g(x) = \frac{1}{a^3} g(ax), \quad \text{for all } x \in X.$$

We prove that J is a strictly contractive mapping with the Lipschitz constant $\frac{\alpha}{a^3}$. In deed, let g, h in E be given such that $d_G(g,h) < \epsilon$. Then

$$N\left(g(x) - h(x), \epsilon t\right) \geq G(x,t), \quad \text{for all } x \in X,\ t > 0.$$

Therefore

$$\begin{aligned} N\left(J_g(x) - J_h(x), \frac{\alpha}{a^3}\epsilon t\right) &= N\left(\frac{g(ax)}{a^3} - \frac{h(2x)}{a^3}, \frac{\alpha}{a^3}\epsilon t\right) \\ &= N(g(ax) - h(ax), \alpha\epsilon t) \\ &\geq G(ax, \alpha t) \\ &= G(x,t), \quad \text{for all } x \in X,\ t > 0 \end{aligned}$$

Hence, it holds that $d_G(J_g, J_h) \leq \frac{\alpha}{a^3}\epsilon$, that is $d_G(J_g, J_h) \leq \frac{\alpha}{a^3} d_G(g,h)$, for all $g, h \in E$.

Next, from $N\left(f(x) - \frac{f(ax)}{a^3}, t\right) \geq G(x,t)$, it follows that $d_G(f, J_f) \leq 1$. From the fixed point alternative, we deduce the existence of a fixed point of J, that is the existence of a mapping $C : X \to Y$ such that $C(ax) = a^3 C(x)$ for each $x \in X$. Moreover, we have $d_G(J^n f, C)$, which implies

$$N - \lim_{n\to\infty} \frac{f(a^n x)}{a^3} = C(x) \quad \text{for all } x \in X.$$

Also, $d_G(f, C) \leq \frac{1}{1-L} d_G(f, J_f)$ implies the inequality

$$d_G(f, C) \leq \frac{1}{1 - \frac{\alpha}{a^3}} = \frac{a^3}{a^3 - \alpha}.$$

If ϵ_n is a decreasing sequence converging to $\frac{a^3}{a^3-\alpha}$, then

$$N(C(x) - f(x), \epsilon_n t) \geq G(x,t), \quad \text{for all } x \in X,\ t > 0,\ n \in N$$

that is (as G is left continuous)

$$\begin{aligned} N(C(x) - f(x), t) &\geq G\left(x, \frac{a^3 - \alpha}{a^3} t\right), \quad \text{for all } x \in X,\ t > 0 \\ &= N'(\psi(x,0), (a^3 - \alpha)2t), \quad \text{for all } x \in X,\ t > 0. \end{aligned}$$

The cubic of C can be proved in a similar fashion as in the proof of Theorem 2.1.

The uniqueness of C follows from the fact that C is the unique fixed point of J with the property that there exists $C \in (0, \infty)$ such that

$$N(C(x) - f(x), Ct) \geq G(x,t), \quad \text{for all } x \in X,\ t > 0. \qquad \square$$

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