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Research Article

Dynamics of Oblate Test Particle under the Influence of Oblate and Radiating Primaries in Elliptic Restricted Three Body Problem

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Abstract. This paper presents a generalized problem of the photogravitational restricted three body, where both the primaries are radiating; in the sense that the eccentricity of the orbits and the oblateness due to both the primaries and infinitesimal are considered. The positions and stability of the equilibrium points of this problem are studied. The stability analysis ensures that, the collinear equilibrium points are unstable in the linear sense while the stability condition for the triangular points is obtained. For illustrative numerical exploration four binary system: Luyten-726, Kruger-60 and Alpha-Centauri are considered, the location and stability of their planar equilibrium points are studied semi-analytically.

Keywords. Elliptical Restricted Three Body Problem; Oblateness; Binary system; Triangular and Collinear equilibrium points

MSC. 70F07; 70E50

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1. Introduction

The *Restricted Three Body Problem* (RTBP) is one of the most rigorously studied branch of celestial mechanics since it plays an important role in studying the motion of not only artificial satellite but also planets, minor-planets, comets and many other celestial bodies. RTBP, however neglects the effect of radiation force acting on the infinitesimal mass if one or both primaries are intense emitter of radiation. It is observed that the motion of cosmic dust when near a star is most affected by the repulsive force of radiation pressure and drag forces. Poynting [16] studied that small meteors or cosmic dust are affected by not only gravitational force but also the radiation force as they come near a luminous body. This effect is prominently apparent by a gradual loss of angular momentum of the infinitesimal mass, as a result a net drag force is active in the direction opposite to that of motion. Later the relativistic form of this problem was given by Robertson [17]. Several studies [2, 9, 11, 12, 21] of the restricted problem have since analyzed the effect of radiation pressure.

Another direction in which the *Classical Restricted Three Body Problem* (CRTBP) has been enriched is the inclusion of additional effects observed when the primaries follows not the circular but elliptical path. This generalization, referred to as *Elliptical Restricted Three Body Problem* (ERTBP), is better equipped in studying the long-time behavior of important dynamical systems. The reason being that in this problem though the position of the primaries are assumed to be fixed the Hamiltonian depends explicitly on time [1, 27, 28].

In RTBP the bodies are assumed to be either point masses or spherical in shape. But it is practically noticed that the celestial bodies are axis-symmetric bodies. Therefore, perturbation effect due to the shape of the bodies on the dynamics of the system should also be taken into account. The replacement of point mass by rigid-body is quite important because of its wide applications in practical problems. The study in this direction was initiated by Nikolaev in 1970. He studied the equilibrium points in the case when the larger primary is an oblate spheroid. The work was extended by Sharma and Subba Rao [18, 19] by studying the characteristic exponents when bigger or both the primaries are oblate spheroid. Since then many authors have undertaken the study of CRTBP and ERTBP taking into account the oblateness of first or second or both primaries and various other perturbing forces [4, 10, 14, 15, 22–25]. The restricted three body problem when the oblateness of the infinitesimal is considered was also studied by some authors [5, 6]. Singh and Haruna [20] investigated the problem considering all the three participating bodies as oblate spheroid and reported the presence of five collinear equilibrium points. Also, they examined the stability of all the planar equilibrium points.

Singh and Umar [26] studied the dynamics of the planar ERTBP considering the oblateness of all three participating bodies and applied the model to binary pulsars. This motivated us to undertake the study of position and stability of triangular and collinear equilibrium points when all the three bodies are oblate spheroid and the massive primaries are radiating. The application of this problem can be found in many binary systems such as Alpha Centuari, Luyten-726, Kruger-60 and many others.

The paper is organized as follows: Section 1 gives the introduction. The formulation of the problem and equations of motion are described in Section 2. In Section 3, which is divided into two subsections the triangular and collinear equilibrium points are obtained. In Section 4 and its subsections, the analysis for stability is conducted. The numerical application of the problem is explored taking for instance four binary systems: Luyten-726, Sirius, Kruger-60 and Alpha Centauri. Finally, discussion and conclusions are included in Section 5.

2. Formulation of Problem and Equations of Motion

Assume that m_1, m_2 and m are the masses of the bigger, smaller and infinitesimal bodies respectively, where m_1 and m_2 have elliptical orbits and m is moving under their gravitational effect but m being too small does not affect the motion of the primaries. Also, let R_1 and R_2 be the distance of the mass m from m_1 and m_2 respectively and R is the distance between the primaries. The Figure 1 shows the position of the three participating bodies with respect to the inertial $(OXYZ)$ and rotating (Oxyz)frame of reference.

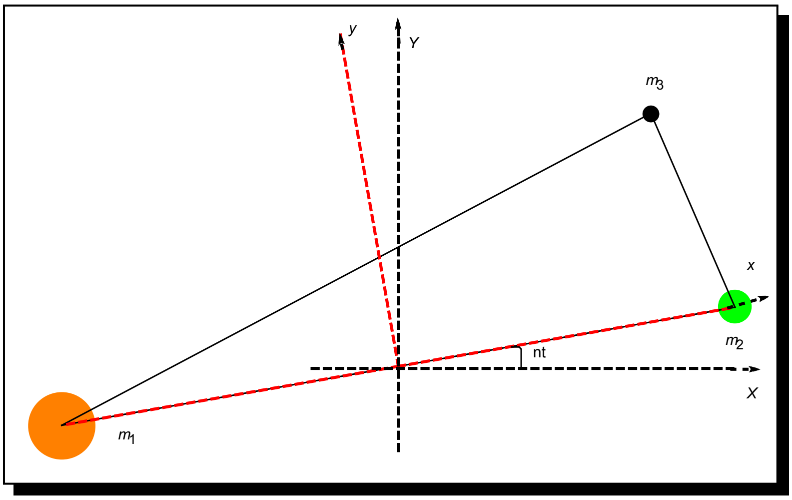


Figure 1. The position of the bodies with respect to the inertial and rotating frame of reference.

The terminologies and notations used are adapted from [27]. In this frame of reference the distance between the primaries and gravitational constant are unity. Also the sum of the masses of the primaries is taken to be unity and mass ratio is given as $\mu = \frac{m_2}{m_1+m_2}$. Since both the primaries are assumed to be luminous bodies, q_1 and q_2 are assumed to be mass reduction factors of the two primaries, where $0 < q_i < 1$, $i = 1,2$. If we assume the oblateness of primary, secondary and infinitesimal bodies are given by the factors A_1, A_2 and A_3 respectively, where $0 < A_i < 1$, $i = 1,2,3$, then we obtain the force function given by (2.2) (Ref. [13] and [7]). The equation of motion in dimension-less pulsating rotating barycentric reference frame is given as:

$$\begin{aligned} x'' - 2y' &= \frac{1}{1+e\cos f} U_x \, ; \\ y'' + 2x' &= \frac{1}{1+e\cos f} U_y \, ; \\ z'' &= \frac{1}{1+e\cos f} U_z \, ; \end{aligned} \tag{2.1}$$

where

$$U = \frac{x^2+y^2-z^2 e\cos f}{2} + \frac{1}{n^2}\left[-(1-\mu)\left(\frac{q_1}{r_1}+\frac{q_1A_1+A_3}{2r_1^3}\right)+\mu\left(\frac{q_2}{r_2}+\frac{q_2A_2+A_3}{2r_2^3}\right)\right]. \tag{2.2}$$

Here, the prime $'$ denotes differentiation with respect to true anomaly f. Using Kepler's law and further simplifying to linear terms, we get the mean motion n is given by:

$$n^2 = \frac{1}{a^3}\left(1+\frac{3}{2}(e^2+A_1+A_2)\right). \tag{2.3}$$

3. Position of Equilibrium Points

The Lagrangian or Libration points in the case of planar three body problem is obtained by solving the equations, $U_x^* = 0$, $U_y^* = 0$ for $x' = x'' = y' = y'' = 0 = z$; that is we need to solve the following two equations:

$$n^2x - \frac{q_1(1-\mu)(x+\mu)}{r_1^3} - \frac{q_2\mu(x+\mu-1)}{r_2^3} - \frac{3(1-\mu)(x+\mu)(A_1q_1+A_3)}{2r_1^5}$$
$$-\frac{3\mu(x+\mu-1)(A_2q_2+A_3)}{2r_2^5} = 0, \tag{3.1}$$

$$n^2y - \frac{q_1(1-\mu)y}{r_1^3} - \frac{q_2\mu y}{r_2^3} - \frac{3(1-\mu)y(A_1q_1+A_3)}{2r_1^5} - \frac{3\mu y(A_2q_2+A_3)}{2r_2^5} = 0. \tag{3.2}$$

It is well known that on solving these two equation five equilibrium points are obtained on the xy-plane which is taken to be the plane of motion of the primaries. These points are: (a) Triangular points (L_4,L_5) when $y \neq 0$ and (b) Collinear points (L_1,L_2,L_3) lying on the line joining the primaries. The two cases are discussed as follows:

3.1 Triangular Points

Multiplying (3.1) by $(x+\mu)$ and multiplying (3.2) by y and subtracting, we get

$$\left(n^2 - \frac{q_2}{r_2^3} - \frac{3(A_2q_2+A_3)}{2r_2^5}\right)\mu y = 0. \tag{3.3}$$

When $A_1 = A_2 = A_3 = 0$, equation (3.2) and (3.3) yields

$$r_1 = \left(\frac{q_1}{n^2}\right)^{1/3} \quad \text{and} \quad r_2 = \left(\frac{q_2}{n^2}\right)^{1/3}. \tag{3.4}$$

Substituting n^2 and simplifying, we get

$$r_1 = \delta_1^{1/3}\left(1-\frac{e^2}{2}\right) \quad \text{and} \quad r_2 = \delta_2^{1/3}\left(1-\frac{e^2}{2}\right), \tag{3.5}$$

where

$$\delta_i = a^3 q_i, \quad i = 1,2. \tag{3.6}$$

So that the triangular equilibrum points on neglecting the oblateness is given by

$$x_0 = \frac{1}{2} - \mu + \frac{1}{2}(\delta_1^{1/3} - \delta_2^{1/3})(1-e^2), \tag{3.7}$$

$$y_0 = \pm\left(\delta_1^{2/3}(1-e^2) - \frac{1}{4}\left(1+(1-e^2)(\delta_1^{2/3}-\delta_2^{2/3})\right)^2\right)^{1/2}. \tag{3.8}$$

Let ϵ_1 and ϵ_2 are the factors denoting perturbation due to oblateness such that

$$r_1 = \delta_1^{1/3}\left(1-\frac{e^2}{2}\right)+\epsilon_1 \quad \text{and} \quad r_2 = \delta_2^{1/3}\left(1-\frac{e^2}{2}\right)+\epsilon_2, \tag{3.9}$$

Solving the equations (3.2), (3.3) on the basis of the assumptions given by (3.9), we get the triangular points as

$$x^* = \frac{1}{2} - \mu + \frac{\Delta_2}{2}(1-e^2) - \frac{A_1}{2}\left(1-\Delta_1\left(1-\frac{5e^2}{2}\right)\right) - \frac{A_2}{2}\left(1+\Delta_1\left(1-\frac{5e^2}{2}\right)\right) - \frac{A_3}{2}\left(\frac{1}{q_1}+\frac{1}{q_2}\right), \tag{3.10}$$

$$\begin{aligned} y^* = \pm\Bigg[& -\frac{1}{4}(1-\Delta_1)^2 + \frac{e^2}{2}(\Delta_2^2-\Delta_1) + \frac{A_1}{2}\left(-(1-\Delta_2(\Delta_1-2)) + \frac{7e^2}{2}\Delta_3\right) \\ & + \frac{A_2}{2}((1-\Delta_2(\Delta_1-2)) + \frac{7e^2}{2}\Delta_3) + \frac{A_3}{2}\left(\frac{1}{q_2}(1+\Delta_2) - \frac{1}{q_1}(1-\Delta_2) - e^2\left(\frac{1}{q_1}+\frac{1}{q_2}\right)\Delta_2\right)\Bigg]^{1/2}, \end{aligned} \tag{3.11}$$

where

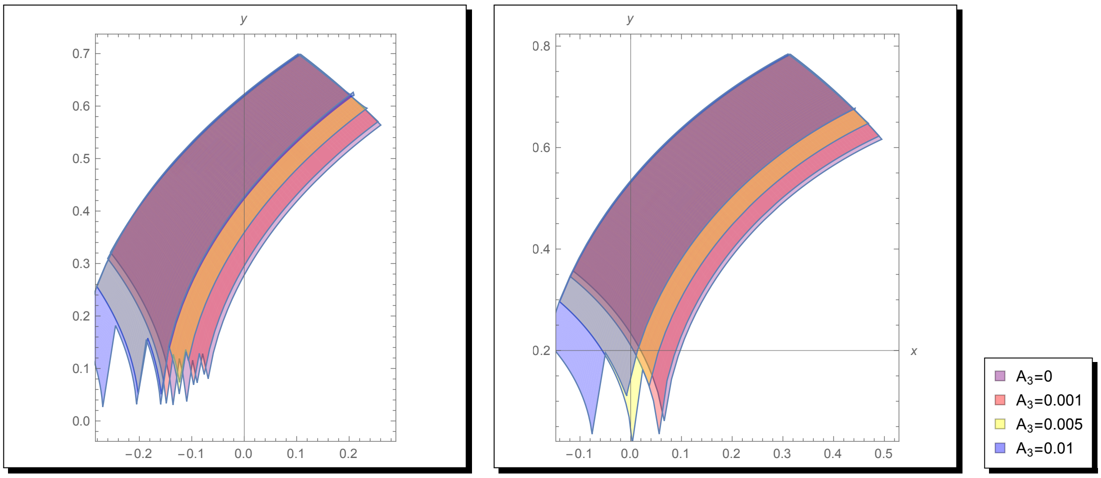
$$\begin{aligned} \Delta_1 &= \delta_1^{2/3} + \delta_2^{2/3}, \\ \Delta_2 &= \delta_1^{2/3} - \delta_2^{2/3}, \\ \Delta_3 &= -\delta_1^{2/3}(1-\delta_1^{2/3}) + \delta_2^{2/3}(1-\delta_2^{2/3}). \end{aligned} \tag{3.12}$$

The problem discussed in this paper is also explored numerically by applying the results obtained to the four binary systems: Luyten-726, Sirius, Kruger-60 and Alpha Centauri, the data used in this paper are presented in Table 1.

Table 1. Some relevant data of the binary systems.

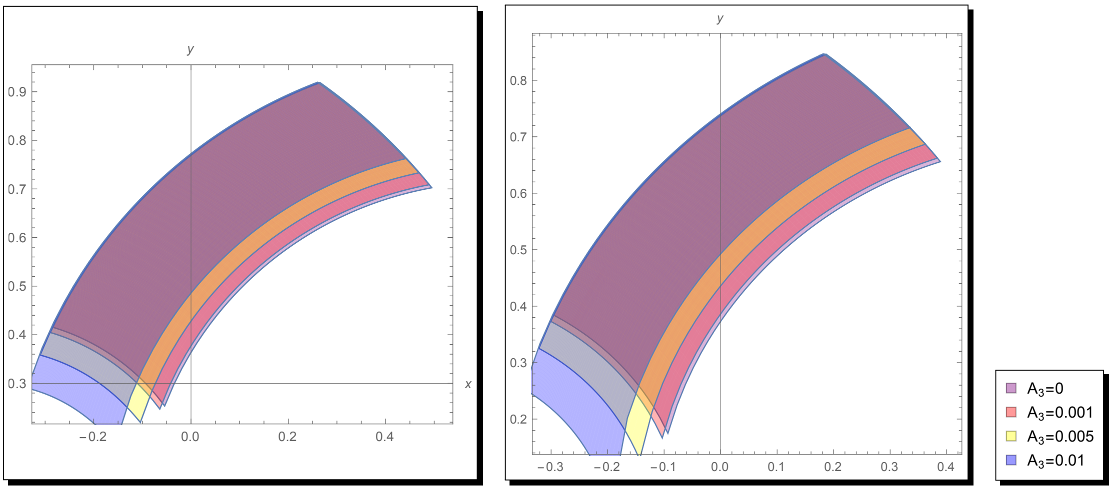
Binary system	$M_1(M_\odot)$	$M_2(M_\odot)$	$a(AU)$	e
Luyten-726	0.109	0.102	2.5	0.62
Sirius	2.02	0.978	6.43	0.59
Kruger-60	0.271	0.176	9.5	0.41
Alpha Centauri	1.1	0.907	10.9	0.5179

Figure 2 and 3 represents the parametric plot of the triangular point L_4, taken as a function of β_1 and β_2, varying the values of A_3 from 0 to 0.01, for the four Binary systems. The oblateness parameter for the primaries are assumed as $A_1 = 0.001$ and $A_2 = 0.002$.



(a) Luyten726-8 (b) Sirius

Figure 2. The parametric plot of the equilibrium point L_4, taken as a function of β_1 and β_2 for the two binary systems Luyten-726 and Sirius. Here the x and y-axis represent the respective coordinate of the triangular point L_4.



(a) Kruger60 (b) Alpha Centauri

Figure 3. The parametric plot of the equilibrium point L_4, taken as a function of β_1 and $\beta_2$for the two binary systems: Kruger 60 and Alpha Centauri.

3.2 Collinear Points

In order to obtain the position of planar equilibrium points lying on the line joining the two primaries, that is the collinear points, along with equations (3.1) and (3.2), the additional condition $y = 0$ is applied. That is the points are obtained by solving the equation:

$$n^2 x - \frac{q_1(1-\mu)(x+\mu)}{|x+\mu|^3} - \frac{q_2\mu(x+\mu-1)}{|x+\mu-1|^3} - \frac{3(1-\mu)(x+\mu)(q_1A_1+A_3)}{2|x+\mu|^5}$$
$$- \frac{3\mu(x+\mu-1)(q_2A_2+A_3)}{2|x+\mu-1|^5} = 0. \tag{3.13}$$

The collinear equilibrium points L_1, L_2 and L_3 are defined as follows:

(1) L_1 lies between the bigger and smaller primary: $-\mu < x < 1-\mu$,

(2) L_2 lies to the right of the smaller primary: $x > 1-\mu$,

(3) L_3 lies to the left of the bigger primary: $x < -\mu$.

3.2.1 Location of the Collinear Point L_1

The position of the collinear point L_1 is given by $-\mu < x < 1-\mu$, then $|x+\mu| = x+\mu$ and $|x+\mu-1| = -(x+\mu-1)$. Then, assuming $x+\mu-1 = -\rho$ and substituting in Equation (3.13), we obtain:

$$\frac{\mu K_2}{5(1-\mu)K_1} = \frac{\rho^5 N_1}{(1-\rho)^4 D_1}, \tag{3.14}$$

where

$$K_1 = \frac{1}{5}(n^2 - 4q_1 - 9(A_1q_1 + A3)),$$

$$K_2 = \frac{-3}{2}(q_2A_2 + A_3),$$

$$N_1 = 1 + \frac{\rho\left(\frac{9A_1q_1}{2} + \frac{9A_3}{2} - n^2 + \frac{14q_1}{5}\right)}{K_1} + \frac{\rho^2\left(-6A_1q_1 - 6A_3 + 2n^2 - 4q_1\right)}{K_1} + \frac{\rho^3\left(\frac{9A_1q_1}{2} + \frac{9A_3}{2} - 2n^2 + 3q_1\right)}{K_1}$$
$$+ \frac{\rho^4\left(-\frac{9A_1q_1}{5} - \frac{9A_3}{5} + n^2 - \frac{6q_1}{5}\right)}{K_1} + \frac{\rho^5\left(\frac{3A_1q_1}{10} + \frac{3A_3}{10} - \frac{n^2}{5} + \frac{q_1}{5}\right)}{K_1},$$

$$D_1 = 1 + \frac{q_2\rho^2}{K_2} - \frac{n^2\rho^5}{K_2}.$$

Assuming,

$$\lambda = \left(\frac{\mu K_2}{5(1-\mu)K_1}\right)^{\frac{1}{5}}.$$

For very small value of ρ, we have $\rho \approx \lambda$. Then a series expansion of ρ can be given by

$$\rho = \lambda(1 + c_1\lambda + c_2\lambda^2 + \cdots). \tag{3.15}$$

The value ρ in the series form is substituted from equation (3.15) into equation (3.14), then on comparing the coefficients, we get:

$$c_1 = \frac{1}{50K_1}\left(4 - 4\beta_1 + 27A_1 + 27A_3 + \frac{1}{a^3}(2 + 3e^2 + 3A_1 + 3A_2)\right),$$
$$c_2 = \frac{1}{625K_1^2K_2}(80(1-2\beta_1-\beta_2) + 360A_1 - 96A_2 + 264A_3$$
$$- \frac{1}{a^3}(40(1-\beta_1-\beta_2) + 60e^2 + 150A_1 + 6A_2 + 36A_3)). \tag{3.16}$$

Thus, substituting equations (3.16) and (3.15) in the equation $x = 1-\mu-\rho$, we get the coordinate for L_1.

3.2.2 Location of the Collinear Point L_2

The position of collinear point L_2 is given by $x > 1-\mu$, which implies $|x+\mu| = x+\mu$ and $|x+\mu-1| = x+\mu-1$. Assuming $x+\mu-1 = \rho$ and substituting in the value of x in terms of ρ in equation (3.13), we obtain:

$$\frac{\mu K_3}{5(1-\mu)K_4} = \frac{\rho^5 N_2}{(1+\rho)^4 D_2}, \tag{3.17}$$

where

$$\begin{aligned}
K_3 =& \frac{1}{5}(11n^2 - 8q_1 - 9(A_1 q_1 + A3)),\\
K_4 =& \frac{3}{2}(q_2 A_2 + A_3),\\
N_2 =& 1 + \frac{\rho\left(\frac{9A_1q_1}{2} + \frac{9A_3}{2} - n^2 + \frac{14q_1}{5}\right)}{K_3} + \frac{\rho^2\left(-6A_1q_1 - 6A_3 + 6n^2 - 4q_1\right)}{K_3}\\
&+ \frac{\rho^3\left(\frac{9A_1q_1}{2} + \frac{9A_3}{2} - 2n^2 + 3q_1\right)}{K_3} + \frac{\rho^4\left(-\frac{9A_1q_1}{5} - \frac{9A_3}{5} + \frac{7n^2}{5} - \frac{6q_1}{5}\right)}{K_3}\\
&+ \frac{\rho^5\left(\frac{3A_1q_1}{10} + \frac{3A_3}{10} - \frac{n^2}{5} + \frac{q_1}{5}\right)}{K_3}.\\
D_2 =& 1 + \frac{q_2\rho^2}{\frac{3A_2q_2}{2} + \frac{3A_3}{2}} - \frac{n^2\rho^5}{\frac{3A_2q_2}{2} + \frac{3A_3}{2}}.
\end{aligned} \tag{3.18}$$

Assuming

$$\lambda = \left(\frac{\mu K_3}{5(1-\mu)K_4}\right)^{1/5}.$$

The value of ρ in series form is substituted from Equation (3.15) into (3.17). Thus, we have obtained the values of the coefficients as:

$$\begin{aligned}
c_1 =& \frac{-1}{(50K_3)}\left(92(1-\beta_1) + 117A_1 + 117A_3 - \frac{1}{a^3}(98 + 147e^2 + 147A_1 + 147A_2)\right),\\
c_2 =& \frac{1}{625K_3^2K_4}(320(1-2\beta_1-\beta_2) + 720A_1 + 3336A_2 + 456A_3\\
&- \frac{1}{a^3}(880(1-\beta_1-\beta_2) + 1320e^2 + 2310A_1 + 5898A_2 + 5568A_3)).
\end{aligned} \tag{3.19}$$

Thus, substituting equation (3.15) and (3.19) in the equation $x = 1-\mu+\rho$, we get the coordinate for L_2.

3.2.3 Location of the Collinear Point L_3

The position of the collinear point L_3 is given by $x < -\mu$, then $|x+\mu| = -(x+\mu)$ and $|x+\mu-1| = -(x+\mu-1)$. Assuming $x+\mu = -\rho$, then substituting in equation (3.13) and rearranging the terms, we obtain:

$$\frac{\mu}{(1-\mu)} = \frac{(1+\rho)^4\left[q_1\rho^2+\frac{3}{2}(q_1A_1+A_3)-n^2\rho^5\right]}{\rho^4\left[n^2(1+\rho)^5-\frac{3}{2}(A_2q_2+A_3)-q_2(1+\rho)^2\right]}. \tag{3.20}$$

Now replacing $\rho = 1+\gamma$ and expanding $1/\rho$ upto $[\rho^3]$ in equation (3.20), we get

$$\frac{\mu}{(1-\mu)} = T_1+T_2u+T_3u^2+T_4u^3+\cdots, \tag{3.21}$$

where

$$T_1 = -\frac{6}{7}\left(1+2e^2+A_2-A_3+\frac{2}{3}\beta_1\right),$$

$$T_2 = 1-\frac{11}{7}e^2-\frac{22}{7}\alpha+\frac{11}{14}A_1-\frac{85}{56}A_2+\frac{135}{56}A_3-\frac{19}{21}\beta_1-\frac{1}{7}\beta_2,$$

$$T_3 = 1-\frac{1277}{672}e^2-\frac{1277}{336}\alpha+\frac{341}{168}A_1-\frac{37}{21}A_2+\frac{1376}{336}A_3-\frac{989}{1008}\beta_1-\frac{2}{7}\beta_2,$$

$$T_4 = \frac{1567}{1728}-\frac{2207}{1152}e^2-\frac{2207}{576}\alpha+\frac{25987}{8064}A_1-\frac{13705}{8064}A_2+\frac{10795}{2016}A_3-\frac{5465}{6048}\beta_1-\frac{4519}{12096}\beta_2,$$

$$u = -\frac{12}{7}\gamma, \quad \text{and}$$

$$\alpha = 1-a.$$

Using Lagrange's inversion integral formula, γ is obtained in terms of $\frac{\mu}{(1-\mu)}$ as

$$\gamma = T_{11}+\frac{\mu}{(1-\mu)}T_{21}+\left(\frac{\mu}{(1-\mu)}\right)^2T_{31}+\left(\frac{\mu}{(1-\mu)}\right)^3T_{41}, \tag{3.22}$$

where

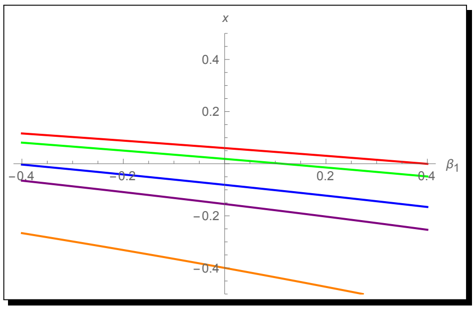
$$T_{11} = -\left(\frac{e^2}{2}+\alpha+\frac{A_2}{2}-\frac{A_3}{2}+\frac{\beta_1}{3}\right),$$

$$T_{21} = -\left(\frac{7}{12}-\frac{e^2}{12}-\frac{\alpha}{6}-\frac{11A_1}{24}-\frac{11A_2}{96}-\frac{13A_3}{32}-\frac{5\beta_1}{36}+\frac{\beta_2}{12}\right),$$

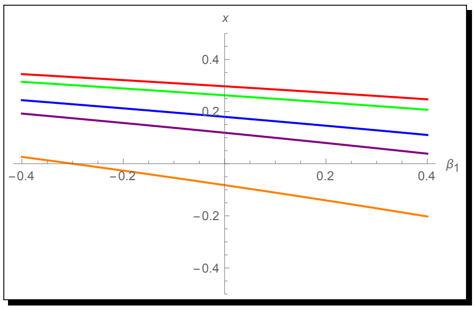
$$T_{31} = \left(\frac{7}{12}+\frac{e^2}{576}+\frac{\alpha}{288}-\frac{55A_1}{288}-\frac{13A_2}{1152}-\frac{79A_3}{384}-\frac{71\beta_1}{864}+\frac{\beta_2}{12}\right),$$

$$T_{41} = \left(\frac{13223}{20736}+\frac{3103e^2}{41472}+\frac{3103\alpha}{20736}-\frac{2677A_1}{41472}-\frac{751A_2}{10368}-\frac{5879A_3}{41472}-\frac{1009\beta_1}{31104}+\frac{569\beta_2}{6912}\right).$$

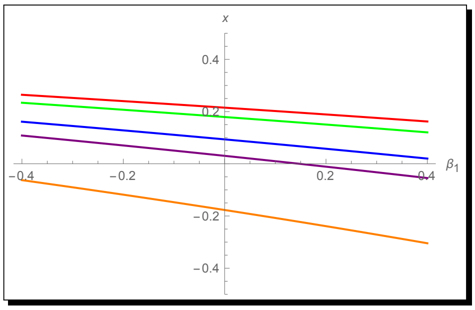
Figures 4, 5 and 6 show the plot of the collinear points L_1, L_2 and L_3, respectively. Here the x-coordinate of the collinear equilibrium point is taken as a function of β_2, varying the values of A_3 from 0 to 0.05, taking the values $\beta_1 = 0.1$ $A_1 = 0.001$ and $A_2 = 0.002$ for the four above mentioned Binary systems.



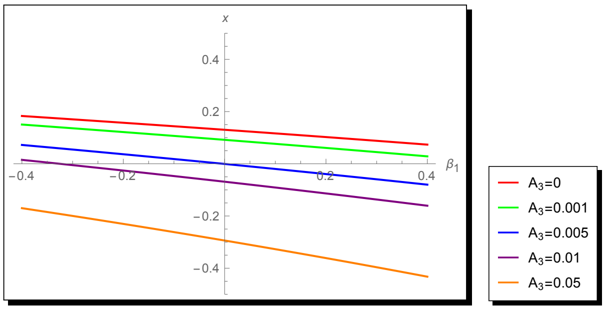
(a) L_1 for Luyten726-8



(b) L_1 for Sirius

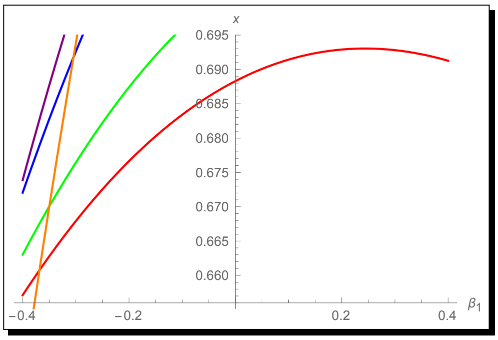


(c) L_1 for Kruger60

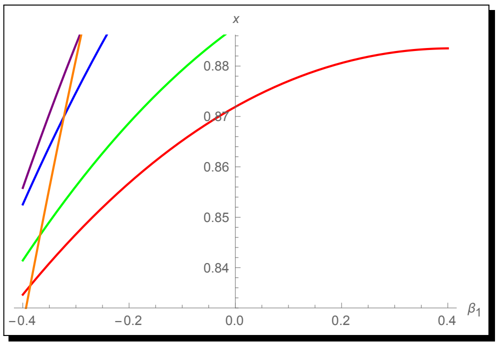


(d) L_1 for Alpha Centauri

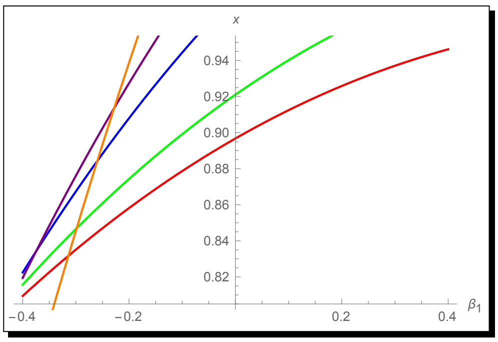
Figure 4. The plot of the collinear points L_1, taken as a function of β_1, varying the values of A_3 from 0 to 0.05, taking the values $\beta_2 = 0.0112$, $A_1 = 0.001$ and $A_2 = 0.002$ for the four Binary systems.



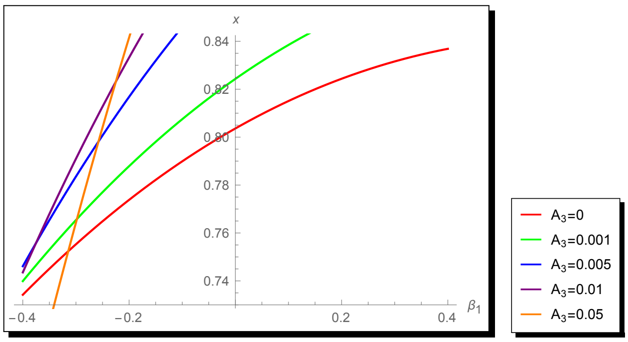
(a) L_2 for Luyten726-8



(b) L_2 for Sirius

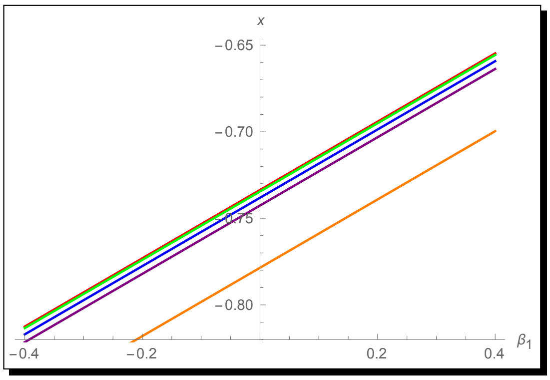


(c) L_2 for Kruger60

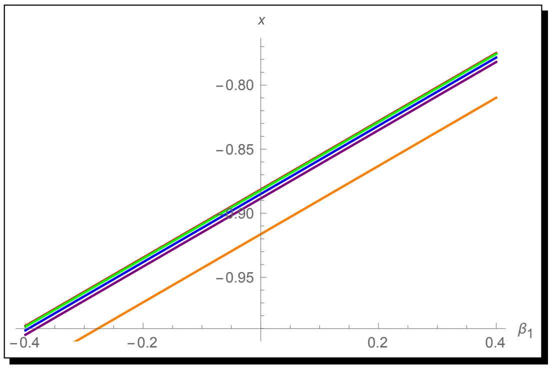


(d) L_2 for Alpha Centauri

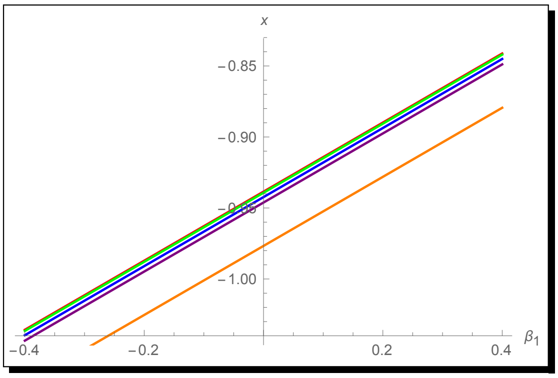
Figure 5. The plot of the collinear points L_2, taken as a function of β_1, varying the values of A_3 from 0 to 0.05, taking the values $\beta_2 = 0.0112$, $A_1 = 0.001$ and $A_2 = 0.002$ for the four Binary systems.



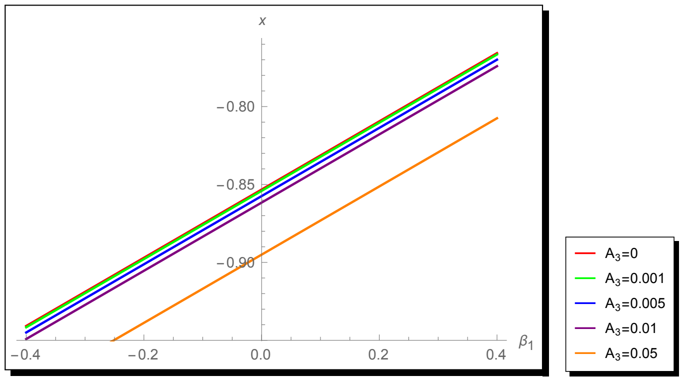
(a) L_3 for Luyten726-8



(b) L_3 for Sirius



(c) L_3 for Kruger60



(d) L_3 for Alpha Centauri

Figure 6. The plot of the collinear points L_3, taken as a function of β_1, varying the values of A_3 from 0 to 0.05, taking the values $\beta_2 = 0.0112$, $A_1 = 0.001$ and $A_2 = 0.002$ for the four Binary systems.

4. Stability of Equilibrium Points

Let the position of the equilibrium points be denoted by (a_0, b_0) and consider a small displacement (u,v) from the point such that $x = a_0 + u$ and $y = b_0 + v$. Substituting these values in (2.1), we obtain the system of equations, taking only linear terms of u and v given below:

$$
\begin{aligned}
u'' - 2v' &= \frac{1}{1+e\cos f}((U^*_{xx})^0 u + (U^*_{xy})^0 v),\\
v'' + 2u' &= \frac{1}{1+e\cos f}((U^*_{yx})^0 u + (U^*_{yy})^0 v),
\end{aligned}
\tag{4.1}
$$

here, the superscript 0 indicates that the derivatives are to be evaluated at the equilibrium points (a_0, b_0). The values of second order derivatives of the function Ω, on taking the averaged values, are as follows:

$$
U^*_{xx} = \left[1 - \frac{1}{n^2}\left(\frac{q_1(1-\mu)}{r_1^5}(r_1^2 - 3(x+\mu)^2) + \frac{q_2\mu}{r_2^5}(r_2^2 - 3(x+\mu-1)^2)\right.\right.
$$

$$+\frac{3(q_1A_1+A_3)(1-\mu)}{2r_1^7}(r_1^2-5(x+\mu)^2)+\frac{3(q_2A_2+A_3)\mu}{2r_2^7}(r_2^2-5(x+\mu-1)^2)\Bigg)\Bigg],$$

$$U^*_{xy}=\Bigg[1-\frac{1}{n^2}\Bigg(\frac{3q_1(1-\mu)(x+\mu)y}{r_1^5}+\frac{3q_2\mu(x+\mu-1)y}{r_2^5}$$
$$+\frac{15(q_1A_1+A_3)(1-\mu)(x+\mu)y}{r_1^7}+\frac{15(q_2A_2+A_3)\mu(x+\mu-1)y}{2r_2^7}\Bigg)\Bigg]=U^*_{yx}, \tag{4.2}$$

$$U^*_{yy}=\Bigg[1-\frac{1}{n^2}\Bigg(\frac{q_1(1-\mu)}{r_1^5}(r_1^2-3y^2)+\frac{q_2\mu}{r_2^5}(r_2^2-3y^2)$$
$$+\frac{3(q_1A_1+A_3)(1-\mu)}{2r_1^7}(r_1^2-5y^2)+\frac{3(q_2A_2+A_3)\mu}{2r_2^7}(r_2^2-5y^2)\Bigg)\Bigg].$$

The characteristic equation for the system of equation (4.1), taking $u=A\exp(\lambda f)$, $v=B\exp(\lambda f)$ is given as

$$\lambda^4-\left(((C_{xx})^0+(C_{xy})^0)-4\right)\lambda^2+\left(C_{xx})^0(C_{yy})^0-(C_{xy})^0(C_{yx})^0\right)=0. \tag{4.3}$$

where

$$C_{xx}=\frac{1}{1+e\cos f}U^*_{xx},\quad C_{xy}=\frac{1}{1+e\cos f}U^*_{xy},$$
$$C_{yx}=\frac{1}{1+e\cos f}U^*_{yx},\quad C_{yy}=\frac{1}{1+e\cos f}U^*_{yy}.$$

4.1 Stability of Triangular Points

In order to study the stability of the equilibrium points, we introduce the following new variables:

$$x_1=x,\quad x_2=y,\quad x_3=\frac{dx}{df},\quad x_4=\frac{dy}{df}. \tag{4.4}$$

Substituting in equation (4.1), the system of equations takes the form:

$$\frac{dx_i}{df}=P_{i1}x_1+P_{i2}x_2+P_{i3}x_3+P_{i4}x_4,\quad i=1,2,3,4, \tag{4.5}$$

where

$$P_{11}=P_{12}=P_{14}=P_{21}=P_{22}=P_{23}=P_{33}=P_{44}=0,$$
$$P_{13}=1,\ P_{24}=1,\ P_{34}=2,\ P_{43}=-2,$$
$$P_{31}=C_{xx},\ P_{32}=P_{41}=C_{xy}=C_{yx}\ \text{and}P_{42}=C_{yy}.$$

Then the coefficients of system of equations (4.5) are periodic function of $'f'$ with period 2π. For further calculation, the averaged values has been considered

$$\frac{dx_i^{(0)}}{df}=P_{i1}^{(0)}x_1^{(0)}+P_{i2}^{(0)}x_2^{(0)}+P_{i3}^{(0)}x_3^{(0)}+P_{i4}^{(0)}x_4^{(0)},\ \ i=1,2,3,4; \tag{4.6}$$

where

$$P_{is}^{(0)} = \int_0^{2f} P_{is}(f)df, \quad i,s = 1,2,3,4.$$

Thus, we get

$$\begin{aligned}
P_{31}^{(0)} =& \frac{3}{4}\left[1+e^2+3A_1-4A_1\mu+2A_2-3A_2\mu-\frac{13A_3}{4}+\frac{15A_3\mu}{2}+\frac{4\beta_2}{3}-2\beta_2\mu-\frac{7\beta_1}{12}+2\beta_1\mu\right.\\
&+\alpha\left(\frac{9}{4}-\frac{47e^2}{8}+4A_1-\frac{67A_1\mu}{8}-\frac{19A_2}{8}+\frac{59A_2\mu}{8}-\frac{47A_3}{8}+\frac{55A_3\mu}{2}\right.\\
&\left.\left.-\frac{19\beta_1}{3}+\frac{29\beta_1\mu}{4}+\frac{5\beta_2}{3}-\frac{29\beta_2\mu}{4}\right)\right],\\
P_{41}^{(0)} =& \frac{3\sqrt{3}}{4}\left[1-2\mu+\frac{e^2}{3}-\frac{1}{3}2e^2\mu+\frac{5A_1}{3}-\frac{4A_1\mu}{3}-\frac{8A_2}{3}+\frac{7A_2\mu}{3}+\frac{7A_3}{4}-5A_3\mu\right.\\
&+\frac{\beta_1}{12}-\frac{5\beta_1\mu}{6}+\frac{4\beta_2}{9}-\frac{2\beta_2\mu}{9}\\
&+\alpha\left(\frac{19}{12}-\frac{19\mu}{6}-\frac{159e^2}{8}+\frac{159e^2\mu}{4}+\frac{28A_1}{3}+\frac{67A_1\mu}{8}+\frac{311A_2}{24}-\frac{463A_2\mu}{24}\right.\\
&\left.\left.+\frac{29A_3}{24}-\frac{245A_3\mu}{12}-\frac{101\beta_1}{6}+\frac{775\beta_1\mu}{36}-\frac{5\beta_2}{9}+\frac{53\beta_2\mu}{4}\right)\right], \qquad (4.7)\\
P_{42}^{(0)} =& \frac{9}{4}\left[1-\frac{e^2}{3}+\frac{A_1}{3}-2A_2+\frac{11A_2\mu}{3}+\frac{11A_3}{4}-\frac{11A_3\mu}{6}+\frac{11\beta_1}{36}-\frac{2\beta_1\mu}{3}-\frac{4\beta_2}{9}+\frac{2\beta_2\mu}{3}\right.\\
&+\alpha\left(-\frac{5}{12}-\frac{1175e^2}{72}+\frac{64A_1}{9}-\frac{601A_1\mu}{24}+\frac{1381A_2}{72}-\frac{415A_2\mu}{24}-\frac{73A_3}{24}+\frac{83A_3\mu}{6}\right.\\
&\left.\left.+\frac{5\beta_2}{27}-\frac{45\beta_2\mu}{4}-\frac{128\beta_1}{9}+\frac{45\beta_1\mu}{4}\right)\right]. \qquad (4.8)
\end{aligned}$$

Then the characteristic equation (4.3) for the system can be represented as

$$\lambda^4 - D\lambda^2 + R = 0, \tag{4.9}$$

where

$$D = P_{31}^{(0)} + P_{42}^{(0)} - 4, \tag{4.10}$$

$$R = P_{31}^{(0)}P_{42}^{(0)} - P_{41}^{(0)}P_{32}^{(0)}. \tag{4.11}$$

So that the characteristic roots are purely imaginary if and only if

$$D < 0, \tag{4.12}$$

$$D^2 - 4R \geq 0. \tag{4.13}$$

Substituting the values of the triangular equilibrium points and using equation (4.2) and (4.7) in (4.11), we get

$$0 < e^2 \leq \frac{7}{16}\left[1-\frac{39A_1}{14}-\frac{15}{14}\alpha+\frac{18A_1\mu}{7}+\frac{33A_2}{14}-\frac{36A_2\mu}{7}+\frac{9A_3\mu}{7}-\frac{9A_3}{2}-\frac{3\beta_1}{14}\right]. \tag{4.14}$$

The condition given by (4.14) coincides with the condition for eccentricity as given by [8], for $\beta_1 = \beta_2 = 0 = A_1 = A_2 = A_3$. From the inequality (4.13), we get the critical value of μ given as μ^*

$$
\begin{aligned}
\mu^* =& \frac{1}{2}\left(1-\sqrt{\frac{23}{27}}\right)+\frac{85\alpha}{9\sqrt{69}}+\frac{e^2}{9\sqrt{69}}\left(22-\left(\frac{703\sqrt{69}}{4}-\frac{10775}{23}\right)\alpha\right) \\
&-\frac{1}{9}\left(\frac{1}{828}\left(83283-7748\sqrt{69}\right)\alpha-\sqrt{\frac{23}{3}}+1\right)A_1-\frac{1}{9}\left(-\left(\frac{665}{12}-\frac{67}{6\sqrt{69}}\right)\alpha+\frac{11}{\sqrt{69}}+4\right)A_2 \\
&+\frac{1}{3}\left(\left(\frac{679\ 3}{4\sqrt{69}}-\frac{3605}{36}\right)\alpha+\frac{5}{6}+\frac{4}{\sqrt{69}}\right)A_3+\frac{1}{9}\left(\frac{5}{\sqrt{69}}-\frac{35}{276}\left(529+4\sqrt{69}\right)\alpha\right)\beta_1 \\
&-\frac{1}{27}\left(\left(\frac{51}{2}-\frac{3977}{12\sqrt{69}}\right)\alpha+\frac{2}{\sqrt{69}}\right)\beta_2\,.
\end{aligned}
\tag{4.15}
$$

The region plot showing the stability region in the $\mu-e$ plane satisfying equation (4.13), taking $A_1 = 0.0001$, $A_2 = 0.001$, $\alpha = 0.99$, $\beta_1 = 0.0002$ and $\beta_2 = 0.0112$ has been plotted and shown in the Figure 7. The points in the plot shows the position in the $\mu-e$ plane for the respective binary systems.

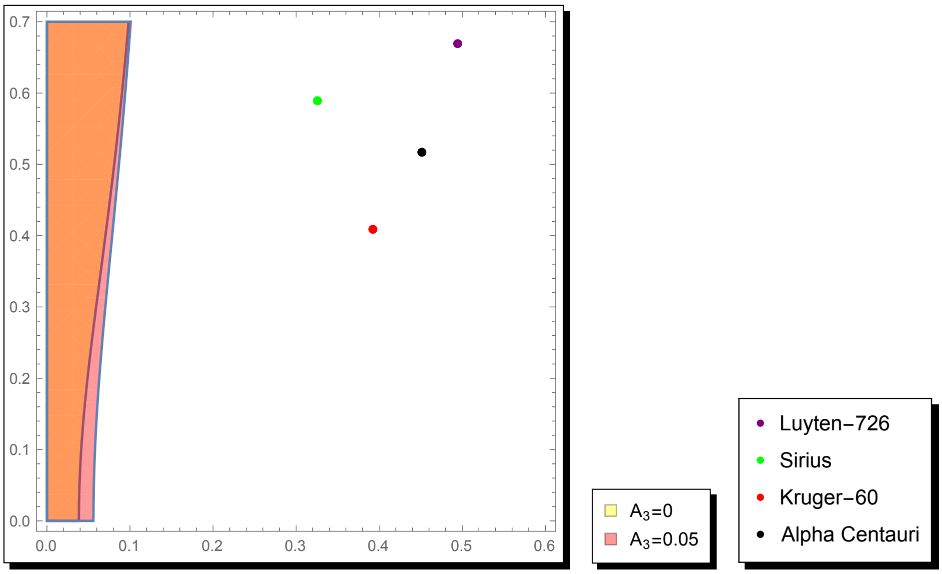


Figure 7. The stability region plot on $\mu-e$ plane for the triangular point L_4. The dots denotes the position corresponding to the known value of μ and e for the four binary systems.

4.2 The Collinear Points

For the collinear points $y = 0$. Consequently, we get the values:

$$
\begin{aligned}
P_{31}^{(0)} =& 1+2(\Phi_1+2\Phi_2)\,, \\
P_{42}^{(0)} =& 1-(\Phi_1+\Phi_2)\,,
\end{aligned}
\tag{4.16}
$$

$$P_{32}^{(0)} = P_{41}^{(0)} = 0,$$

where

$$\Phi_1 = \frac{1}{n^2}\left(\frac{(1-\mu)q_1}{r_1^3} + \frac{\mu q_2}{r_2^3}\right),$$

$$\Phi_2 = \frac{3}{2n^2}\left(\frac{(1-\mu)(q_1 A_1 + A_3)}{r_1^5} + \frac{\mu(q_2 A_2 + A_3)}{r_2^5}\right). \tag{4.17}$$

Let the two square roots of the biquadratic equation given by (4.3) be λ_1^2 and λ_2^2. Then using relation between roots and coefficient we have,

$$\lambda_1^2 \lambda_2^2 = \frac{1}{(1-e^2)}(1+2\Phi_1+4\Phi_2)(1-\Phi_1-\Phi_2) \tag{4.18}$$

and

$$\lambda_1^2 + \lambda_2^2 = -\left(4 - \frac{1}{\sqrt{1-e^2}}(2+\Phi_1+3\Phi_2)\right). \tag{4.19}$$

Now the system will be stable if the roots of characteristic equation are purely imaginary, that is the roots λ_1^2 and λ_2^2 are negative, thus we get the conditions

$$4 - \frac{1}{\sqrt{1-e^2}}(2+\Phi_1+3\Phi_2) > 0$$

$$\Rightarrow \quad \Phi_1 + 3\Phi_2 < 2(1-e^2) \tag{4.20}$$

and

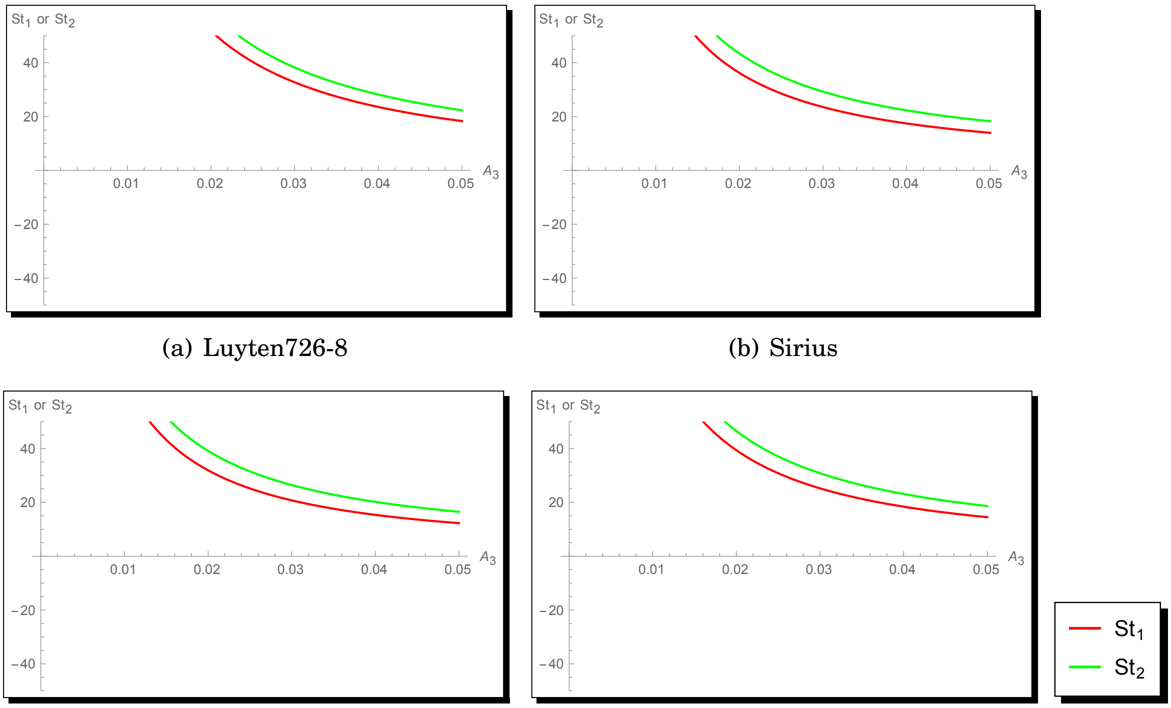
$$(1+2\Phi_1+4\Phi_2)(1-\Phi_1-\Phi_2) > 0. \tag{4.21}$$

Now, taking both the brackets of the Equation (4.21) negative yields contradictory condition. Therefore, taking both the brackets positive, we get, the condition for stability of collinear point as

$$-\frac{1}{2} < \Phi_1 + \Phi_2 < 1. \tag{4.22}$$

Therefore, the stability condition for the collinear points comprises of two inequalities (4.20) and (4.22) which need to be satisfied simultaneously. We have analyzed the stability condition for each of the collinear points graphically.

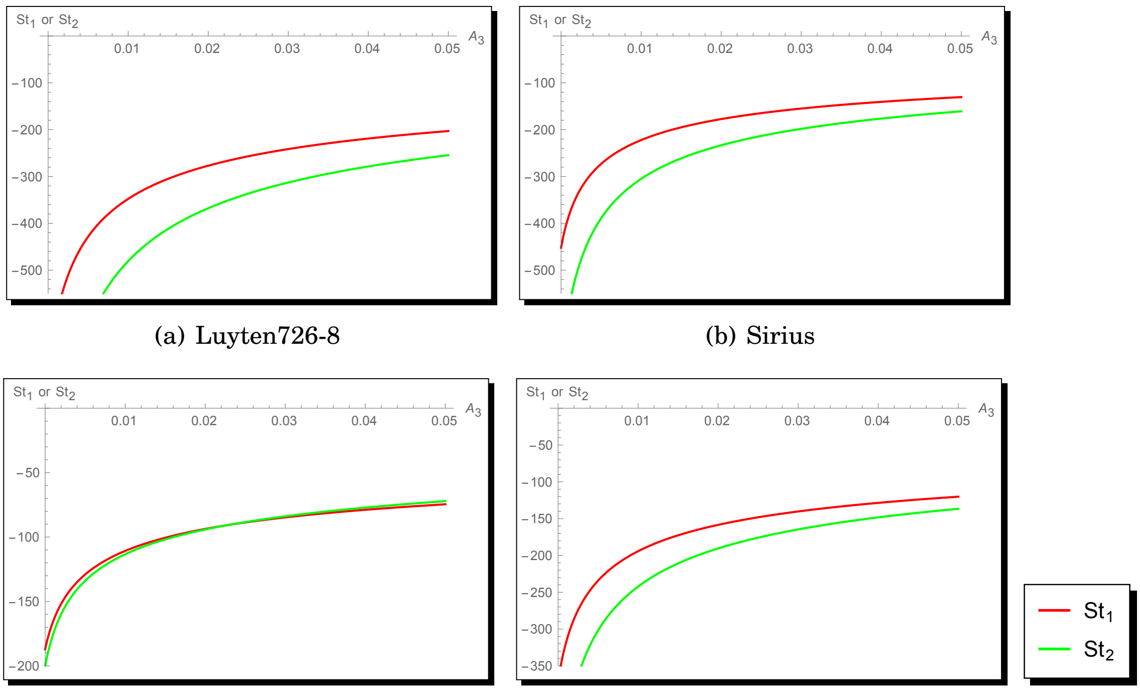
Assuming $St_1 = \Phi_1 + \Phi_2$ and $St_2 = \Phi_1 + 3\Phi_2 - 2(1-e^2)$, we plot St_1 and St_2 as functions of A_3 for three collinear points around the respective binary system. Figure 8 shows the variation of St_1 and St_2 for the collinear point L_1 for the four binary systems. Similarly, Figures 9 and 10 shows the variation of St_1 and St_2 for the collinear point L_2 and L_3 respectively for the four binary systems.



(a) Luyten726-8 (b) Sirius

(c) Kruger60 (d) Alpha Centauri

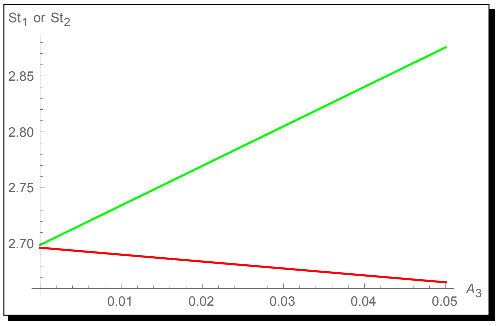
Figure 8. The plot of the stability conditions Δ_1 and Δ_2 for the collinear points L_1 taking the values $\beta_1 = 0.0002$, $\beta_2 = 0.0112$, $A_1 = 0.001$ and $A_2 = 0.002$ for the four Binary systems.



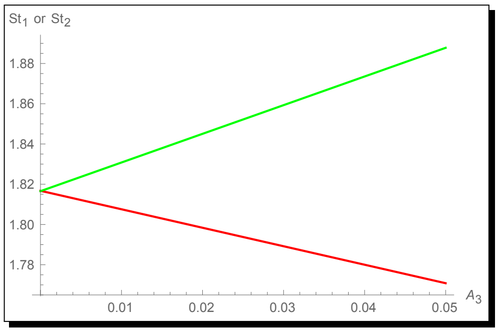
(a) Luyten726-8 (b) Sirius

(c) Kruger60 (d) Alpha Centauri

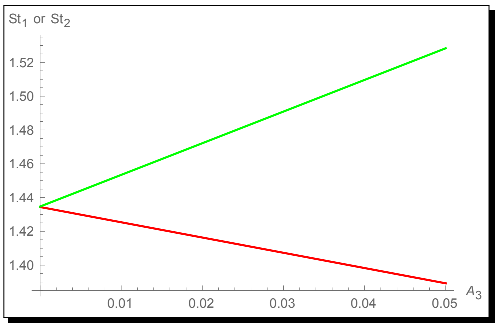
Figure 9. The plot of the stability conditions Δ_1 and Δ_2 for the collinear points L_2 taking the values $\beta_1 = 0.0002$, $\beta_2 = 0.0112$, $A_1 = 0.001$ and $A_2 = 0.002$ for the four Binary systems.



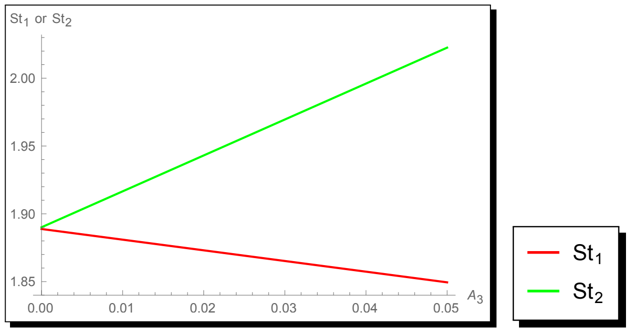
(a) Luyten726-8



(b) Sirius



(c) Kruger60



(d) Alpha Centauri

Figure 10. The plot of the stability conditions Δ_1 and Δ_2 for the collinear points L_3 taking the values $\beta_1 = 0.0002$, $\beta_2 = 0.0112$, $A_1 = 0.001$ and $A_2 = 0.002$ for the four Binary systems.

5. Discussion and Conclusions

The three dimensional elliptical restricted three-body problem is investigated, considering the effect of radiation pressure due to luminous primaries and the oblateness of all the participating bodies. The problem models the system of an infinitesimal particle of irregular shape orbiting a pair of stars. In particular, the binary systems: Luyten-726, Sirius, Kruger-60 and Alpha Centauri has been studied. From the computations, it is evident that the position of the Lagrangian points are affected by the oblateness of all three bodies. Figures 2 and 3 shows the parametric plot of the triangular point L_4 taken as a function of β_1 and β_2 for varying values of the oblateness factor A_3 of the infinitesimal body and it was observed that the triangular points are affected by the oblateness of the infinitesimal. Figures 4, 5 and 6 show the plot of the collinear points L_1, L_2 and L_3 taken as a function of β_1, varying the values of A_3 from 0 to 0.05, for the four Binary systems. From these graphs it is evident that the collinear points are also affected by the oblateness of the infinitesimal. It was observed that the collinear point L_2 is most affected by A_3, especially for the binary system Alpha Centauri.

Taking $q_1 = q_2 = 1$, our problem reduces to the problem discussed by [26] and the values of the coordinates of the L_4 and L_5 points agrees with their result when the product of oblateness term with eccentricity and radiation factor is neglected. However, the collinear points are obtained using the method developed in [29].

The stability of the triangular and collinear equilibrium points was analyzed by Lyapunov's first method. The value of the critical value of μ is obtained for the triangular points. The region plot showing the stability region in the $\mu - e$ plane satisfying equation (4.13) is shown in the Figure 7. The points in the plot shows the position in the $\mu - e$ plane for the respective binary systems. From the graph it is evident that the increase in the value of oblateness of the infinitesimal increases the possible stable region. However, the four binary systems considered in this paper are still beyond the stable region. Therefore, we conclude that the triangular equilibrium points around these binary system are unstable.

The Stability criteria for the collinear points are obtained and presented by equation (4.20) and (4.22). The Figures 8, 9 and 10 shows the plot of the two stability conditions taken as a function of the oblateness factor A_3 of the infinitesimal. From equation (4.20) and (4.22), it is evident that the collinear points will be stable if $\frac{-1}{2} < St_1 < 1$ and $St_2 < 0$. But it is observed in the graphs plotted that the values of both St_1 and St_2 is well beyond the range for the collinear points L_1 and L_3 around all four binary systems considered. Whereas, for the collinear point L_2 the stability condition (4.20) is satisfied but the stability condition (4.22) is not satisfied. Thus, we conclude that all the three collinear equilibrium points around these binary system are also unstable.

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Competing Interests

The author declares that he has no competing interests.

Authors' Contributions

The author wrote, read and approved the final manuscript.

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