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Research Article

On Geodesic Equation and Main Scalar in Two Dimensional Finsler Space with Matsumoto Metrics

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Abstract. In this paper, we consider the two dimensional Finsler space with generalized Matsumoto metric and derived the geodesics equation in the Weierstrass form of differential equation. Then we find the main scalar for generalized Matsumoto metric in the two dimensional Finsler space.

Keywords. (α,β)-metrics; Geodesics; Two-dimensional Finsler space; Main scalar; Matsumoto metric

MSC. 53B20; 53B40; 53C60

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1. Introduction

The geometry of two dimensional Finsler space is an interesting research area in Finsler geometry, specially differential equation of geodesics in two dimensional Finsler space derived by many authors. In 1997, M. Matsumoto and H.S. Park, have obtained the differential equations of geodesics in two-dimensional Randers spaces and Kropina space in the most clear form $y'' = f(x,y,y')$, where (x,y) are co-ordinate of two-dimensional Finsler space [7]. In 2004, I.Y. Lee and H.S. Park, studied the geodesic equations of a two dimensional Finsler space with respect to an isothermal coordinate system [5]. In 2013, V.K. Chaubey, B.N. Prasad and D.D. Tripathi, has found out the equation of geodesic for general (α,β)-metric as compared to Randers, Kropina and Matsumoto metric under the same conditions [2].

In this paper, we are obtained the differential equations of geodesic in a two dimensional Finsler space with generalized Matsumoto metric, and also we discus the condition for a main scalars in two dimensional Finsler space with Matsumoto metric.

2. Preliminaries

Let (M,F) be an n-dimensional Finsler space. We denote the tangent space at $x \in M$ by T_xM and the tangent bundle of M by TM. Each element of TM has the form (x,y), where $x \in M$ and $y \in T_xM$. The formal definition of Finsler space as follows:

Definition 2.1 ([3])**.** A Finsler space is a triple $F^n = (M,D,L)$, where M is an n-dimensional manifold, D is an open subset of a tangent bundle TM and L is a Finsler metric defined as a function $L : TM \to [0,\infty)$ with the following properties:

(i) *Regular*: L is C^∞ on the entire tangent bundle $TM\backslash\{0\}$.

(ii) *Positive homogeneous*: $L(x,\lambda y) = \lambda L(x,y)$.

(iii) *Strong convexity*: The $n \times n$ Hessian matrix

$$g_{ij} = \frac{1}{2}[L^2]_{y^i y^j}$$

is positive definite at every point on $TM\backslash\{0\}$, where $TM\backslash\{0\}$ denotes the tangent vector y is non zero in the tangent bundle TM.

Now, consider a two dimensional Finsler space, denoted by $F^2 = (M^2, L(x,y))$. Since the homogeneity of $L(x,y)$ in y, we have

$$L_{(1)(1)}y^1 + L_{(1)(2)}y^2 = L_{(2)(1)}y^1 + L_{(2)(2)}y^2 = 0. \tag{2.1}$$

Then there exist a function $W(x,y)$ satisfying

$$L_{(1)(1)} = W(y^2)^2, \quad L_{(1)(2)} = -Wy^1y^2, \quad L_{(2)(2)} = W(y^1)^2, \tag{2.2}$$

which is called the Weierstrass invariant ([1], [6]).

The associated Riemannian space $R^2 = (M^2, \alpha)$ for a two dimensional case with respect to $L = \alpha$ and α^2, Weierstrass invariant W_r of R^2 is written as:

$$W_r = \frac{1}{\alpha^3}\{a_{11}a_{22} - (a_{12})^2\}. \tag{2.3}$$

And, also the geodesic equation in a two dimensional Finsler space is given by the differential equation in Weierstrass form as follows [1]:

$$L_{1(2)} - L_{2(1)} + (y^1\dot{y}^2 - y^2\dot{y}^1)W = 0; \quad \dot{y}^i = \frac{dy^i}{dt}. \tag{2.4}$$

For a two dimensional Finsler space, $\Gamma = \{\gamma^i_{jk}(x^1,x^2)\}$ be the Levi-Civita connection of the associated Riemannian space R^2, and the Finsler connection $\Gamma^* = (\gamma^i_{jk}, \gamma^i_{0j}, 0)$ and h and v-covariant differentiation in Γ^* are denoted by $(;i,(i))$ respectively, where the index (0) means the contraction with y^i. Then $y^i_{;j} = 0$; $\alpha_{;i} = 0$ and $\alpha_{(i);j} = 0$.

According to [4], the main scalar I of two dimensional Finsler space F^2 with the (α,β)-metric is given by

$$\epsilon I^2 = \left(\frac{L}{\alpha}\right)^4 \left[\frac{\gamma^2 (T_2)^2}{4T^3}\right], \tag{2.5}$$

where ϵ is signature of the space, $\gamma^2 = b^2\alpha^2 - \beta^2$,

$$T = P(P + P_0 b^2 + P_{-1}\beta) + \{P_0 P_{-2} - (P_{-1})^2\}\gamma^2 \text{ and } T_2 = \frac{\partial T}{\partial \beta}, \tag{2.6}$$

$$P = LL_1\alpha^{-1},\ P_0 = LL_{22} + (L_2)^2,\ P_{-1} = (LL_{12} + L_1L_2)\alpha^{-1},$$

$$P_{-2} = L\alpha^{-2}(L_{11} - L_1\alpha^{-1}) + L_1^2\alpha^{-2}. \tag{2.7}$$

3. Differential Equation of Geodesic in Two Dimensional Finsler Space

In this section, we consider the two dimensional Finsler space and derive the differential equation of geodesics in two dimensional Finsler space.

Consider a two dimensional Finsler space $F^2 = (M^2, L(\alpha,\beta))$. Then the derivatives of the Finsler metric function $L(\alpha,\beta)$ are given by,

$$L_{;i} = L_\beta \beta_{;i}, \quad L_{(i)} = L_\alpha \alpha_{(i)} + L_\beta b_i, \tag{3.1}$$

where $\alpha_{(i)} = \frac{\alpha_{(ir)}y^r}{\alpha}$ and L_α, L_β are the partial derivatives of L with respect to α, β, respectively. Then with respect Γ^* we have,

$$L_{(j);i} = L_{(j)i} - L_{(j)(r)}\gamma^r_{oi} - L_{(r)}\gamma^r_{ji}, \tag{3.2}$$

from which

$$L_{1(2)} - L_{2(1)} = L_{(2);1} - L_{(1);2} + L_{(2)(r)}\gamma^r_{01} - L_{(1)(r)}\gamma^r_{02}. \tag{3.3}$$

From (2.2) and (3.3)

$$L_{1(2)} - L_{2(1)} = L_{(2);1} - L_{(1);2} + (y^1\gamma^2_{00} - y^2\gamma^i_{00})W. \tag{3.4}$$

Again from (3.1) we get

$$L_{(j);i} = L_{\alpha\beta}\beta_{;i}\alpha_{(j)} + L_{\beta\beta}\beta_{;i}b_j + L_\beta b_{j;i}. \tag{3.5}$$

Then, Weierstrass invariant $\omega(\alpha,\beta)$ for $L(\alpha,\beta)$ which is similar to $L(x^1,x^2;y^1y^2)$ and $\alpha(x^2,x^2)$ as follows,

$$\omega = \frac{L_{\alpha\alpha}}{\beta^2} = -\frac{L_{\alpha\beta}}{\alpha\beta} = \frac{L_{\beta\beta}}{\alpha^2}. \tag{3.6}$$

Plugging (3.6) in (3.5), which yields

$$L_{(j);i} = \alpha\omega\beta_{;i}(\alpha b_j - \beta\alpha_{(j)}) + L_\beta b_{j;i}. \tag{3.7}$$

Again from (3.4) and (3.7), we have

$$L_{1(2)} - L_{2(1)} = \alpha\omega\{\beta_{;1}(\alpha b_2 - \beta\alpha_{(2)}) - \beta_{;2}(\alpha b_1 - \beta\alpha_{(1)})\} - L_\beta(b_{1;2} - b_{2;1}) + (y^1\gamma^2_{00} - y^2\gamma^1_{00})W. \tag{3.8}$$

Suppose $Y^i_{;0} = \dot{y}^i + \gamma^i_{00}$, then

$$y^1\dot{y}^2 - y^2\dot{y}^1 = y^1y^2_{;0} - y^2y^1_{;0} + (y^1\gamma^2_{00} - y^2\gamma^1_{00}). \tag{3.9}$$

Again plugging (3.8) and (3.9) in (2.4), we have

$$\alpha\omega\{\beta_{;1}(\alpha b_2 - \beta\alpha_{(2)}) - \beta_{;2}(\alpha b_1 - \beta\alpha_{(1)})\} - L_\beta\left(\frac{\partial b_1}{\partial x^2} - \frac{\partial b_2}{\partial x^1}\right) + (y^1y^2_{;0} - y^2y^1_{;0})W = 0, \tag{3.10}$$

where $\beta_{;i} = b_{r:i}y^r$.

Now, the relation between W, W_r and ω is

$$W = (L_\alpha + \alpha\omega\gamma^2)W_r, \tag{3.11}$$

where $\gamma^2 = b^2\alpha^2 - \beta^2$ and $b^2 = a^{ij}b_ib_j$. Then (3.10) implies that

$$(L_\alpha + \alpha\omega\gamma^2)(y^1y^2_{;0} - y^2y^1_{;0})W_r - L_\beta\left(\frac{\partial b_1}{\partial x^2} - \frac{\partial b_2}{\partial x^1}\right) + \alpha\omega\{b_{0;1}(\alpha b_2 - \beta\alpha_{(2)}) - b_{0;2}(\alpha b_1 - \beta\alpha_{(1)})\} = 0. \tag{3.12}$$

In [9], H.S. Park and I.Y. Lee proved that:

Theorem 3.1. *The differential equation of a geodesic in a two dimensional Finsler space $F^2 = (M^2, L)$ can be written in the form of* (3.12).

Now from [3] and [9], with respect to an isothermal coordinate system (x, y), we use the following result,

Theorem 3.2. *The differential equation of a geodesic in a two dimensional Finsler space $F^2 = (M^2, L)$ with respect to an isothermal coordinate system (x, y) such that $\alpha = aE$ can be written in the form of,*

$$\{L_\alpha + aE\omega(b_1\dot{y} - b_2\dot{x})^2\}\{a(\dot{x}\ddot{y} - \dot{y}\ddot{x}) + E^2(a_x\dot{y} - a_y\dot{x})\} - E^3L_\beta(b_{1y} - b_{2x}) - E^3a^2\omega(b_1\dot{y} - b_2\dot{x})b_{0;0} = 0, \tag{3.13}$$

where

$$b_{0;0} = b_{r;s}y^ry^s = (b_{1x}\dot{x} + b_{1y}\dot{y})\dot{x} + (b_{2x}\dot{x} + b_{2y}\dot{y})\dot{y} + \frac{1}{a}\{(\dot{x} + \dot{y})^2(a_xb_1 + a_yb_2) - 2(b_1\dot{x} + b_2\dot{y})(a_x\dot{x} + a_y\dot{y})\}, \tag{3.14}$$

where $b_{ix} = \frac{\partial b_i}{\partial x}$ and $b_{iy} = \frac{\partial b_i}{\partial y}$.

4. Differential Equation of Geodesics of Finsler Space with (α, β)-metric

In this section, we extend the study of differential equation of a geodesic of a Finsler spaces in two dimensional spaces with (α, β)-metrics such as generalized Matsumoto (α, β)-metric and first approximation Matsumoto metric.

4.1 Geodesic equation of a Finsler space with the generalized Matsumoto metric

Consider generalized Matsumoto metric

$$L = \frac{\alpha^{(m+1)}}{(\alpha-\beta)^m}. \tag{4.1}$$

The partial derivatives of (4.1) with respect to α and β are as follows

$$L_\alpha = \frac{(m+1)\alpha^m}{(\alpha-\beta)^m} - \frac{m\alpha^{m+1}}{(\alpha-\beta)^{m+1}},\quad L_\beta = \frac{m\alpha^{(m+1)}}{(\alpha-\beta)^{(m+1)}},\quad \omega = \frac{L_{\alpha\alpha}}{\beta^2} = \frac{m(m+1)\alpha^{(m-1)}}{\beta^{(m+2)}}. \tag{4.2}$$

Plugging (4.2) in (3.13), we have

$$\{(m+1)\alpha^m(\alpha-\beta)^2 - m\alpha^{m+1}(\alpha-\beta) - aEm(m+1)\alpha^{(m-1)}(b_1\dot{y}-b_2\dot{x})^2\}\{a(\dot{x}\ddot{y}-\dot{y}\ddot{x}) \tag{4.3}$$

$$+E^2(a_x\dot{y}-a_y\dot{x})\} - E^3\{m\alpha^{m+1}(\alpha-\beta)(b_1y-b_2x) + a^2m(m+1)\alpha^{(m-1)}(b_1\dot{y}-b_2\dot{x})b_{0;0}\} = 0. \tag{4.4}$$

Suppose t be the parameter of a curve C, then $\dot{x}=1$, $\dot{y}=y'$, $\ddot{x}=0$, $\ddot{y}=y''$, $E^2 = 1+(y')^2$. Then (4.3) reduces to:

$$\{(m+1)a^m(a-\beta)^2 - ma^{m+1}(a-\beta) - a^m m(m+1)(b_1y'-b_2)^2\}\{ay''+E^2(a_xy'-a_y)\}$$
$$-E^2\{Em(a^{(m+2)}-a^{(m+1)}\beta)(b_{1y}-b_{2x}) + Em(m+1)a^{(m+1)}(b_1y'-b_2)b_{0;0}\} = 0, \tag{4.5}$$

where

$$b_{0;0} = (b_{1x}+b_{1y}y') + (b_{2x}+b_{2y}y')y' + \frac{1}{a}\{[1+(y')^2](a_xb_1+a_yb_2) - 2(b_1+b_2y')(a_x+a_yy')\}. \tag{4.6}$$

In a local coordinate system (x,y), the associated Riemannian space which is Euclidean with an orthonormal coordinate system, then $a=1, a_x=a_y=0$, so that (4.5) is reduced to

$$y'' = \frac{[1+(y')^2]m(m+1)(b_1y'-b_2)b_{0;0}}{(1+(y')^2) + (m+1)(b_1+b_2y')^2 - (2m+1)(1+(y')^2)^{1/2}(b_1+b_2y') + m(m+1)(b_1y'-b_2)^2\}}, \tag{4.7}$$

where

$$b_{0;0} = (b_{1x}+b_{1y}y') + (b_{2x}+b_{2y}y')y'. \tag{4.8}$$

If we take b_1 and b_2 such that $b_1=b_x$ and $b_2=b_y$ for a scalar b, then $b_{1y}-b_{2x}=0$. Therefore, the above equation (4.7) reduces to,

$$y'' = \frac{m(m+1)[A_0+A_1(y')+A_2(y')^2+A_3(y')^3+A_4(y')^4]}{1+(m+1)(b_x)^2+m(m+1)(b_y)^2+B_0y'+B_1(y')^2+B_2}, \tag{4.9}$$

where

$A_0 = b_{xx}b_{xy} - b_yb_{xx}$, $A_1 = 2b^2_{xy} - 2b_yb_{xy}$, $A_2 = (b_{xy}b_y)(b_{yy}+b_{xx})$, $A_3 = 2b^2_{xy} - 2b_yb_{xy}$, $A_4 = b_{yy}(b_{xy}-b_y)$,
$B_0 = 1+(m+1)b^2_x + m(m+1)b^2_y$, $B_1 = 2(m+1)(1-m)b_xb_y$, $B_2 = -(2m+1)(b_x+b_yy')(1+(y'))^{1/2}$.

Thus, we state the result as follows:

Theorem 4.3. *Let F^2 be the two dimensional Finsler space with an generalized Matsumoto metric $L = \frac{\alpha^{(m+1)}}{(\alpha-\beta)^m}$. If we refer to an orthonormal coordinate system (x,y) with respect to α and $b_1 = \partial b/\partial x$, $b_2 = \partial b/\partial y$ for a scalar b, then the differential equation of a geodesic $y=y(x)$ of F^2 is given by* (4.9).

Example. If we take $m = 1$ in (4.1), then (4.1) becomes $L = \frac{\alpha^2}{\alpha-\beta}$. Then the differential equation of geodesic of a Finsler space can be written as a polynomial in (y') of degree at most 4, i.e., it can be written as

$$y'' = \frac{2[A'_0 + A'_1(y') + A'_2(y')^2 + A'_3(y')^3 + A'_4(y')^4]}{1 + 2(b_x)^2 + 2(b_y)^2 + B'_0(y')^2 + B'_1}, \tag{4.10}$$

where

$$A'_0 = b_{xx}b_{xy} - b_y b_{xx}, \quad A'_1 = 2b^2_{xy} - 2b_y b_{xy}, \quad A'_2 = (b_{xy}b_y)(b_{yy} + b_{xx}),$$
$$A'_3 = 2b^2_{xy} - 2b_y b_{xy}, \quad A'_4 = b_{yy}(b_{xy} - b_y), \quad B'_0 = 1 + 2b^2_x + 2b^2_y,$$
$$B'_1 = -3(b_x + b_y y')(1 + (y')^2)^{1/2}.$$

4.2 Geodesic equation of a Finsler space with the first approximation Matsumoto metric

Consider the first approximation Matsumoto metric,

$$L = \alpha + \beta + \frac{\beta^2}{\alpha}. \tag{4.11}$$

The partial derivatives of (4.11) with respect to α and β are as follows

$$L_\alpha = 1 - \frac{\beta^2}{\alpha^2}, L_\beta = 1 + \frac{2\beta}{\alpha}, \omega = \frac{2}{\alpha^3}. \tag{4.12}$$

Plugging (4.12) in (3.13), we have

$$\begin{aligned}&\{\alpha^3 - \alpha\beta^2 + 2aE(b_1\dot{y} - b_2\dot{x})^2\}\{a(\dot{x}\ddot{y} - \dot{y}\ddot{x}) + E^2(a_x\dot{y} - a_y\dot{x})\}\\ &\qquad - E^3\{(\alpha^3 + 2\alpha^2\beta)(b_1y - b_2x) + 2a^2(b_1\dot{y} - b_2\dot{x})b_{0;0}\} = 0.\end{aligned} \tag{4.13}$$

Suppose 't' be the parameter of a curve C, then $\dot{x} = 1$, $\dot{y} = y'$, $\ddot{x} = 0$, $\ddot{y} = y''$, $E^2 = 1 + (y')^2$. Then, (4.3) reduces to

$$\begin{aligned}&\{\alpha^3E^2 - \alpha\beta^2 + 2a(b_1y' - b_2)^2\}\{a(y'') + E^2(a_xy' - a_y)\}\\ &\qquad - E^2\{(\alpha^3E^3 + 2a^2E^2\beta)(b_1y - b_2x) + 2a^2(b_1y' - b_2)b_{0;0}\} = 0,\end{aligned} \tag{4.14}$$

where

$$b_{0;0} = (b_{1x} + b_{1y}y') + (b_{2x} + b_{2y}y')y' + \frac{1}{a}\{[1 + (y')^2](a_xb_1 + a_yb_2) - 2(b_1 + b_2y')(a_x + a_yy')\}. \tag{4.15}$$

In a local coordinate system (x, y), the associated Riemannian space which is Euclidean with an orthonormal coordinate system, then $a = 1$, $a_x = a_y = 0$, so that (4.14) is reduced to

$$y'' = \frac{2[1 + (y')^2](b_1y' - b_2)b_{0;0}}{(1 + (y')^2) - (b_1 + b_2y')^2 + 2(b_1y' - b_2)^2}, \tag{4.16}$$

where

$$b_{0;0} = (b_{1x} + b_{1y}y') + (b_{2x} + b_{2y}y')y'. \tag{4.17}$$

If we take b_1 and b_2 such that $b_1 = b_x$ and $b_2 = b_y$ for a scalar b, then $b_{1y} - b_{2x} = 0$. From (4.16) and (4.17), we state that

Theorem 4.4. *Let F^2 be the two dimensional Finsler space with an first approximation Matsumoto metric $L = \alpha + \beta + \frac{\beta^2}{\alpha}$. If we refer to an orthonormal coordinate system (x, y) with respect to α and $b_1 = \partial b/\partial x$, $b_2 = \partial b/\partial y$ for a scalar b, then the differential equation of a geodesic $y = y(x)$ of F^2 is given by*

$$y'' = \frac{A_0 + A_1(y') + A_2(y')^2 + A_3(y')^3 + A_4(y')^4 + A_5(y')^5}{B_0 + B_1 y' + B_2(y')^2}, \tag{4.18}$$

where

$$A_0 = -2b_y b_{xx}, \quad A_1 = (2b_x - 4b_y)b_{xx} - 4b_y b_{xy},$$

$$A_2 = (4b_x - 8b_y)b_{xy} + (4b_x - 2b_y)b_{xy} - 2b_y b_{yy},$$

$$A_3 = 2b_x b_{xx} + 2b_x b_{yy} + (8b_x - 4b_y)b_{xy} - 4b_y b_{yy},$$

$$A_4 = 4b_x b_{xy} + (4b_x - 2b_y)b_{yy}, \quad A_5 = 2b_x b_{yy}$$

$$B_0 = 1 - b_x^2 + 2b_y^2, \quad B_1 = 1 - 6b_x b_y,$$

$$B_2 = 2b_x^2 - b_y^2.$$

5. Main Scalar of Two Dimensional Finsler Space with (α, β)-metric

In this section, we derived the expression for the main scalar of two dimensional Finsler spaces with Matsumoto (α, β)-metric.

Main Scalar I of Two dimensional Finsler Space with Matsumoto metric

Let us consider Matsumoto metric,

$$L = \frac{\alpha^2}{\alpha - \beta}. \tag{5.1}$$

Now, the partial derivatives of (5.1)with respect to α and β are as follows;

$$L_\alpha = \frac{\alpha^2 - 2\alpha\beta}{(\alpha-\beta)^2}, \quad L_{\alpha\alpha} = \frac{2(\alpha-\beta)^3 - 4\alpha(\alpha-\beta)^2 + 2\alpha^2(\alpha-\beta)}{(\alpha-\beta)^4},$$

$$L_\beta = -\frac{\alpha^2}{(\alpha-\beta)^2}, \quad L_{\beta\beta} = \frac{2\alpha^2}{(\alpha-\beta)^3}, \quad L_{\alpha\beta} = \frac{2\alpha^2 - 2\alpha(\alpha-\beta)}{(\alpha-\beta)^3}. \tag{5.2}$$

Plugging (5.2) in (2.7), we have

$$P = \frac{\alpha^3 - 2\alpha^2\beta}{(\alpha-\beta)^3}, \quad P_0 = \frac{3\alpha^4}{(\alpha-\beta)^4},$$

$$P_{-1} = \frac{4\alpha^2\beta - \alpha^3}{(\alpha-\beta)^3}, \quad P_{-2} = \frac{-\alpha^4 + 3\alpha^3\beta - \alpha^2 - 2\alpha^2\beta^2 - 4\alpha\beta + 6\beta^2 + 2\alpha}{(\alpha-\beta)^4}. \tag{5.3}$$

Again Plugging (5.3) in to (2.6), we have

$$T = \frac{C_0\alpha^{10} + C_1\alpha^9 + C_2\alpha^8 + C_3\alpha^7 + C_6\alpha^6 + C_5\alpha^5 + C_6\alpha^4}{(\alpha-\beta)^8}, \tag{5.4}$$

where

$$C_0 = -4b^2, \quad C_1 = 19\beta b^2 - \beta, \quad C_2 = 13\beta^2 12b^4 - 84b^6)\beta^4,$$

$$C_3 = -48\beta^3 - 6\beta - 21\beta b^2 + 40\beta^3 b^2 + 6b^2,$$

$$C_4 = 82\beta^4 + 16\beta^2 + 24\beta^2 b^2 - 16\beta^4 b^2,$$

$$C_5 = -30\beta^5 - 12\beta^3 - 40\beta^5 b^2 - 12\beta^3 b^2 - 6\beta^2 b^2,$$

$$C_6 = 24\beta^6 - 14\beta^4$$

and

$$T_2 = \frac{D_0\alpha^{10} + D_1\alpha^9 + D_2\alpha^8 + D_3\alpha^7 + D_4\alpha^6 + D_5\alpha^5 + D_6\alpha^4}{(\alpha-\beta)^9}, \tag{5.5}$$

where

$$D_0 = -13b^2, \;\; D_1 = 55\beta b^2 + 18\beta, \;\; D_2 = -66\beta^2 - 114\beta^2 b^2 - 21b^2 + 2,$$

$$D_3 = 88\beta^3 + 136\beta^3 b^2 - 99\beta b^2 - 10\beta - 48b^2,$$

$$D_4 = 178\beta^4 - 264\beta^4 b^2 + 60\beta^2 + 108\beta^2 b^2 - 12\beta b^2,$$

$$D_5 = 54\beta^5 - 120\beta^5 b^2 - 116\beta^3 - 60\beta^3 b^2 - 36\beta^2 b^2,$$

$$D_6 = 48\beta^6 - 56\beta^4.$$

Now, the the main scalar of two dimensional space becomes

$$\epsilon I^2 = \frac{(b^2\alpha^4 - \beta^2\alpha^2)}{4(\alpha-\beta)^4}\frac{[D_0\alpha^{10} + D_1\alpha^9 + D_2\alpha^8 + D_3\alpha^7 + D_4\alpha^6 + D_5\alpha^5 + D_6\alpha^4]^2}{[C_0\alpha^{10} + C_1\alpha^9 + C_2\alpha^8 + C_3\alpha^7 + C_6\alpha^6 + C_5\alpha^5 + C_6\alpha^4]^3} \tag{5.6}$$

Thus, we have the following theorem:

Theorem 5.5. *The main scalar of two dimensional Finsler space with* (α,β)*-metric* (5.1) *is given by* (5.6).

6. Conclusion

In two dimensional Finsler space $F^2 = (M^2;L)$, the differential equations of geodesics in a two-dimensional Finsler space with an (α,β)-metric is an interesting geometric quantity and has useful applications in physics. The geodesics of F^2 are the curves of an associated Riemannian space $R^2 = (M^2;\alpha)$ which are bent by the differential 1-form β. In the present paper, we give the derivation of the differential equations of geodesic in two dimensional Finsler space $F^2 = (M^2,L)$ with respect to an isothermal coordinate system. Then, we derived the geodesic equation by considering Finsler metrics to be generalized Matsumoto metric and First approximation matsumoto metric. Also, we obtained the main scalar for Matsumoto metric. Further, by using the main scalar we can prove that the space is Landsberg or not.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

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