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Research Article

Solving Non-Homogeneous Coupled Linear Matrix Differential Equations in Terms of Matrix Convolution Product and Hadamard Product

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Abstract. We investigate a system of coupled non-homogeneous linear matrix differential equations. By applying the diagonal extraction operator, this system is reduced to a simple vector-matrix differential equation. An explicit formula of the general solution is then obtained in terms of matrix convolution product, Hadamard product, and elementary matrix functions. Moreover, we discuss certain special cases of the main system when initial conditions are imposed.

Keywords. Matrix differential equation; Matrix convolution product; Hadamard product; Diagonal extraction operator; Matrix exponential

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1. Introduction

Linear matrix differential equations play important roles in various branches of science which including mathematical physics, statistics, game theory, econometrics, control and system theory (see e.g., [5, 6, 12, 13]). Theory of linear matrix differential equations has been developed by many researchers (see e.g., [1–3, 7, 8, 10, 11, 14] and references therein). The simplest form of non-homogeneous matrix differential equation is the equation

$$X'(t) = AX(t) + U(t). \tag{1}$$

Here, A is a given square matrix, $U(t)$ is a given integrable matrix-valued function and $X(t)$ is an unknown integrable matrix-valued function. In fact, (1) has a general solution given by a one-parameter matrix-valued function [4]

$$X(t) = e^{(t-t_0)A}X(t_0) + e^{(t-t_0)A} * U(t),$$

here $*$ denotes the matrix convolution product. A non-homogeneous Sylvester differential equation

$$X'(t) = AX(t) + X(t)B + U(t)$$

was investigated in [1] under the situation that $X(t)$ is diagonal.

A system of coupled linear matrix differential equations generally takes the form

$$\begin{aligned} X'(t) &= AX(t)B + CY(t)D + U(t), \\ Y'(t) &= EX(t)F + GY(t)H + V(t). \end{aligned} \tag{2}$$

Here, A, B, C, D, E, F, G, H are given constant square matrices, $U(t), V(t)$ are given integrable matrix-valued functions and $X(t)$, $Y(t)$ are unknown integrable matrix-valued functions. The general solution of the system (2) when $E = A$, $F = D$, $G = A$, $H = B$ was obtained in [1] under the assumptions $X(t)$, $Y(t)$ are diagonal and $(A \circ B^T)(C \circ B^T) = (C \circ B^T)(A \circ B^T)$.

In this paper, we investigate the linear system (2) when $E = D$, $F = C$, $G = B$, $H = A$ and the unknowns $X(t)$ and $Y(t)$ are diagonal. Our strategy is to reduce our problem to the simplest form (1) by applying the diagonal extraction operator. Under certain assumptions on constant matrices, an explicit formula of the general solution to this system is obtained in terms of matrix series concerning exponentials and hyperbolic functions. Moreover, we discuss certain special cases of the main system when initial conditions are imposed.

This paper is structured as follows. We provide tools for solving a system of matrix differential equations in Section 2. These includes Hadamard product, the diagonal extraction operator, functions of matrices defined by power series, and matrix convolution product. The main results of the paper will be discussed in Section 3. Conclusion is provided in Section 4.

2. Preliminaries

For any natural numbers m and n, denote by $M_{m,n}$ the set of m-by-n real matrices. When $m = n$, the set $M_{n,n}$ will be written as M_n.

2.1 Hadamard Product and Diagonal Extraction Operator

The Hadamard product of two matrices $A = [a_{ij}] \in M_{m,n}$ and $B = [b_{ij}] \in M_{m,n}$ is defined to be the entry wise product

$$A \circ B = [a_{ij}b_{ij}] \in M_{m,n}.$$

In contrast to the usual product, the Hadamard product is commutative. The diagonal extraction operator of A is defined by

$$\operatorname{Vecd}(A) = [a_{11}\, a_{22}\, \dots\, a_{nn}]^T.$$

This operator is linear, and it is injective on the set of diagonal matrices.

Lemma 1 (see e.g., [1]). *Let $A, B, X \in M_n$ be such that X is diagonal. Then*

$$\operatorname{Vecd}(AXB) = (A \circ B^T)\operatorname{Vecd}(X).$$

2.2 Functions of Matrices Defined by Power Series

Let A be a complex square matrix and let f be an analytic function defined on a region containing the origin and all eigenvalues of A. Then there exists a radius $R > 0$ such that for any $|z| < R$, the following McLaurin series converges:

$$f(z) = \sum_{n=0}^{\infty} \alpha_n z^n,$$

where $\alpha_n = \frac{f^{(n)}(0)}{n!}$ for any $n \in \mathbb{N} \cup \{0\}$. It follows that the matrix series

$$f(A) := \sum_{n=0}^{\infty} \alpha_n A^n$$

converges. Since the exponential function and the hyperbolic sine/cosine functions are entire functions, each of the following matrix series converges:

$$e^A := \sum_{n=0}^{\infty} \frac{1}{n!} A^n,$$

$$\sinh(A) := \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} A^{2n+1},$$

$$\cosh(A) := \sum_{n=0}^{\infty} \frac{1}{(2n)!} A^{2n}.$$

Lemma 2 (see e.g., [15]). *If (X, Y) is a pair of commuting complex square matrices, then $e^{X+Y} = e^X e^Y$.*

2.3 Matrix Convolution Product

In what follows, let Ω be any interval of the form $[0, \infty)$ or $[0, b]$ for some $b > 0$. Recall that a matrix-valued function $A : \Omega \to M_{m,n}$, $A(t) = [a_{ij}(t)]$ is said to be integrable if the real-valued function a_{ij} is integrable for each $i = 1, \dots, m$ and $j = 1, \dots, n$. The convolution of two integrable functions $f : \Omega \to \mathbb{R}$ and $g : \Omega \to \mathbb{R}$ is defined by

$$(f * g)(t) = \int_0^t f(\tau) g(t - \tau)\, d\tau, \quad t \in \Omega.$$

Definition 1. The (usual) convolution product of two integrable matrix-valued functions $A : \Omega \to M_{m,n}$, $A(t) = [a_{ij}(t)]$ and $B : \Omega \to M_{n,p}$, $B(t) = [b_{ij}(t)]$ is defined to be the matrix-valued function $A * B : \Omega \to M_{m,p}$,

$$(A * B)(t) = \left[\sum_{k=1}^{n} a_{ik}(t) * b_{kj}(t)\right] \in M_{m,p}, \quad \text{for each } t \in \Omega.$$

For convenience, we may write $A(t) * B(t)$ instead of $(A * B)(t)$.

See more information about matrix convolution product in [4, 9].

3. Main Results

In this section, we investigate a system of coupled non-homogeneous linear matrix differential equations. By applying the diagonal extraction operator, this system is reduced to a simple vector-matrix differential equation. Under certain assumptions of constant matrices, an explicit formula of the general solution is obtained in terms of matrix convolution product and elementary matrix functions. Moreover, we discuss certain special cases of the main system when initial conditions are imposed. In such case, its solution is uniquely determined.

To begin with, we consider the general system (2).

Lemma 3. *The general solution of the system* (2) *satisfies*

$$\begin{bmatrix} \operatorname{Vecd} X(t) \\ \operatorname{Vecd} Y(t) \end{bmatrix} = e^{(t-t_0)Q} \begin{bmatrix} \operatorname{Vecd} X(t_0) \\ \operatorname{Vecd} Y(t_0) \end{bmatrix} + e^{(t-t_0)Q} * \begin{bmatrix} \operatorname{Vecd} U(t) \\ \operatorname{Vecd} V(t) \end{bmatrix}, \tag{3}$$

where $Q = \begin{bmatrix} A \circ B^T & C \circ D^T \\ E \circ F^T & G \circ H^T \end{bmatrix}$.

Proof. From the system (2), we obtain by Lemma 1 that

$$\begin{aligned} \operatorname{Vecd} X'(t) &= \operatorname{Vecd}\{AX(t)B + CY(t)D + U(t)\} \\ &= (A \circ B^T)\operatorname{Vecd} X(t) + (C \circ D^T)\operatorname{Vecd} Y(t) + \operatorname{Vecd} U(t), \\ \operatorname{Vecd} Y'(t) &= \operatorname{Vecd}\{EX(t)F + GY(t)H + V(t)\} \\ &= (E \circ F^T)\operatorname{Vecd} X(t) + (G \circ H^T)\operatorname{Vecd} Y(t) + \operatorname{Vecd} V(t). \end{aligned}$$

By denoting

$$z(t) = \begin{bmatrix} \operatorname{Vecd} X(t) \\ \operatorname{Vecd} Y(t) \end{bmatrix} \quad \text{and} \quad w(t) = \begin{bmatrix} \operatorname{Vecd} U(t) \\ \operatorname{Vecd} V(t) \end{bmatrix},$$

we have $z'(t) = Qz(t) + w(t)$. Hence, we get

$$z(t) = e^{(t-t_0)Q} z(t_0) + e^{(t-t_0)Q} * w(t),$$

and the formula (3) follows. □

The following lemma is useful for deriving an explicit formula of the solution in the main theorem.

Lemma 4. *For any* $A \in M_n$ *and* $B \in M_m$, *we have*

$$e^{\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}} = \begin{bmatrix} \cosh(A) & \sinh(A) \\ \sinh(B) & \cosh(B) \end{bmatrix}. \tag{4}$$

Proof. Standard techniques in matrix analysis reveal that

$$
\begin{aligned}
e^{\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}} &= \sum_{k \text{ is even}} \frac{1}{k!} \begin{bmatrix} A^k & 0 \\ 0 & B^k \end{bmatrix} + \sum_{k \text{ is odd}} \frac{1}{k!} \begin{bmatrix} 0 & A^k \\ B^k & 0 \end{bmatrix} \\
&= \begin{bmatrix} \sum\limits_{k \text{ is even}} \frac{1}{k!} A^k & \sum\limits_{k \text{ is odd}} \frac{1}{k!} A^k \\ \sum\limits_{k \text{ is odd}} \frac{1}{k!} B^k & \sum\limits_{k \text{ is even}} \frac{1}{k!} B^k \end{bmatrix} \\
&= \frac{1}{2} \begin{bmatrix} \sum\limits_{k=0}^{\infty} \frac{(1+(-1)^k)A^k}{k!} & \sum\limits_{k=0}^{\infty} \frac{(1-(-1)^k)A^k}{k!} \\ \sum\limits_{k=0}^{\infty} \frac{(1-(-1)^k)B^k}{k!} & \sum\limits_{k=0}^{\infty} \frac{(1+(-1)^k)B^k}{k!} \end{bmatrix} = \begin{bmatrix} \cosh(A) & \sinh(A) \\ \sinh(B) & \cosh(B) \end{bmatrix}. \qquad \square
\end{aligned}
$$

From now on, let $A, B, C, D, E, F \in M_n$ be given matrices. We wish to solve certain systems of linear matrix differential equations. In these problems, we are given integrable matrix-valued functions $U, V : \Omega \to M_n$. Unknown matrix-valued functions $X, Y : \Omega \to M_n$ are assumed to be integrable and diagonal.

Theorem 1. *Assume that*

$$
\begin{aligned}
(A \circ B^T)(C \circ D^T) &= (C \circ D^T)(A^T \circ B), \\
(A^T \circ B)(C^T \circ D) &= (C^T \circ D)(A \circ B^T).
\end{aligned} \tag{5}
$$

Then the general solution of the following non-homogeneous system

$$
\begin{aligned}
X'(t) &= AX(t)B + CY(t)D + U(t), \\
Y'(t) &= DX(t)C + BY(t)A + V(t)
\end{aligned} \tag{6}
$$

is given by

$$
\begin{aligned}
\operatorname{Vecd} X(t) = {} & e^{(t-t_0)(A\circ B^T)} \{P_1(t) \operatorname{Vecd} X(t_0) + P_2(t) \operatorname{Vecd} Y(t_0) \\
& + P_1(t) * \operatorname{Vecd} U(t) + P_2(t) * \operatorname{Vecd} V(t)\}, \\
\operatorname{Vecd} Y(t) = {} & e^{(t-t_0)(A^T\circ B)} \{P_2(t)^T \operatorname{Vecd} X(t_0) + P_1(t)^T \operatorname{Vecd} Y(t_0) \\
& + P_2(t)^T * \operatorname{Vecd} U(t) + P_1(t)^T * \operatorname{Vecd} V(t)\}.
\end{aligned} \tag{7}
$$

Here, $P_1(t) = \cosh(t - t_0)(C \circ D^T)$ *and* $P_2(t) = \sinh(t - t_0)(C \circ D^T)$.

Proof. For simplicity, let us denote

$$
z(t) = \begin{bmatrix} \operatorname{Vecd} X(t) \\ \operatorname{Vecd} Y(t) \end{bmatrix}, \quad w(t) = \begin{bmatrix} \operatorname{Vecd} U(t) \\ \operatorname{Vecd} V(t) \end{bmatrix}.
$$

By Lemma 3, the general solution of the system (6) is given by

$$
z(t) = e^{(t-t_0)Q} z(t_0) + e^{(t-t_0)Q} * w(t).
$$

In order to obtain an explicit formula of $e^{(t-t_0)Q}$, denote

$$
R = \begin{bmatrix} A \circ B^T & 0 \\ 0 & A^T \circ B \end{bmatrix} \quad \text{and} \quad S = \begin{bmatrix} 0 & C \circ D^T \\ C^T \circ D & 0 \end{bmatrix}.
$$

Observe that

$$
RS = \begin{bmatrix} 0 & (A \circ B^T)(C \circ D^T) \\ (A^T \circ B)(C^T \circ D) & 0 \end{bmatrix}, \quad SR = \begin{bmatrix} 0 & (C \circ D^T)(A^T \circ B) \\ (C^T \circ D)(A \circ B^T) & 0 \end{bmatrix}.
$$

The assumption (5) now implies that $RS = SR$. Since $Q = R + S$, it follows from Lemmas 2 and 4 that

$$\begin{aligned} e^{(t-t_0)Q} &= e^{(t-t_0)R} e^{(t-t_0)S} \\ &= \begin{bmatrix} e^{(t-t_0)(A\circ B^T)} & 0 \\ 0 & e^{(t-t_0)(A^T\circ B)} \end{bmatrix} \begin{bmatrix} P_1(t) & P_2(t) \\ P_2(t)^T & P_1(t)^T \end{bmatrix} \\ &= \begin{bmatrix} e^{(t-t_0)(A\circ B^T)} P_1(t) & e^{(t-t_0)(A\circ B^T)} P_2(t) \\ e^{(t-t_0)(A^T\circ B)} P_2(t)^T & e^{(t-t_0)(A^T\circ B)} P_1(t)^T \end{bmatrix}. \end{aligned}$$

Thus, an explicit formula for $e^{(t-t_0)Q} * w(t)$ is given by

$$\begin{bmatrix} e^{(t-t_0)(A\circ B^T)} \{P_1(t) * \operatorname{Vecd} U(t) + P_2(t) * \operatorname{Vecd} V(t)\} \\ e^{(t-t_0)(A^T\circ B)} \{P_2(t)^T * \operatorname{Vecd} U(t) + P_1(t)^T * \operatorname{Vecd} V(t)\} \end{bmatrix}$$

Therefore, the general solution of the system (6) is given by (7) □

Once we know $\operatorname{Vecd} X(t)$, we can obtain $X(t)$ due to the injectivity of the diagonal extraction operator. When $U(t) = V(t) = 0$, the system (6) becomes a homogeneous one, and its general solution is simplified to

$$\begin{aligned} \operatorname{Vecd} X(t) &= e^{(t-t_0)(A\circ B^T)} \{[\cosh(t-t_0)(C\circ D^T)] \operatorname{Vecd} X(t_0) + [\sinh(t-t_0)(C\circ D^T)] \operatorname{Vecd} Y(t_0)\}, \\ \operatorname{Vecd} Y(t) &= e^{(t-t_0)(A^T\circ B)} \{[\sinh(t-t_0)(C^T\circ D)] \operatorname{Vecd} X(t_0) + [\cosh(t-t_0)(C^T\circ D)] \operatorname{Vecd} Y(t_0)\}. \end{aligned}$$

In the rest of paper, we discuss certain initial value problems related to the system (7).

Corollary 1. *The solution of the initial value problem*

$$\begin{aligned} X'(t) &= X(t) + CY(t)D + U(t), \\ Y'(t) &= DX(t)C + Y(t) + V(t), \\ X(0) &= E, Y(0) = F \end{aligned}$$

is given by

$$\begin{aligned} \operatorname{Vecd} X(t) = e^t \{ & [\cosh t(C\circ D^T)] \operatorname{Vecd} E + [\sinh t(C\circ D^T)] \operatorname{Vecd} F \\ & + [\cosh t(C\circ D^T)] * \operatorname{Vecd} U(t) + [\sinh t(C\circ D^T)] * \operatorname{Vecd} V(t)\}, \\ \operatorname{Vecd} Y(t) = e^t \{ & [\sinh t(C^T\circ D)] \operatorname{Vecd} E + [\cosh t(C^T\circ D)] \operatorname{Vecd} F \\ & + [\sinh t(C^T\circ D)] * \operatorname{Vecd} U(t) + [\cosh t(C^T\circ D)] * \operatorname{Vecd} V(t)\}. \end{aligned}$$

Proof. This is a special case of Theorem 1 when $A = B = I_n$ and $t_0 = 0$. □

Corollary 2. *Assume that* $(A^T \circ A)(D \circ I_n) = (D \circ I_n)(A^T \circ A)$*. Then, the solution of the initial value problem*

$$\begin{aligned} X'(t) &= AX(t)A + Y(t)D + U(t), \\ Y'(t) &= DX(t) + AY(t)A + V(t), \\ X(0) &= E, Y(0) = F \end{aligned}$$

is given by

$$\operatorname{Vecd} X(t) = e^{t(A^T\circ A)} \{K_1(t) \operatorname{Vecd} E + K_2(t) \operatorname{Vecd} F + K_1(t) * \operatorname{Vecd} U(t) + K_2(t) * \operatorname{Vecd} V(t)\},$$

$$\operatorname{Vecd} Y(t) = e^{t(A^T \circ A)}\{K_2(t)\operatorname{Vecd} E + K_1(t)\operatorname{Vecd} F + K_2(t) * \operatorname{Vecd} U(t) + K_1(t) * \operatorname{Vecd} V(t)\}.$$

Here, $K_1(t) = \operatorname{diag}(\cosh tc_{11}, \dots, \cosh tc_{nn})$ *and* $K_2(t) = \operatorname{diag}(\sinh tc_{11}, \dots, \sinh tc_{nn})$.

Proof. This is a special case of Theorem 1 when $A = B$, $C = I_n$ and $t_0 = 0$. Note that $\cosh t(D \circ I_n) = K_1(t)$ and $\sinh t(D \circ I_n) = K_2(t)$. □

Corollary 3. *Assume that* $(A^T \circ A)(I_n \circ C) = (I_n \circ C)(A^T \circ A)$. *Then, the solution of the initial value problem*

$$X'(t) = AX(t)A + CY(t) + U(t),$$
$$Y'(t) = X(t)C + AY(t)A + V(t),$$
$$X(0) = E, Y(0) = F$$

is given by

$$\operatorname{Vecd} X(t) = e^{t(A^T \circ A)}\{K_1(t)\operatorname{Vecd} E + K_2(t)\operatorname{Vecd} F + K_1(t) * \operatorname{Vecd} U(t) + K_2(t) * \operatorname{Vecd} V(t)\},$$
$$\operatorname{Vecd} Y(t) = e^{t(A^T \circ A)}\{K_2(t)\operatorname{Vecd} E + K_1(t)\operatorname{Vecd} F + K_2(t) * \operatorname{Vecd} U(t) + K_1(t) * \operatorname{Vecd} V(t)\}.$$

Here, $K_1(t)$ *and* $K_2(t)$ *are defined as in the previous corollary.*

Proof. This is a special case of Theorem 1 when $A = B$, $D = I_n$ and $t_0 = 0$. Note that $\cosh t(C \circ I_n) = K_1(t)$ and $\sinh t(C \circ I_n) = K_2(t)$. □

Corollary 4. *The solution of the initial value problem*

$$X'(t) = CY(t) + U(t),$$
$$Y'(t) = X(t)C + V(t),$$
$$X(0) = E, Y(0) = F$$

is given by

$$\operatorname{Vecd} X(t) = K_1(t)\operatorname{Vecd} E + K_2(t)\operatorname{Vecd} F + K_1(t) * \operatorname{Vecd} U(t) + K_2(t) * \operatorname{Vecd} V(t),$$
$$\operatorname{Vecd} Y(t) = K_2(t)\operatorname{Vecd} E + K_1(t)\operatorname{Vecd} F + K_2(t) * \operatorname{Vecd} U(t) + K_1(t) * \operatorname{Vecd} V(t).$$

Here, $K_1(t) = \operatorname{diag}(\cosh tc_{11}, \dots, \cosh tc_{nn})$ *and* $K_2(t) = \operatorname{diag}(\sinh tc_{11}, \dots, \sinh tc_{nn})$.

Proof. This is a special case of Corollary 3 when $A = 0$. □

4. Conclusion

We investigate the following system of coupled non-homogeneous linear matrix differential equations:

$$X'(t) = AX(t)B + CY(t)D + U(t), \qquad Y'(t) = DX(t)C + BY(t)A + V(t),$$

where $(A \circ B^T)(C \circ D^T) = (C \circ D^T)(A^T \circ B)$ and $(A^T \circ B)(C^T \circ D) = (C^T \circ D)(A \circ B^T)$. We obtain an explicit formula of the general solution of this system in terms of the matrix convolution product, the diagonal extraction operator, and elementary matrix functions as follows:

$$\operatorname{Vecd} X(t) = e^{(t-t_0)(A \circ B^T)}\{P_1(t)\operatorname{Vecd} X(t_0) + P_2(t)\operatorname{Vecd} Y(t_0)$$

$$
\begin{aligned}
&\quad + P_1(t) * \operatorname{Vecd} U(t) + P_2(t) * \operatorname{Vecd} V(t)\},\\
\operatorname{Vecd} Y(t) &= e^{(t-t_0)(A^T \circ B)}\{P_2(t)^T \operatorname{Vecd} X(t_0) + P_1(t)^T \operatorname{Vecd} Y(t_0)\\
&\quad + P_2(t)^T * \operatorname{Vecd} U(t) + P_1(t)^T * \operatorname{Vecd} V(t)\}.
\end{aligned}
$$

Here, $P_1(t) = \cosh(t-t_0)(C \circ D^T)$ and $P_2(t) = \sinh(t-t_0)(C \circ D^T)$. In particular, many interesting special cases of the main system are studied.

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Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All the authors contributed significantly in writing this article. The authors read and approved the final manuscript.

References

[1] Z. Al-Zhour, A computationally-efficient solutions of coupled matrix differential equations for diagonal unknown matrices, *J. Math. Sci. Adv. Appl.* **1**(2) (2008), 373 – 387.

[2] Z. Al-Zhour, Efficient solutions of coupled matrix and matrix differential equations, *Intell. Cont. Autom.* **3**(2) (2012), 176 – 187.

[3] Z. Al-Zhour, The general (vector) solution of such linear (coupled) matrix fractional differential equations by using Kronecker structures, *Appl. Math. Comp.* **232** (2014), 498 – 510.

[4] Z. Al-Zhour and A. Kilicman, Some applications of the convolution and Kronecker products of matrices, in *Proceedings of the Simposium Kebangsaan Sains Matematik ke XIII*, 551 – 562 (2005).

[5] G. N. Boshnakov, The asymptotic covariance matrix of the multivariate serial correlations, *Stoch. Proc. Appl.* **65** (1996), 251 – 258.

[6] T. Chen and B. A. Francis, *Optimal Sampled-Data Control Systems*, Springer, London (1995).

[7] L. Jódar and H. Abou-Kandil, Kronecker products and coupled matrix Riccati differential systems, *Linear Algebra Appl.* **121** (1989), 39 – 51.

[8] A. Kilicman and Z. Al-Zhour, The general common exact solutions of coupled linear matrix and matrix differential equations, *J. Anal. Comput.* **1**(1) (2005), 15 – 30.

[9] A. Kilicman and Z. Al-Zhour, On the connection between Kronecker and Hadamard convolution products of matrices and some applications, *Abstr. Appl. Anal.* **2009** (2009), 10 pages, doi:10.1155/2009/736243.

[10] A. Kilicman and Z. Al-Zhour, Note on the numerical solutions of the general matrix convolution equations by using the iterative methods and box convolution products, *Abstr. Appl. Anal.* **2010** (2010), 16 pages, doi:10.1155/2010/106192.

[11] R. Kongyaksee and P. Chansangiam, Solving systems of nonhomogeneous coupled linear matrix differential equations in terms of Mittag-Leffler matrix functions, *J. Comp. Appl. Anal.* **27**(7) (2019), 1150 – 1160.

[12] J.R. Magnus and H. Neudecker, *Matrix Differential Calculus with Applications in Statistics and Econometrics*, John Wiley & Sons, Chichester (2007).

[13] C.R. Rao and M.B. Rao, *Matrix Algebra and Its Applications to Statistics and Econometrics*, World Scientific, Singapore (1998).

[14] S. Saechai and P. Chansangiam, General exact solution to a system of coupled linear matrix differential equations, *Adv. Appl. Math. Sci.* **16**(5) (2017), 151 – 161.

[15] W.H. Steeb and Y. Hardy, *Matrix Calculus and Kronecker Product: A Practical Approach to Linear and Multilinear Algebra*, World Scientific, Singapore (2011).